Asset Trading and Valuation with Uncertain Exposure*

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Abstract

This paper considers an asset market where investors have private information not only about asset payoffs, but also about their own exposure to an aggregate risk factor. In equilibrium, rational investors disagree about asset payoffs: those with higher exposure to the risk factor are more optimistic about claims on the risk factor, which leads to less risk sharing than under symmetric information. Moreover, uncertainty about exposure amplifies the effect of aggregate exposure on asset prices, and can thereby help explain the excess volatility of prices and the predictability of excess returns.

*Preliminary and incomplete! Comments are welcome.

1 Introduction

A lot of modern financial market activity consists of agents trading claims to aggregate risk factors. Examples include secondary markets for government debt, exchange traded index funds, and the burgeoning markets for interest rate, exchange rate and stock index derivatives. According to conventional wisdom, such markets perform two jobs. First, they aggregate dispersed information that market participants have about risk factors. Second they improve the allocation of aggregate risk across market participants. For example, when an investor who initially has a high exposure to some aggregate risk factor sells a claim on the risk factor to an otherwise identical agent with less exposure, the trade can make both agents better off.

In the typical market, participants do not know each other's *exposure* to aggregate risk. Indeed, a market participant cannot easily observe the portfolios and risk attitudes of other participants; he may not even know their identities. This paper examines the effect of uncertain exposures on welfare and asset valuation. We show that uncertain exposures inhibit both jobs commonly assigned to asset markets. On the one hand, dispersed information about asset payoffs is not perfectly aggregated. Instead, in our framework investors rationally disagree about asset payoffs: those with higher exposure to the risk factor are more optimistic about claims on the risk factor. On the other hand, risk is not shared efficiently: optimistic high exposure agents sell too few claims on the risk factor to agents with lower exposure. Moreover, uncertainty about exposure amplifies the effect of aggregate exposure on asset prices.

Our model describes an economy where agents have private information not only about (i) the future realization of a tradable risk factor, but also about (ii) their own current exposure to that risk factor. However, no single agent knows all the relevant information available in the economy, and no single agent knows the aggregate exposure of the economy to the risk factor. In equilibrium, investors disagree about expected future asset payoffs and also about the appropriate risk premia on assets. This is because asset prices cannot simultaneously reveal all information about the risk factor that is relevant for asset payoffs, *and* the aggregate exposure that would dictate risk premia if everyone had the same information.

Investors with higher exposure to the risk factor are more optimistic about assets with high betas (that is, high covariance with the factor) than investors with lower exposure. This is because investors rely on both private and public information to estimate expected asset payoffs and risk premia. On the one hand, investors take individual exposure as a signal of aggregate exposure: high exposure agents believe risk premia are higher than low exposure agents. On the other hand, agents extract from the price a signal about payoffs. Since risk premia drive down prices, high exposure investors rationally view any given price as a better signal about future payoffs than low exposure investors.

There are two key implications. First, optimistic high exposure agents prefer to sell less claims on the risk factor compared to what they would sell in the fully revealing equilibrium, while pessimistic low exposure agents prefer buy less claims compared to the fully revealing equilibrium. In equilibrium there is too little trade, and aggregate risk is not shared efficiently. A second implication is that asset prices reflect less of the information that is actually available in the economy than would be the case with known exposures. At the same time, aggregate exposure itself has a powerful effect on prices regardless of the information structure, since it works through the direct effect of individual endowments and risk attitudes on demand.

A long standing puzzle in empirical asset pricing is why movements in asset prices are not more correlated with movements in expected future payoffs. This puzzle is especially pronounced for aggregate risk prices such as equity indices and long term interest rates. One attractive hypothesis is that there are sizeable changes in the exposure of the marginal investor that move prices, but not expected payoffs. Standard representative agent analysis of this hypothesis has tried to trace changes in exposure to changes in the conditional volatility of fundamentals, but has had only mixed success. This paper shows that uncertain exposures give rise to an amplification mechanism that strengthens the effect of changes in conditional volatility.

2 Model

There are two dates and a continuum of agents of measure 1. At date 1, nature draws a distribution of agent types μ, where an agent's type θ determines his endowment, preferences and information. When agents trade assets at date 1, they know their own type, but not the distribution μ. At date 2, assets pay off a single consumption good, and agents consume. Tradeable risk is captured by an event $\tau \in \{1, 2\}$, drawn by nature at date 2. The assets traded at date 1 are contingent claims on the event τ.

Types and exposure

There is a finite number of agent types, indexed by $\theta \in \Theta$. Types are iid across agents. As a result, $\mu(\theta)$ is both the probability that an individual agent is of type θ and the fraction of agents of type θ in the population. Agents have expected utility preferences over consumption in the good and bad state, with felicity

$$u(c; \theta),$$

where u is continuously differentiable, strictly increasing and strictly concave. An agent's endowment is a vector $\omega(\theta) = (\omega_1(\theta), \omega_2(\theta))' \in \Re^2_{++}$, where $\omega_\tau(\theta)$ is the endowment that agent θ receives when event τ occurs.

Definition 1 *An type θ agent's* exposure *to the event $\tau = 1$ is the log ratio of marginal utilities at the endowment:*

$$e_1(\theta) := \log\left(\frac{u'(\omega_2(\theta); \theta)}{u'(\omega_1(\theta); \theta)}\right)$$

An agent has positive (negative) exposure to the event $\tau = 1$ if his endowment is higher (lower) when the event $\tau = 1$ occurs than when it does not occur. An agent has zero exposure if his endowment is independent of τ. In general, differences in exposure across types can be due to differences in either endowments or risk attitudes.

In our model, the point of having differences in endowments and felicities across types is only to generate differences in exposure. Accordingly, we assume that two agents cannot have different endowments and felicities if they have the same exposure, that is, for all $\theta, \tilde{\theta} \in \Theta$, $e(\theta) = e(\tilde{\theta})$ implies $u(.; \theta) = u(.; \tilde{\theta})$ and $\omega(\theta) = \omega(\tilde{\theta})$. We also assume that exposure takes one of two values: $e_1(\theta) \in \{\bar{e}, \underline{e}\}$ with $\bar{e} > \underline{e}$. The economy is thus always populated by high and low exposure agents.

In addition to determining exposure, an agent's type plays a second role in our setup: it serves as a signal about the event τ. For example, suppose the uncertain distribution μ is correlated with τ in a way that, say, $\mu(\theta)$ is higher than $\mu(\tilde{\theta})$ whenever $\tau = 1$ is more likely than $\tau = 2$. In this case, an agent who learns that his type is θ and not $\tilde{\theta}$ thereby receives a noisy signal that $\tau = 1$ is more likely than $\tau = 2$. In other words, the type is informative about τ. Importantly, our setup allows for type to be informative about τ even if the distribution of endowments and felicities is not correlated with τ. For example, we could have that the type θ contains a pure signal component s that is orthogonal to $e_1(\theta)$ but correlated with the event τ.

Uncertainty about type distributions: aggregate news and average exposure

It remains to specify the joint distribution of the aggregates, that is, the type distribution $\mu \in \Delta(\Theta)$ drawn at date 1 and the event $\tau \in (1, 2)$ drawn at date 2. We define the aggregate state space as $(X, \Xi, \Pr)$, where $X = \Delta(\Theta) \times \{1, 2\}$. The probability Pr governs the joint distribution of the aggregates μ and τ as well as agents' individual types θ. We assume that agents have rational expectations, so that Pr also describes every agent's individual belief. Agents can disagree only if they have different information.

We consider distributions μ that are parametrized by two numbers. First, let $\delta(\mu)$ denote the information carried by μ about the event τ,

$$\delta(\mu) = \Pr\{\tau = 1|\mu\}$$

Agents care only about the event τ. If they were to pool their information, and thus knew the distribution μ, only the parameter $\delta(\mu)$ would be relevant to them. In other words, δ represents the "aggregate news" about τ available at date 1.

Second, let $\varepsilon(\mu)$ denote the fraction of high exposure agents in the population (or, equivalently the probability that an individual agent has high exposure):

$$\varepsilon(\mu) = \Pr\{e_1(\theta) = \bar{e}\}$$

Since endowments and felicities differ only if exposures differ, the parameter ε summarizes the effect of the distribution of types on the distribution of endowments and felicities. In particular, it determines the average exposure in the economy.

We assume that, *given the distribution μ, the pair $(\delta(\mu), \varepsilon(\mu))$ is a sufficient statistic for forecasting an individual's type θ.* This parametrization rules out distributions where agents have unequal information quality. For example, suppose that endowments are constant ($\varepsilon(\mu) = 1$, say), and that some random fraction $\alpha(\mu)$ of the agents is told what the event τ is, whereas the other half receives no signal. It follows that $\delta(\mu) = 1$ if $\tau = 1$ and $\delta(\mu) = 0$ otherwise. Given μ, $(\delta(\mu), \varepsilon(\mu))$ is then not a sufficient statistic for forecasting θ. This is because θ can take the value "no signal". Forecasting θ given μ thus involves $\alpha(\mu)$; this information is lost when attention is restricted to $(\delta(\mu), \varepsilon(\mu))$.

Agent Problem

Since there are two states, only one relative price must be determined in equilibrium. We normalize prices such that $p \in [0, 1]$ is the price of a contingent claim that pays one unit of consumption in state 1, and $1 - p$ is the price of a claim that pays one unit in state 2. Individual information sets at date 1 contain an agent's own type θ as well as the price p. A key feature of the model is that the price not only enters the budget constraint, but also serves as a signal. If $\hat{\delta}$ is an agent's subjective probability of event $\tau = 1$, the agent solves

$$\begin{aligned} &\max_{(c_1,c_2)} \quad \hat{\delta} u(c_1, \theta) + (1 - \hat{\delta}) u(c_2, \theta) \\ &s.t. \quad p c_1 + (1-p) c_2 = w(\theta, p) := p\omega_1(\theta) + (1-p)\omega_2(\theta) \end{aligned} \tag{1}$$

We denote the optimal policy function for this problem by $c^*(\hat{\delta}, \theta, p)$.

Equilibrium

Since there is a continuum of each type of agent, the aggregate demand for assets, and hence the equilibrium price, depend only on the distribution of types μ. We can write the price as $p = P(\mu)$. For any price function P, let

$$\hat{\delta}(\theta, p; P)$$

denote the posterior probability that an agent of type θ assigns to the event $\tau = 1$ if he observes the price function P take the value p. This probability can be derived by Bayes ' rule from the joint distribution of μ, τ and θ, given knowledge of the price function P.

Definition. *A rational expectations equilibrium (REE) consists of a price function* $P : \Delta(\Theta) \to [0,1]$ *and a consumption allocation* $c : \Theta \times \Delta(\Theta) \to \mathbb{R}^2$ *such that:*

1. *The individual consumption plan* $c(\theta,\mu)$ *solves the problem (1) for the price* $p = P(\mu)$ *and the belief* $\delta = \hat{\delta}(\theta, p; P)$*, that is, for every* μ *and* θ*,*

$$c(\theta,\mu) = c^*(\hat{\delta}(\theta, P(\mu); P), \theta; P(\mu)).$$

2. *Markets clear: for every* μ

$$\sum_{\theta\in\Theta} \mu(\theta)\, c(\theta,\mu) = \sum \mu(\theta)\, \omega(\theta).$$

The Revelation of Information by Prices

Since the pair $(\delta(\mu), \varepsilon(\mu))$ is a sufficient statistic for θ, the distribution of agents' individual demands depends only on these parameters. The same is true for the aggregate excess demand at some price p, which can be written as

$$\sum_{\theta\in\Theta} \mu(\theta) \left(c^*(\hat{\delta}(\theta, p; P), \theta, p) - \omega(\theta) \right) =: Z(p; \delta(\mu), \varepsilon(\mu)).$$

It follows that the equilibrium price can also be represented as a function of the parameters, that is, $P(\mu) = \tilde{P}(\delta(\mu), \varepsilon(\mu))$ where $\tilde{P}$ is defined by

$$Z(\tilde{P}(\delta,\varepsilon); \delta, \varepsilon) = 0. \tag{2}$$

Equation (2) illustrates why a fully revealing equilibrium – where agents' beliefs all agree at δ – need not exist in our model. The existence of such an equilibrium requires that agents can infer the relevant aggregate news δ from observing the price and their own type. However, the unobservable distribution μ has two unknown parameters δ and ε, whereas there is only one price signal $\tilde{P}$. In general, inferring δ from $\tilde{P}$ need not be possible. At the same time, there may be a partially revealing equilibrium in which different parameter pairs (δ, ε) lead to the same solution for the price.

Section 3 considers the simplest possible example of a partially revealing equilibrium. We assume that there are only two possible distributions μ, and choose two pairs of parameters $(\delta(\mu), \varepsilon(\mu))$ such that one price solves (2) for both distributions. The example shows when the separate influence of beliefs and exposure on demand allows such a choice, and what beliefs and trades emerge in equilibrium. While this example is useful, it is not sufficient for our purposes. On the one hand, it implies a constant price and we cannot talk about volatility of asset prices. On the other hand, it cannot be robust: it must rely on a knife edge choice of parameters which small perturbations to endowments or preferences would destroy.(reference Radner here.

In Section 4, we consider a version of the model where the parameter δ varies continuously. This implies that there cannot exist a fully revealing equilibrium. The intuition is apparent from (2). Suppose that Z is continuous in δ and that variations in beliefs δ lead to sufficient variation in Z for given average exposure ε and price p. We can then find, for every ε and p, a δ such that $p = \tilde{P}(\delta, \varepsilon)$. But then the price can never reveal δ. In contrast, a partially revealing equilibrium is a robust feature of the economy. In this context we explore the relative importance of news and exposure for asset prices.

Asset Pricing, Beliefs and Aggregate Exposure

Suppose that agents know the distribution μ, so that individual beliefs agree at $\hat{\delta} = \delta(\mu)$ for all types θ. Since there is a complete set of contingent claims for τ-risk and agents maximize expected utility with common beliefs, standard arguments imply the existence of a representative agent with expected utility preferences and the same belief who prices all claims.[1] Let $v_{FI}(c;\mu)$ denote the felicity of this representative agent. It depends on μ, which governs both beliefs and the distribution of endowments and preferences. Since the parameter $\varepsilon(\mu)$ summarizes the distribution of endowments, the aggregate endowment depends on μ only through $\varepsilon(\mu)$: we write

$$\sum_{\theta\in\Theta} \mu(\theta)\,\omega_\tau(\theta) =: \Omega_\tau(\varepsilon(\mu))$$

for the aggregate endowment if the distribution μ is drawn and the event τ occurs.

The equilibrium price with pooled information $P_{FI}(\mu)$ can be read off the representative

[1] See, for example, Duffie (1997), Section 1.4.

agent's marginal rate of substitution at the aggregate endowment:

$$\frac{P_{FI}(\mu)}{1-P_{FI}(\mu)} = \frac{\delta(\mu)}{1-\delta(\mu)} \frac{v'_{FI}(\Omega_1(\varepsilon(\mu));\mu)}{v'_{FI}(\Omega_2(\varepsilon(\mu));\mu)} = \frac{\delta(\mu)}{1-\delta(\mu)} \exp(-E_1(\mu)), \tag{3}$$

where $E_1(\mu)$ is the exposure of the representative agent to the event $\tau = 1$. In general, asset prices depend on both beliefs and aggregate exposure. Other things equal, the relative price of a claim on the event $\tau = 1$ is higher if state 1 is more likely or if the representative agent is less exposed to the event $\tau = 1$.

Aggregate exposure drives risk premia. If $E_1(\theta) = 0$, then there is risk neutral pricing; contingent claims prices are simply the probabilities of their payoff events. More generally, the price of an asset that pays off more in the event $\tau = 1$ than in the event $\tau = 2$ incorporates a risk premium if aggregate exposure to the event $\tau = 1$ is positive. Equation (3) offers another way to think about the information revealed by prices. Agents in a standard full information environment agree on the probability of $\tau = 1$, and therefore also on the risk premium contained in the price. In contrast, agents in a world with uncertain exposure are not sure about either.

3 Inefficient trading: a discrete example

In this section, we explore a discrete version of the model to show how uncertain exposure leads to inefficiency in asset markets. There are two types of agents, with high or low exposure: $\Theta = \{\bar{\theta}, \underline{\theta}\}$, and $e_1(\bar{\theta}) > e_1(\underline{\theta})$. A type distribution is therefore summarized by the fraction of high exposure agents. There are only two possible type distributions. With probability η, nature draws the distribution μ^h with a high number of high exposure agents $\varepsilon^h = \varepsilon(\mu^h)$. With probability $1-\eta$, nature draws the type distribution is μ^l, with a low fraction of high exposure agents $\varepsilon^l = \varepsilon(\mu^l) < \varepsilon^h$.

Since exposure is iid across agents, pooling all agents' information about their own exposure reveals the distribution μ. Moreover, since there are only two states, the news δ carried by the type distribution must be perfectly correlated with ε. We use $\delta^j = \delta(\mu^j)$ for $j = h, l$ to denote the aggregate news about the date 2 event τ that is carried by the distribution μ^j. To summarize, an economy is described

$$\mathcal{E} = \left(\delta^h, \delta^l, \varepsilon^h, \varepsilon^l, \omega\left(\bar{\theta}\right), \omega\left(\underline{\theta}\right), u\left(.;\bar{\theta}\right), u\left(.;\underline{\theta}\right)\right)$$

Since the number of states in $\Delta(\Theta) \times \{1,2\}$ is finite, REE prices are fully revealing for a generic economy $\mathcal{E}$. To illustrate the effect of uncertain exposure on trading, we thus construct nongeneric economies that have nonrevealing equilibria. The economic mechanisms that emerge here are also relevant in the model of Section 4, where $\Delta(\Theta)$ is uncountable and revealing equilibria do not exist. What is special about the example is that, in a nonrevealing equilibrium, the price is constant across distributions μ^j and carries no information at all. Individual beliefs are thus independent of μ^j and depend only on agents' individual types; we write $\hat{\delta}(\theta)$ for the probability that type θ assigns to the event $\tau = 1$. Individual consumption is also independent of μ; we write $c_\tau(\theta)$, suppressing the dependence on μ.

The following proposition constructs economies that have nonrevealing equilibria and characterizes their properties.

Proposition 3.1.

1. *An economy $\mathcal{E}$ can have at most one nonrevealing REE. In a nonrevealing REE,*

   (a) $c_\tau(\theta) = \omega_\tau(\theta)$ *for all* θ, τ *(there is no trade)*

   (b) $\hat{\delta}\left(\bar{\theta}\right) > \hat{\delta}\left(\underline{\theta}\right)$ *(agents with higher exposure to the event* $\tau = 1$ *believe that this event is more likely).*

   (c) $\delta^h > \delta^l$ *(the aggregate news about the event* $\tau = 1$ *is better when there are more agents with higher exposure to the event* $\tau = 1$*)*

2. *The following conditions are equivalent:*

   (a) *there exist* $\delta^h, \delta^l \in (0,1)$ *such that the economy*

   $$\mathcal{E} = \left(\delta^h, \delta^l, \varepsilon^h, \varepsilon^l, \omega\left(\bar{\theta}\right), \omega\left(\underline{\theta}\right), u\left(.;\bar{\theta}\right), u\left(.;\underline{\theta}\right)\right)$$

   has a nonrevealing REE.

(b) *the endowments, felicities and distribution parameters ε satisfy*

$$e_1\left(\bar{\theta}\right) - e_1\left(\underline{\theta}\right) \leq \log\left(\frac{\varepsilon^h}{1-\varepsilon^h}\frac{1-\varepsilon^l}{\varepsilon^l}\right). \tag{4}$$

The proof of part 1 is straightforward. The fact that there cannot be trade in equilibrium is due to the fact that there are two distributions and two types. Indeed, market clearing requires that, for $j = h, l$ and $\tau = 1, 2$,

$$\varepsilon^j\left(c_\tau\left(\bar{\theta}\right) - \omega_\tau\left(\bar{\theta}\right)\right) + \left(1-\varepsilon^j\right)\left(c_\tau\left(\underline{\theta}\right) - \omega_\tau\left(\underline{\theta}\right)\right) = 0.$$

For fixed τ, this equation can only hold for both $j = h$ and $j = l$ if net demands $c_\tau(\theta) - \omega_\tau(\theta)$ are zero for both types, which shows part 1.a.

Under our assumptions on felicities, agents' first order conditions must hold in equilibrium. If agents consume their endowments, this means

$$\frac{p}{1-p} = \frac{\hat{\delta}(\theta)}{1-\hat{\delta}(\theta)}\frac{u'(\omega_1(\theta),\theta)}{u'(\omega_2(\theta),\theta)} = \frac{\hat{\delta}(\theta)}{1-\hat{\delta}(\theta)}\exp(-e(\theta)). \tag{5}$$

In an autarkic equilibrium the type with higher initial exposure to the event $\tau = 1$ must also believe that that event is more likely, than the type with lower exposure, that is $\hat{\delta}\left(\bar{\theta}\right) > \hat{\delta}\left(\underline{\theta}\right)$ (part 1.b). Otherwise, high and low exposure agents cannot both rationalize the observed price.

The existence of an equilibrium requires further that the beliefs in (5) can be derived by Bayesian updating from agents' individual types, which serve as noisy signals of the type distribution. In particular, an agent's subjective probability that $\tau = 1$ must be his conditional expectation of the aggregate news δ, conditional on his type:

$$\begin{aligned}\hat{\delta}\left(\bar{\theta}\right) &= \eta\frac{\varepsilon^h}{\bar{\varepsilon}}\delta^h + (1-\eta)\frac{\varepsilon^l}{\bar{\varepsilon}}\delta^l,\\ \hat{\delta}\left(\underline{\theta}\right) &= \eta\frac{1-\varepsilon^h}{1-\bar{\varepsilon}}\delta^h + (1-\eta)\frac{1-\varepsilon^l}{1-\bar{\varepsilon}}\delta^l,\end{aligned} \tag{6}$$

where we have defined $\bar{\varepsilon} := \eta\varepsilon^h + (1-\eta)\varepsilon^l$, the unconditional probability of type $\bar{\theta}$. Since $\varepsilon^h > \varepsilon^l$ and $\hat{\delta}\left(\bar{\theta}\right) > \hat{\delta}\left(\underline{\theta}\right)$, we must have $\delta^h > \delta^l$ (part 1.c). Moreover, given a pair of δ^js in $(0,1)$ and hence an economy, the formulas (6) deliver a unique pair of posteriors, and (5) the unique nonrevealing equilibrium price.

Intuitively, existence of a nonrevealing equilibrium requires that the probability δ is higher when more agents have high exposure to the event $\tau = 1$. High exposure agents then interpret their type as a signal that the event $\tau = 1$ is more likely. Since they are more optimistic about the event $\tau = 1$, they are happy to consume their endowment at the price p, even though they have higher exposure than other agents (so that gains from trade would exist with symmetric information).

The proof of part 2 is provided in the appendix. It starts from the fact that for given $\hat{\delta}(\theta)$s, (6) can be viewed as a pair of linear equations in $\left(\delta^h, \delta^l\right)$ with a unique solution. The existence problem then amounts to finding a price such that, if the posteriors satisfy (5), then the solutions $\left(\delta^h, \delta^l\right)$ to (6) are indeed between zero and one. Such a price exists if and only if condition (4). The condition requires that requires that there should not be "too much" heterogeneity in individual exposure, relative to the differences in type distributions across states.

4 Uncertain Exposure and Asset Pricing

In this section, we explore the role of uncertain exposure under a more general distribution of aggregate news and aggregate exposure that gives rise to asset price volatility. This is in contrast to the previous section where news and exposure were perfectly correlated, and the price was constant. We retain the assumption that the number of high exposure agents is either ε^h or ε^l, where the former occurs with probability η. However, the parameter $\delta(\mu) = \Pr(\tau = 1|\mu)$, the news contained in agents' pooled information, can now vary over the whole unit interval. The distribution of δ conditional on ε is described by a pair of continuous strictly positive densities $f\left(\delta|\varepsilon^j\right)$ on $[0, 1]$.

We assume that the pooled information is reflected in private signals observed by agents. In particular, every agent receives a private signal $s(\theta) \in \{\bar{s}, \underline{s}\}$ about the event $\tau = 1$. The signals are iid across agents, and independent of exposure. The probability of a "good" signal $\bar{s}$ about $\tau = 1$ is $\delta(\mu)$. By the law of large numbers, a fraction δ of agents thus receive a good signal, while a fraction $1 - \delta$ receive a bad signal. As a result, the value of $\delta(\mu)$ can be recovered if all signals are pooled. In contrast to the previous section, an agent's type θ is now not only identified with an exposure $e(\theta)$, but also with a signal $s(\theta)$. To simplify the notation below, we will simply write $\theta = (s, e)$.

Preferences are restricted to the Linear Risk Tolerance (LRT) class. Felicities are

$$u(c;\theta) = \begin{cases} \frac{\sigma}{\sigma-1}(\alpha(\theta)+\sigma c)^{1-\frac{1}{\sigma}} & \text{if } \sigma \notin \{0,1\} \\ \log(\alpha(\theta)+c) & \text{if } \sigma = 1 \\ -\alpha(\theta)\exp(-c/\alpha(\theta)) & \text{if } \sigma = 0 \text{ and } \alpha(\theta) > 0 \end{cases}$$

The common denominator of these preferences is that risk tolerance $-u'/u''$ (the inverse of the coefficient of absolute risk aversion) is given by the linear function $\alpha(\theta)+\sigma c$. Important special cases of LRT preferences are CRRA utility ($\alpha(\theta) = 0$, with $1/\sigma > 0$ the coefficient of relative risk aversion), CARA utility ($\sigma = 0$, with $\alpha(\theta) > 0$ the coefficient of absolute risk aversion), and quadratic utility ($\sigma = -1$). The coefficient of marginal risk tolerance σ must be equal across agents. However, there can be differences in risk attitude independent of income that are captured by differences in $\alpha(\theta)$. For the case $\sigma > 0$, an intuitive way to think about the coefficient $\alpha(\theta)$ in our context is as a riskless endowment that cannot be traded away.

Full Information Benchmark and Non-Revelation

A convenient feature of the LRT family of preferences is that the price function $P_{FI}(\mu) = \tilde{P}_{FI}(\delta(\mu), \varepsilon(\mu))$ in the full information case is available in closed form. Indeed, there exists a representative agent who has an LRT felicity function with the same coefficient of marginal risk tolerance σ as the individual agents and the average coefficient α:

$$\sum_{\theta\in\Theta} \mu(\theta)\,\alpha(\theta) =: \bar{\alpha}(\varepsilon(\mu)).$$

Here $\bar{\alpha}$ is well defined as a function of ε only, because we have assumed that types with the same exposure have the same felicity function.

For $\sigma \neq 0$, the full information price (3) can now be written as

$$\frac{\tilde{P}_{FI}(\delta,\varepsilon)}{1-\tilde{P}_{FI}(\delta,\varepsilon)} = \frac{\delta}{1-\delta}\left(\frac{\bar{\alpha}(\varepsilon)+\sigma\Omega_1(\varepsilon)}{\bar{\alpha}(\varepsilon)+\sigma\Omega_2(\varepsilon)}\right)^{-\frac{1}{\sigma}} = \frac{\delta}{1-\delta}\exp(-E_1(\varepsilon)). \tag{7}$$

With LRT utility, aggregate exposure E_1 to the event $\tau = 1$ thus depends on the type distribution μ only via the fraction of agents who have high exposure to the event $\tau = 1$; the beliefs δ do not matter. Moreover, it follows from (7) that aggregate exposure is strictly increasing in ε. In other words, the parameter ε can be taken as a measure of aggregate exposure.(NOTE: add exponential case here, or postpone)

The formula (7) also implies that there cannot be a fully revealing REE in which agents learn the relevant pooled information δ. Indeed, suppose there is such a fully revealing equilibrium. Since agents' beliefs agree at δ, the REE price function must be $\tilde{P}_{FI}$. Moreover, agents must have been able to arrive at the belief δ based on their information, that is, their own types and the price. Since an individual's type θ alone does not reveal δ, agents must have been able to invert the price function $\tilde{P}_{FI}$ to infer δ. However, (7) implies that for every price $p \in (0,1)$, there exist $\delta^h, \delta^l \in (0,1)$ with $\delta^h \neq \delta^l$ such that

$$\tilde{P}_{FI}\left(\delta^h, \varepsilon^h\right) = \tilde{P}_{FI}\left(\delta^l, \varepsilon^l\right) = p$$

In other words, agents who see the price p (and their own type) cannot know whether the relevant information carried by the distribution of types is δ^h or δ^l. Since the densities f^j are continuous and strictly positive on $[0,1]$, Bayes' Rule says that agents place positive probability on both distributions, contradicting the fact that their beliefs agree at a single number δ. It follows that P_{FI} cannot be an equilibrium price function when there is asymmetric information.

Beliefs in Partially Revealing Equilibrium

In the simple example of the previous section, the price is constant and conveys no information, so that individual beliefs depend only on individual types. In this section, the price will generally convey some information. To describe inference from prices, it is helpful to define, for $j = h, l$, functions $\tilde{P}^j : [0,1] \to [0,1]$ by $\tilde{P}^j(\delta) := \tilde{P}(\delta, \varepsilon^j)$.We focus on partially revealing equilibria in which the $\tilde{P}_j$ are continuous and strictly increasing in δ. In this case, there are well-defined inverse functions defined by $\delta_j(p) = \tilde{P}_j^{-1}(p)$.

Individual agents want to forecast the event τ and thus the relevant aggregate news δ. If they observe a price p, their knowledge of the price function tells them that the probability of $\tau = 1$ is either $\delta_h(p)$ (if $\varepsilon = \varepsilon^h$) or else $\delta_l(p)$. Let $\hat{\eta}(\theta, p)$ denote the probability that a type θ

agent observing a price p assigns to the event $\varepsilon = \varepsilon^h$. It is given by

$$\hat{\eta}(\theta, p) = \frac{\eta_p \mu_h(\theta; p)}{\eta_p \mu_h(\theta; p) + (1 - \eta_p) \mu_l(\theta; p)}.$$

where η_p reflects the information carried by the price:

$$\eta_p = \frac{\eta \delta_h'(p) f\left(\delta_h(p); \varepsilon^h\right)}{\eta \delta_h'(p) f\left(\delta_h(p); \varepsilon^h\right) + (1 - \eta) \delta_l'(p) f\left(\delta_l(p); \varepsilon^l\right)}.$$

Type θ places higher probability on $\varepsilon = \varepsilon^h$ if his own type or the observed price are more likely to occur when $\varepsilon = \varepsilon^h$ than when $\varepsilon = \varepsilon^l$. His posterior posterior probability of the event $\tau = 1$ is then

$$\hat{\delta}(\theta, p) = \hat{\eta}(\theta, p) \delta^h(p) + (1 - \hat{\eta}(\theta, p)) \delta^l(p).$$

The following proposition shows that a partially revealing equilibrium has two key properties that were already present in the simple example of the previous section. On the one hand, agents with higher exposure to the event $\tau = 1$ believe that the event $\tau = 1$ is more likely. High exposure agents find it more likely that other agents also have high exposure, which would lower the price. As a result, they extract from any given price a better signal of the event $\tau = 1$ than do low exposure agents.

On the other hand, holding fixed the price, the aggregate news about the event $\tau = 1$ is better when there are more agents with higher exposure to the event $\tau = 1$. In the example of the previous section, this property held unconditionally and was assumed exogenously – it was a result of our "reverse engineering" a discrete economy with a nonrevealing equilibrium. Here, in contrast, it is a property of the endogenous price function and holds only conditional on the price. Unconditionally, δ and ε may or may not be negatively correlated, depending on the properties of the densities f.

Proposition 4.1. Consider a nonrevealing equilibrium with a price function $\tilde{P}$ that is continuous and increasing in δ. Then

1. Individual beliefs are ranked by

   (a) $\hat{\delta}(s, \bar{e}, p) > \hat{\delta}(s, \underline{e}, p)$ (for a given signal, agents with higher exposure to the event $\tau = 1$ believe that event to be more likely)

(b) $\hat{\delta}(\bar{s}, e, p) > \hat{\delta}(\underline{s}, e, p)$ (for given exposure, agents with a better signal about the event $\tau = 1$ believe that event to be more likely).

2. $\delta^h(p) > \delta^l(p)$ (holding fixed the price, aggregate news about $\tau = 1$ is better when more agent have high exposure to $\tau = 1$).

Exposure Shocks and Asset Prices

Asset price behavior in a partially revealing equilibrium is best explained in the case of log utility. The proof of Proposition 4.1 shows that, in the log case, the price can be represented by

$$\frac{\tilde{P}(\delta, \varepsilon)}{1 - \tilde{P}(\delta, \varepsilon)} = \frac{\bar{\delta}_2(\varepsilon)}{1 - \bar{\delta}_1(\varepsilon)} \exp(-E_1(\varepsilon)), \tag{8}$$

where $\bar{\delta}_\tau$ is an average of individual agents' beliefs $\hat{\delta}(\theta, p)$ weighted by agents' endowments in the event τ. These averages depend on ε since the latter affects the distribution of endowments.

The functional form is thus very similar to the full information case; in fact, if all agents agree on δ, we are back to (7). The key difference is in the impact of the "true" unobservable news δ on the price. In the full information case, a change in δ directly changes the price. With asymmetric information, δ affects the price only to the extent that it shifts the mean of the (now nondegenerate) distribution of individual beliefs. Since agents' signals of δ are imperfect, the sensitivity of price to δ is typically smaller than in the full information case. As the opposite extreme from full information, consider the case where δ and ε are independent and individual signals are uninformative, i.e. $\bar{s} = \underline{s}$. In this case, all agents would agree at the prior η. The news δ then has no effect, but the effect of ε on the price is the same as in the full information case.

The bottom line is that, with asymmetric information, aggregate news tends to matter less, whereas aggregate exposure tends to matter more for prices. The following proposition shows that that this general intuition holds for the whole LRT class.

Proposition 4.2 Consider a nonrevealing equilibrium with price function $\tilde{P}$ that is continuous and strictly increasing in δ. The equilibrium price depends more strongly on aggregate exposure than in the full information case: for every $\delta \in (0,1)$,

$$\tilde{P}\left(\delta,\varepsilon^{l}\right) > \tilde{P}_{FI}\left(\delta,\varepsilon^{l}\right) > \tilde{P}_{FI}\left(\delta,\varepsilon^{h}\right) > \tilde{P}\left(\delta,\varepsilon^{h}\right).$$

Measured Risk Premia

Consider what would happen if a standard empirical asset pricing study is performed on data from a nonrevealing equilibrium of our asymmetric information model. For this subsection, we assume that all agents' preference can be represented by log utility. Suppose an econometrician assumes a representative agent with log utility. He observes the joint distribution of (d, C, p): dividends, aggregate consumption and the price. The econometrician does not know a priori the information structure of the agent. He is aware of this, and therefore estimates the model by maximum likelihood, allowing for prices to depend on signals about future aggregate consumption and dividends that agents receive at date 1.

An unrestricted estimation will recover true joint distribution, summarized by the number η, the distribution of δ and the price function. In particular, the econometrician will find that movements in ε (i.e. changes in aggregate consumption that are not in dividends) are reflected in the price. He infers from this that the representative agent receives a signal that reveals ε.

However, when the econometrician imposes the cross equation restrictions implied by log preferences, he will reject the model. Satisfying the cross equation restrictions would require that, for all p,

$$\frac{p}{1-p} = \frac{\delta^{j}(p)}{1-\delta^{j}(p)} \frac{\Omega_b\left(\varepsilon^{j}\right)}{\Omega_g\left(\varepsilon^{j}\right)} \quad \text{for } j = h, l.$$

Proposition 4.2 implies that this condition is violated.

Now suppose the econometrician introduces preference shocks to fit the data exactly using his representative agent model. We capture the preference shock by specifying subjective beliefs $\tilde{\delta}^{j}(p)$ which depend on the price as well as on the state j. The econometrician thus determines $\tilde{\delta}^{j}(p)$ such that

$$\frac{p}{1-p} = \frac{\tilde{\delta}^{j}(p)}{1-\tilde{\delta}^{j}(p)} \frac{\Omega_b\left(\varepsilon^{j}\right)}{\Omega_g\left(\varepsilon^{j}\right)} \quad \text{for } j = h, l.$$

Proposition 4.2 implies that $\tilde{\delta}^h(p) = \delta^h(p)$ and $\tilde{\delta}^l(p) < \tilde{\delta}^h(p)$. In other words, the econometrician's model will make agents more optimistic (about $\tau = 1$) in times of high aggregate exposure to the event $\tau = 1$, and more pessimistic in times of low aggregate exposure.

It also interesting to consider the econometrician's belief conditional on the price only. The following proposition shows that the econometrician concludes agents are pessimistic at a price p if aggregate wealth $W(\varepsilon, p)$ is positively correlated with aggregate exposure conditional on the price.

Proposition 4.3 With log utility, the econometrician's belief is more pessimistic conditional on the price if and only if

$$W\left(\varepsilon^l, p\right) > W\left(\varepsilon^h, p\right),$$

that is, there is more wealth in states with less aggregate exposure.

The condition in Proposition 4.3 will in general depend on the price. We have $W\left(\varepsilon^l, p\right) > W\left(\varepsilon^h, p\right)$ iff

$$p\left(\Omega_g\left(\varepsilon^l\right) - \Omega_g\left(\varepsilon^h\right)\right) + (1-p)\left(\Omega_b\left(\varepsilon^l\right) - \Omega_b\left(\varepsilon^h\right)\right) > 0 \tag{9}$$

However, in economies where

$$\left(\Omega_g\left(\varepsilon^l\right) - \Omega_g\left(\varepsilon^h\right)\right)\left(\Omega_b\left(\varepsilon^l\right) - \Omega_b\left(\varepsilon^h\right)\right) > 0, \tag{10}$$

the price does not matter. In such economies, the covariance of aggregate wealth and aggregate exposure determines whether the econometrician finds the asset prices looking "too pessimistic" on average or not.

5 Numerical results

This section presents an example where a partially revealing equilibrium can be found using numerical techniques. The computed equilibrium price function is continuous and strictly increasing in the news parameter δ. We use the numerical solution to explore further the asset pricing properties of the model. In particular, we show that the presence of incomplete information may help to reduce the correlation between stock prices and dividends generated by the model, compared to the correlation that is obtained in a representative agent model—with no informational frictions.

Consider the following interpretation of our endowment and asset structure. There is a tree, shares of which are traded in the market at date 1. The tree pays low dividends (d_1) with probability δ and high dividends (d_2) with probability $1-\delta$. Agents are endowed with trees, but also expect some riskless endowment of goods in period 2. In particular, a fraction ϕ of the population – "the rich" – is endowed with $a+(1-\lambda)\Delta_a$ shares of the tree and a riskless endowment of $b+\lambda\Delta_b$. The other $1-\varepsilon$ agents – "the poor" – are endowed with $a-(1-\lambda)\Delta_a$ shares of the tree and a riskless endowment of $b-\lambda\Delta_b$. To ensure that the rich have more endowment than the poor whatever event τ realizes, we assume that $\Delta_a \in (0,a)$, $\Delta_b \in (0,b)$, and $\lambda \in [0,1]$.

Individual preferences are represented by a logarithmic felicity functions. As a result, the rich have exposure to $\tau = 1$ equal to

$$\log \frac{\omega_2(\theta)}{\omega_1(\theta)} = \log \left(\frac{b+\lambda\Delta_b + (a+(1-\lambda)\Delta_a)\, d_1}{b+\lambda\Delta_b + (a+(1-\lambda)\Delta_a)\, d_2} \right),$$

while the poor's exposure to state 1 is

$$\log \frac{\omega_2(\theta)}{\omega_1(\theta)} = \log \left(\frac{b-\lambda\Delta_b + (a-(1-\lambda)\Delta_a)\, d_1}{b-\lambda\Delta_b + (a-(1-\lambda)\Delta_a)\, d_2} \right).$$

The value of λ determines whether the rich or the poor have higher exposure to the event $\tau = 1$. In particular, for $\lambda < \lambda^* = \frac{a\Delta_a}{a\Delta_a + s\Delta_s}$, poor agents are more exposed to $\tau = 1$ than rich agents. In contrast, for $\lambda > \lambda^* = \frac{a\Delta_a}{a\Delta_a + s\Delta_s}$, poor agents have lower exposure to $\tau = -1$ than rich agents. For $\lambda = \lambda^*$, the exposure of both groups of agents is the same. At that point, the value of ϕ has no effect on prices, and therefore, agents can infer the actual realization of δ from the observed price, that is, prices become fully revealing.

The fraction of rich agents is drawn from a two-point distribution with support $\{\phi_l, \phi_h\}$. We assume $\phi_h = 1 - \phi_l$, so that the fraction of high exposure agents is also either ϕ_h or ϕ_l. We assume $d_1 < d_2$ and thus call the event $\tau = 1$ the "bad" – low dividend – state. The probability δ of a low dividend realization is drawn from a uniform distribution with support $[0,1]$, and is independent of ϕ. The population of agent types thus consists of

d_1	d_2	a	Δ_a	b	Δ_b	ϕ_l	ϕ_h	$Pr(\phi_h)$
0.1	1.0	1.0	0.4	0.7	0.35	0.15	0.85	0.5

Table 1: Parameter values.

1. $\delta\phi$ of agents with high endowments and a bad signal,

2. $(1-\delta)\phi$ of agents with high endowments and a good signal,

3. $\delta(1-\phi)$ of agents with low endowments and a bad signal, and

4. $(1-\delta)(1-\phi)$ of agents with low endowments and a good signal.

The particular parameter values chosen to solve for the model are specified in the table below.

Figure 1 compares the equilibrium prices between an economy with full information and an economy with imperfect information in the case when $\lambda = 1$, that is, when rich agents have a higher exposure to the low dividend state. It illustrates that for the parameterization described in Table 1, a partially revealing equilibrium with a well behaved price function can be found. It also illustrates that in the partially revealing equilibrium, prices are more responsive to the fraction of high exposure agent, compared to the equilibrium with perfect information.

Figure 4 illustrates individual beliefs. As it is shown in proposition 4.1, the beliefs of high exposure agents lie above the beliefs of low exposure agents. Similarly, the beliefs of agents with good signals about state 1 lie above the beliefs of agents with bad signals about state 1.

The result that individuals with high (low) exposure are optimistic (pessimistic) reduces their willingness to decrease (increase) their exposure. This explains why the economy with asymmetric information displays lower trading volume than the economy with full information. Trading volume is shown in Figure 3. The average trading volume under imperfect information is lower than the trading volume under perfect information, except at λ^*, where there is no trade.

Figure 4 illustrates the result presented in proposition 4.3. The graph shows the average belief estimated by an econometrician who observes preferences and a sufficiently large number of realizations of prices, dividends, and exposure distribution across agents, but reads the data

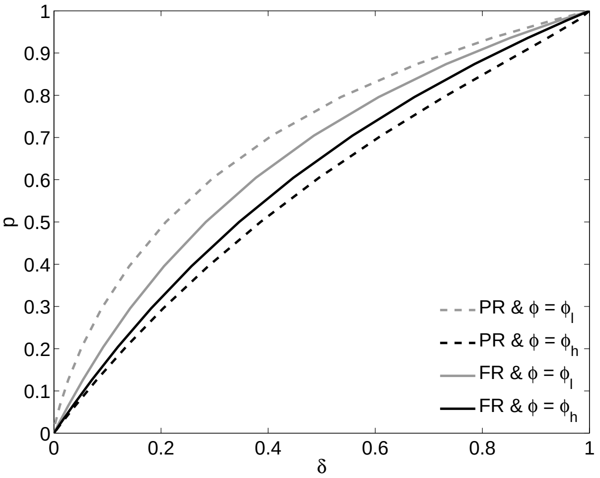

Figure 1: Price functions $p(\delta, \phi)$ in the fully and partially revealing equilibria. The equilibria were found assuming $\lambda = 1$, which means that the fraction of agents with high exposure coincide with the fraction of agents that are highly endowed (ϕ).

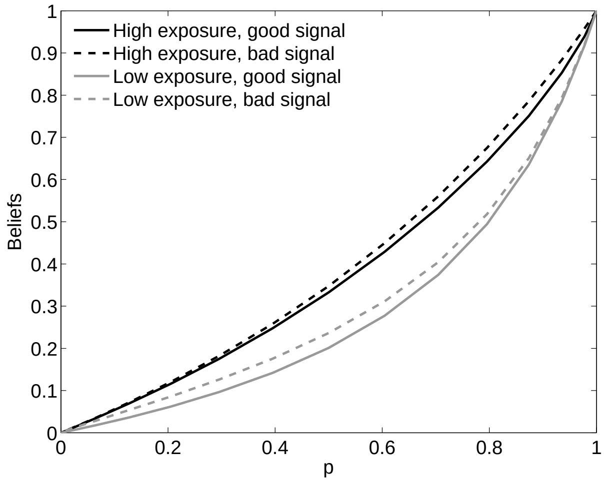

Figure 2: Beliefs about δ in the economy with imperfect information and $\lambda = 1$.

using a representative agent model, assuming that agents have homogeneous beliefs. On average, this approach would underestimate (overestimate) the probability of the low dividend state when the correlation between exposure and endowments is negative (positive). In other words, the econometrician would mistakenly infer that agents are excessively optimistic about the risky asset in an economy with a negative correlation between endowments and exposure ($\lambda < \lambda^*$), while he would infer that agents are excessively pessimistic in an economy with a positive correlation between endowments and exposure ($\lambda > \lambda^*$).

In the economy with imperfect information, prices are more responsive to the magnitude of the fraction of agents with high exposure, which increases the importance of shocks unrelated to the actual dividend probability distribution, and may reduce the correlation between prices and cash flows promised by each asset. Figure 5 shows the correlation between dividends and the price of the tree shares for two economy specifications: with fully revealing prices and with partially revealing prices. The economy with partially revealing prices display a lower correlation between prices and dividends compared to economies with full revelation. The lower correlation

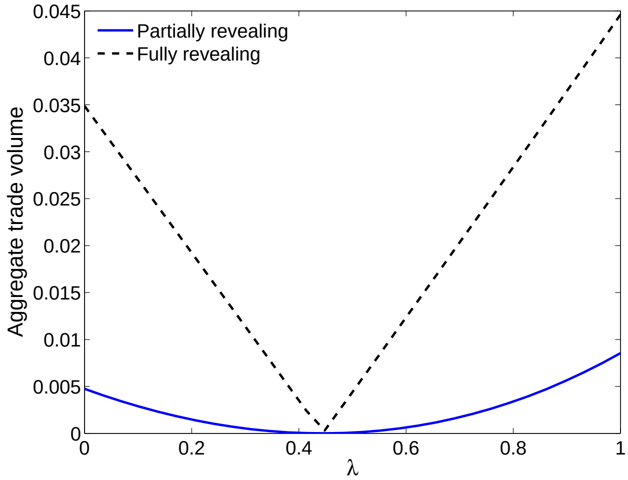

Figure 3: Average aggregate trading volume for different values of λ.

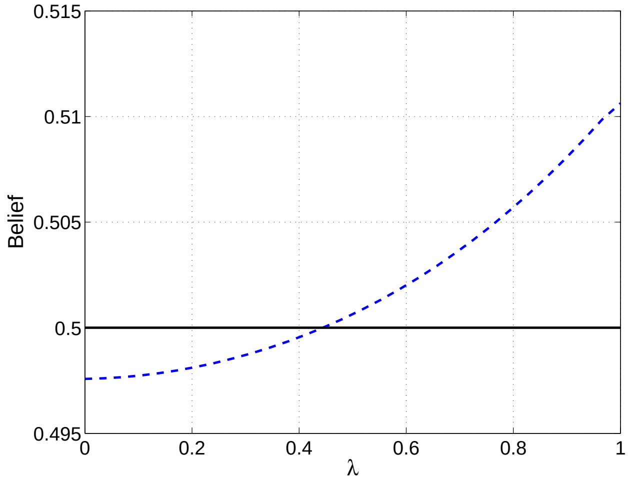

Figure 4: Average belief .

between prices and dividends tends to induce a higher volatility of the price-dividend ratio, as shown in Figure 6.

6 Appendix

Proof of Proposition 3.1, part 2.

We need to establish the existence of $\left(\delta^h, \delta^l\right)$ and a price p such that a price function that is constant at p together with the autarkic allocation constitute an equilibrium in the economy parametrized by the δs.

Given our assumptions on utility, it is optimal for agents to consume their endowment at the price p and for belief $\hat{\delta}(\theta)$ if and only if the the first order conditions

$$\frac{\hat{\delta}(\theta)}{1-\hat{\delta}(\theta)} \exp\left(-e_1(\theta)\right) = \frac{p}{1-p}.$$

hold for every θ. In other words, equilibrium posteriors must be

$$\hat{\delta}(\theta) = \left(1 + \frac{1-p}{p} \exp\left(-e_1(\theta)\right)\right)^{-1}. \tag{11}$$

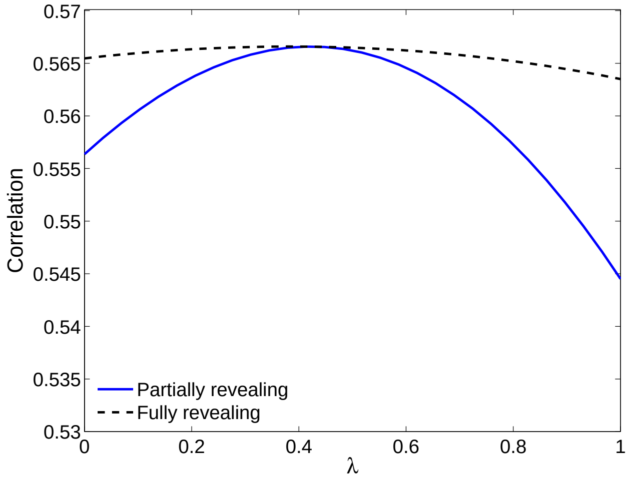

Figure 5: Correlation between aggregate dividends and the tree price.

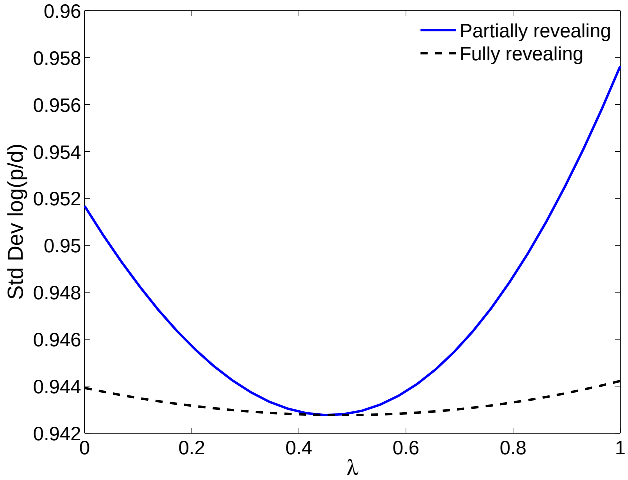

Figure 6: Standard deviation of the logarithm of the price-dividend ratio.

We are done if we can show that there exist $\left(\delta^h, \delta^l\right)$ and p such that the posteriors $\hat{\delta}\left(\theta\right)$ not only satisfy (11), but are also derived from agents' individual types by Bayes' Rule. If this is true, then the $\hat{\delta}\left(\theta\right)$ are also posteriors given a constant, and hence uninformative, price function. (11) thus says that the autarkic allocation is optimal in every state given the price p. Finally, markets clear in all states if each consumer chooses his endowment.

Consider agents' updating given their individual type. Bayes' Rule says

$$\begin{aligned}
\hat{\delta}\left(\bar{\theta}\right) &= \frac{\eta\varepsilon^h\delta^h + \left(1-\eta\right)\varepsilon^l\delta^l}{\eta\varepsilon^h + \left(1-\eta\right)\varepsilon^l} = \eta\frac{\varepsilon^h}{\bar{\varepsilon}}\delta^h + \left(1-\eta\right)\frac{\varepsilon^l}{\bar{\varepsilon}}\delta^l, \\
\hat{\delta}\left(\underline{\theta}\right) &= \frac{\eta\left(1-\varepsilon^h\right)\delta^h + \left(1-\eta\right)\left(1-\varepsilon^l\right)\delta^l}{\eta\left(1-\varepsilon^h\right) + \left(1-\eta\right)\left(1-\varepsilon^l\right)} = \eta\frac{1-\varepsilon^h}{1-\bar{\varepsilon}}\delta^h + \left(1-\eta\right)\frac{1-\varepsilon^l}{1-\bar{\varepsilon}}\delta^l,
\end{aligned}$$

where we have defined $\bar{\varepsilon} = \eta\varepsilon^h + \left(1-\eta\right)\varepsilon^l$.

For fixed p, this can be viewed as a linear equation in $\left(\delta^h, \delta^l\right)$ with unique solution

$$\begin{aligned}
\delta^h &= \frac{\left(1-\varepsilon^l\right)\bar{\varepsilon}\hat{\delta}\left(\bar{\theta}\right) - \varepsilon^l\left(1-\bar{\varepsilon}\right)\hat{\delta}\left(\underline{\theta}\right)}{\eta\left(\varepsilon^h-\varepsilon^l\right)}, \\
\delta^l &= \frac{\varepsilon^h\left(1-\bar{\varepsilon}\right)\hat{\delta}\left(\underline{\theta}\right) - \left(1-\varepsilon^h\right)\bar{\varepsilon}\hat{\delta}\left(\bar{\theta}\right)}{\left(1-\eta\right)\left(\varepsilon^h-\varepsilon^l\right)}.
\end{aligned} \tag{12}$$

We must ensure that δ^h and δ^l are between zero and one. The inequalities $\delta^h > 0$ and $\delta^l > 0$

are equivalent to the two inequalities in

$$\frac{\varepsilon^l}{1-\varepsilon^l} < \frac{\bar{\varepsilon}}{1-\bar{\varepsilon}} \frac{\hat{\delta}\left(\bar{\theta}\right)}{\hat{\delta}\left(\underline{\theta}\right)} < \frac{\varepsilon^h}{1-\varepsilon^h}, \tag{13}$$

respectively. Moreover, the inequalities $\delta^h < 1$ and $\delta^l < 1$ are equivalent to the inequalities in

$$\frac{\varepsilon^l}{1-\varepsilon^l} < \frac{\bar{\varepsilon}}{1-\bar{\varepsilon}} \frac{1-\hat{\delta}\left(\bar{\theta}\right)}{1-\hat{\delta}\left(\underline{\theta}\right)} < \frac{\varepsilon^h}{1-\varepsilon^h}, \tag{14}$$

respectively.

To simplify notation, we write $\bar{\rho} = \exp\left(e_1\left(\bar{\theta}\right)\right)$ and $\underline{\rho}=\exp\left(e_1\left(\underline{\theta}\right)\right)$. From agents' first order conditions, we know

$$\begin{aligned} \frac{\hat{\delta}\left(\bar{\theta}\right)}{\hat{\delta}\left(\underline{\theta}\right)} &= \frac{p+(1-p)/\underline{\rho}}{p+(1-p)/\bar{\rho}} =: f(p) \\ \frac{1-\hat{\delta}\left(\bar{\theta}\right)}{1-\hat{\delta}\left(\underline{\theta}\right)} &= \frac{1-p+p\underline{\rho}}{1-p+p\bar{\rho}} =: g(p) \end{aligned}$$

We want to show that there exists a price $p \in (0,1)$ such that

$$f(p), g(p) \in \left[\frac{\varepsilon^l}{1-\varepsilon^l}\frac{1-\bar{\varepsilon}}{\bar{\varepsilon}}, \frac{\varepsilon^h}{1-\varepsilon^h}\frac{1-\bar{\varepsilon}}{\bar{\varepsilon}}\right] =: \left[\underline{b}, \bar{b}\right]$$

If such a price exists, then the δs in (12) are between zero and one, and therefore p is a nonrevealing equilibrium price for the economy parameterized by those δs. By construction, we have $\bar{b} > 1 > \underline{b}$. This already shows that there exists an equilibrium price if the differences in exposure are not "too large": if $\bar{\rho} = \underline{\rho}$, then $f(p) = g(p) = 1$ for any price. By continuity, an equilibrium also exists for "small enough" heterogeneity. We now establish that condition (4) provides tight bounds for this heterogeneity.

Using the fact that $\bar{\rho} > 1$ and $\bar{\rho} > \underline{\rho}$, it can be verified that the function f is continuous and strictly decreasing for all $p > p_f$, where

$$p_f = -\frac{\bar{\rho}}{\bar{\rho}-1}$$

Furthermore $f(0) = \bar{\rho}/\underline{\rho} > 1$ and $f(1) = 1$ and f tends to $+\infty$ as p tends to p_f from above. It follows that $f(p) \geq \underline{b}$ for all $p \in (0,1)$. and that there exists a unique price $p^u > p_f$ such that $f(p^u) \leq \bar{b}$ for all $p \geq p^u$.

It can also be verified that the function g is continuous and strictly decreasing for all $p > p_g$, where

$$p_g = -\frac{1}{\bar{\rho} - 1} > p_f.$$

Furthermore $g(0) = 1$ and $g(1) = \underline{\rho}/\bar{\rho} < 1$ and g tends to $+\infty$ as p tends to p_g from above. It follows that $g(p) \leq \bar{b}$ for all $p \in (0, 1)$. We also know that $f(p) > g(p)$ for all $p \in (0, 1)$.

It follows that there exists a price in $p \in (0, 1)$ such that $f(p), g(p) \in \left[\underline{b}, \bar{b}\right]$ if and only if $g(p^u) \geq \underline{b}$. Indeed, suppose that $g(p^u) \geq \underline{b}$. Since $f(1) = 1$, we know that $p^u < 1$. If $p^u \in (0, 1)$, then $f(p^u), g(p^u) \in \left[\underline{b}, \bar{b}\right]$. If $p^u < 0$, then $f(0) < \bar{b}$. But we also have $f(0) > g(0) = 1 > \underline{b}$. Using continuity of f and g, we can therefore pick a small positive price p such that $f(p), g(p) \in \left[\underline{b}, \bar{b}\right]$. To show the converse, suppose that $g(p^u) < \underline{b}$. Since $g(0) = 1$, it must be that $p^u > 0$. Since g is decreasing, we have $g(p) < \underline{b}$ for all $p \in [p^u, 1)$. But at the same time, $f(p) > \bar{b}$ for all $p \in (0, p^u)$. As a result there exists no price in the unit interval such that $f(p), g(p) \in \left[\underline{b}, \bar{b}\right]$.

We now show that the condition $g(p^u) \geq \underline{b}$ is equivalent to condition (4). We first solve for p^u from the equation $f(p^u) = \bar{b}$ to find

$$\frac{p^u}{1 - p^u} = \frac{\underline{\rho}^{-1} - \bar{b}\bar{\rho}^{-1}}{\bar{b} - 1}$$

The condition $g(p^u) \geq \underline{b}$ is

$$\frac{\frac{1-p^u}{p^u} + \underline{\rho}}{\frac{1-p^u}{p^u} + \bar{\rho}} \geq \underline{b}.$$

Substituting in for $\frac{p^u}{1-p^u}$ and multiplying the numerator and denominator by $\underline{\rho}^{-1} - \bar{b}\bar{\rho}^{-1}$, we obtain equivalently

$$\frac{\left(\bar{b} - 1\right) + \underline{\rho}\left(\underline{\rho}^{-1} - \bar{b}\bar{\rho}^{-1}\right)}{\left(\bar{b} - 1\right) + \bar{\rho}\left(\underline{\rho}^{-1} - \bar{b}\bar{\rho}^{-1}\right)} \geq \underline{b},$$

which simplifies to

$$\frac{\bar{b}\left(1 - \underline{\rho}/\bar{\rho}\right)}{\bar{\rho}/\underline{\rho} - 1} \geq \underline{b}.$$

and further to

$$\bar{\rho}/\underline{\rho} \leq \bar{b}/\underline{b}$$

Using the definitions of $\bar{\rho}$, $\underline{\rho}$, $\bar{b}$ and $\underline{b}$ we arrive at the condition (4).■

Proof of Proposition 4.1.

As a preliminary step, we establish

Lemma 1. *(a)* $\hat{\delta}(s, \bar{e}, p) > \hat{\delta}(s, \underline{e}, p)$ *for all* s *if and only if* $\delta^h(p) > \delta^l(p)$.
(b) $\hat{\delta}(\bar{s}, e, p) > \hat{\delta}(\underline{s}, e, p)$ *for all* e.

Proof. The individual belief $\hat{\delta}$ can be viewed as an average of δ^h and δ^l,

$$\hat{\delta}(s, e, p) = \hat{\eta}(s, e, p)\,\delta^h(p) + (1 - \hat{\eta}(s, e, p))\,\delta^l(p) \tag{15}$$

where the individual weights are

$$\hat{\eta}(s, e, p) = \frac{\eta_p \mu_h(s, e)}{\eta_p \mu_h(s, e) + (1 - \eta_p)\,\mu_l(s, e)}.$$

By independence of s and e, the population weights are,

$$\begin{aligned}
\mu_j(\bar{s}, \bar{e}; p) &= \delta^j(p)\,\varepsilon^j \\
\mu_j(\bar{s}, \underline{e}; p) &= \delta^j(p)\left(1 - \varepsilon^j\right) \\
\mu_j(\underline{s}, \bar{e}; p) &= \left(1 - \delta^j(p)\right)\varepsilon^j \\
\mu_j(\underline{s}, \underline{e}; p) &= \left(1 - \delta^j(p)\right)\left(1 - \varepsilon^j\right)
\end{aligned}$$

The implication (a) follows from the fact that $\hat{\eta}(s, \bar{e}, p) > \hat{\eta}(s, \underline{e}, p)$. Indeed, that statement is equivalent to

$$\frac{\mu_h(s, \bar{e}; p)}{\mu_l(s, \bar{e}; p)} > \frac{\mu_h(s, \underline{e}; p)}{\mu_l(s, \underline{e}; p)} \tag{16}$$

which is in turn equivalent to

$$\frac{\mu_h(s, \bar{e}; p)}{\mu_h(s, \underline{e}; p)} = \frac{\varepsilon^h}{1 - \varepsilon^h} > \frac{\varepsilon^l}{1 - \varepsilon^l} = \frac{\mu_l(s, \bar{e}; p)}{\mu_l(s, \underline{e}; p)}.$$

To show implication (b), consider first the case $\delta^h > \delta^l$. We want to show that $\hat{\eta}(\bar{s}, e, p) > \hat{\eta}(\underline{s}, e, p)$, which is equivalent to

$$\frac{\mu_h(\bar{s}, e; p)}{\mu_l(\bar{s}, e; p)} > \frac{\mu_h(\underline{s}, e; p)}{\mu_l(\underline{s}, e; p)}. \tag{17}$$

For any e, this is equivalent to

$$\frac{\delta^h}{\delta^l} > \frac{1-\delta^h}{1-\delta^l},$$

and thus holds if and only if $\delta^h > \delta^l$.

In the case $\delta^h < \delta^l$, we want to show that $\hat{\eta}(\bar{s}, e, p) < \hat{\eta}(\underline{s}, e, p)$, that is, the reverse of (17), which holds iff $\delta^h < \delta^l$.■

We now establish **Part 2** of the proposition. **Part 1** then follows immediately from Lemma 1

We begin with the case $\sigma \neq 0$. Start from the market clearing condition for the claim on $\tau = 1$:

$$p \sum_{\theta} \mu_j(\theta; p)\, c_1(\theta; \mu_j) = p\Omega_1\left(\varepsilon^j\right)$$

Multiplying the equation by σ, adding $\bar{\alpha}(\varepsilon^j)$ and rearranging, we obtain

$$p \sum_{\theta} \mu_j(\theta; p)\left(\alpha(\theta) + \sigma c_1(\theta; \mu_j)\right) = p\left(\bar{\alpha}\left(\varepsilon^j\right) + \sigma\Omega_1\left(\varepsilon^j\right)\right)$$

The first order conditions and budget constraint for agent θ can be written as

$$\begin{aligned} \frac{\hat{\delta}(\theta, p)}{1-\hat{\delta}(\theta, p)}\left(\frac{\alpha(\theta)+\sigma c_1}{\alpha(\theta)+\sigma c_2}\right)^{-\frac{1}{\sigma}} &= \frac{p}{1-p} \\ p\left(\alpha(\theta)+c_1\right)+(1-p)\left(\alpha(\theta)+c_2\right) &= w(\theta, p)+\alpha(\theta) \end{aligned}$$

Define the expenditure share in the case of power utility with belief δ by

$$\psi(\delta, p) := \frac{1}{1+\left(\frac{1-p}{p}\right)^{1-\sigma}\left(\frac{1-\delta}{\delta}\right)^{\sigma}}.$$

Combining the first order conditions and the definition of ψ, we can then rewrite the market clearing conditions as

$$\sum_{\theta} \mu_j(\theta; p)\, \psi\left(\hat{\delta}(\theta, p), p\right)\left(\alpha(\theta) + \sigma w(\theta, p)\right) = p\left(\bar{\alpha}\left(\varepsilon^j\right) + \sigma\Omega_g\left(\varepsilon^j\right)\right)$$

Intuitively, the function ψ acts as an "adjusted" expenditure share in a world where endowments and consumption have been linearly translated using the parameters σ and α. Using the definition of wealth, we now have

$$\frac{p}{1-p} = \left(\frac{\bar{\psi}_2\left(\varepsilon^j, p\right)}{1-\bar{\psi}_1(\varepsilon^j, p)}\right)\left(\frac{\bar{\alpha}\left(\varepsilon^j\right)+\sigma\Omega_2\left(\varepsilon^j\right)}{\bar{\alpha}\left(\varepsilon^j\right)+\sigma\Omega_1\left(\varepsilon^j\right)}\right) \quad \text{for } j = h, l. \tag{18}$$

where

$$\bar{\psi}_i\left(\varepsilon^j, p\right) = \sum \mu_j\left(\theta\right)\frac{\alpha\left(\theta\right)+\sigma\omega_i\left(\theta\right)}{\bar{\alpha}\left(\varepsilon^j\right)+\sigma\Omega_i\left(\varepsilon^j\right)}\psi\left(\hat{\delta}\left(\theta, p\right), p\right)$$

is an average of the adjusted expenditure shares ψ formed by weighting individual adjusted expenditure shares by adjusted endowments in states i and j.

Now suppose towards a contradiction that $\delta^h\left(p\right) \le \delta^l\left(p\right)$. By Lemma 1, the individual beliefs are ordered as $\hat{\delta}\left(s, \bar{e}, p\right) \le \hat{\delta}\left(s, \underline{e}, p\right)$ and $\hat{\delta}\left(\bar{s}, e, p\right) \ge \hat{\delta}\left(\underline{s}, e, p\right)$.

The effect of beliefs on the adjusted expenditure shares ψ depends on the sign of σ. We begin with the case $\sigma > 0$. To simplify notation, write $\hat{\psi}\left(e, s, p\right) := \psi\left(\hat{\delta}\left(s, \bar{e}, p\right), p\right)$. If $\sigma > 0$, the function ψ is strictly increasing in δ, which implies $\hat{\psi}\left(s, \bar{e}, p\right) \le \hat{\psi}(s, \underline{e}, p)$ and $\hat{\psi}\left(\bar{s}, e, p\right) \ge \hat{\psi}\left(\underline{s}, e, p\right)$.

The averages $\bar{\psi}_i\left(\varepsilon^j\right)$ can now be ranked, for $i = g, b$:

$$\begin{aligned}
\bar{\psi}_i\left(\varepsilon^h\right) &= \frac{\varepsilon^h\omega_i\left(\bar{e}\right)}{\Omega_i\left(\varepsilon^h\right)}\left[\delta^h\hat{\psi}\left(\bar{s}, \bar{e}, p\right)+\left(1-\delta^h\right)\hat{\psi}\left(\underline{s}, \bar{e}, p\right)\right] \\
&\quad +\frac{\left(1-\varepsilon^h\right)\omega_i\left(\underline{e}\right)}{\Omega_i\left(\varepsilon^h\right)}\left[\delta^h\hat{\psi}\left(\bar{s}, \underline{e}, p\right)+\left(1-\delta^h\right)\hat{\psi}\left(\underline{s}, \underline{e}, p\right)\right] \\
&\le \frac{\varepsilon^l\omega_i\left(\bar{e}\right)}{\Omega_i\left(\varepsilon^l\right)}\left[\delta^h\hat{\psi}\left(\bar{s}, \bar{e}, p\right)+\left(1-\delta^h\right)\hat{\psi}\left(\underline{s}, \bar{e}, p\right)\right] \\
&\quad +\frac{\left(1-\varepsilon^l\right)\omega_i\left(\underline{e}\right)}{\Omega_i\left(\varepsilon^l\right)}\left[\delta^h\hat{\psi}\left(\bar{s}, \underline{e}, p\right)+\left(1-\delta^h\right)\hat{\psi}\left(\underline{s}, \underline{e}, p\right)\right] \\
&\le \frac{\varepsilon^l\omega_i\left(\bar{e}\right)}{\Omega_i\left(\varepsilon^l\right)}\left[\delta^l\hat{\psi}\left(\bar{s}, \bar{e}, p\right)+\left(1-\delta^l\right)\hat{\psi}\left(\underline{s}, \bar{e}, p\right)\right] \\
&\quad +\frac{\left(1-\varepsilon^l\right)\omega_i\left(\underline{e}\right)}{\Omega_i\left(\varepsilon^l\right)}\left[\delta^l\hat{\psi}\left(\bar{s}, \underline{e}, p\right)+\left(1-\delta^l\right)\hat{\psi}\left(\underline{s}, \underline{e}, p\right)\right] \\
&= \bar{\psi}_i\left(\varepsilon^l\right),
\end{aligned} \tag{19}$$

where the first inequality holds because $\hat{\psi}\left(s, \bar{e}, p\right) \le \hat{\psi}(s, \underline{e}, p)$ and $\varepsilon^h > \varepsilon^l$, and where the second inequality holds because $\hat{\psi}\left(\bar{s}, e, p\right) \ge \hat{\psi}\left(\underline{s}, e, p\right)$ and $\delta^h \le \delta^l$.

We also know that aggregate exposure is strictly increasing in ε. If $\sigma > 0$, it thus follows that

$$\frac{\bar{\alpha}\left(\varepsilon^{l}\right)+\sigma\Omega_{2}\left(\varepsilon^{l}\right)}{\bar{\alpha}\left(\varepsilon^{l}\right)+\sigma\Omega_{1}\left(\varepsilon^{l}\right)} > \frac{\bar{\alpha}\left(\varepsilon^{h}\right)+\sigma\Omega_{2}\left(\varepsilon^{h}\right)}{\bar{\alpha}\left(\varepsilon^{h}\right)+\sigma\Omega_{1}\left(\varepsilon^{h}\right)} \tag{20}$$

Putting together inequalities (19) and (20), we have

$$\begin{aligned}\frac{p}{1-p} &= \frac{\bar{\psi}_{2}\left(\varepsilon^{h}\right)}{1-\bar{\psi}_{1}(\varepsilon^{h})}\frac{\bar{\alpha}\left(\varepsilon^{h}\right)+\sigma\Omega_{2}\left(\varepsilon^{h}\right)}{\bar{\alpha}\left(\varepsilon^{h}\right)+\sigma\Omega_{1}\left(\varepsilon^{h}\right)} \\ &< \frac{\bar{\psi}_{2}\left(\varepsilon^{l}\right)}{1-\bar{\psi}_{1}(\varepsilon^{l})}\frac{\bar{\alpha}\left(\varepsilon^{l}\right)+\sigma\Omega_{2}\left(\varepsilon^{l}\right)}{\bar{\alpha}\left(\varepsilon^{l}\right)+\sigma\Omega_{1}\left(\varepsilon^{l}\right)},\end{aligned}$$

which contradicts the equilibrium condition (18).

Now suppose instead that $\sigma < 0$. In this case, the function ψ is decreasing in δ. As a result, we also have $\hat{\psi}\left(s,\bar{e},p\right) \geq \hat{\psi}(s,\underline{e},p)$ and $\hat{\psi}\left(\bar{s},e,p\right) \leq \hat{\psi}\left(\underline{s},e,p\right)$. The inequalities in (19) are thus reversed, and we have $\bar{\psi}_{i}\left(\varepsilon^{h}\right) \geq \bar{\psi}_{i}\left(\varepsilon^{l}\right)$. At the same time, the fact that aggregate exposure is increasing in ε implies that (20) is reversed as well. But then

$$\begin{aligned}\frac{p}{1-p} &= \frac{\bar{\psi}_{2}\left(\varepsilon^{h}\right)}{1-\bar{\psi}_{1}(\varepsilon^{h})}\frac{\bar{\alpha}\left(\varepsilon^{h}\right)+\sigma\Omega_{2}\left(\varepsilon^{h}\right)}{\bar{\alpha}\left(\varepsilon^{h}\right)+\sigma\Omega_{1}\left(\varepsilon^{h}\right)} \\ &> \frac{\bar{\psi}_{2}\left(\varepsilon^{l}\right)}{1-\bar{\psi}_{1}(\varepsilon^{l})}\frac{\bar{\alpha}\left(\varepsilon^{l}\right)+\sigma\Omega_{2}\left(\varepsilon^{l}\right)}{\bar{\alpha}\left(\varepsilon^{l}\right)+\sigma\Omega_{1}\left(\varepsilon^{l}\right)},\end{aligned}$$

which again contradicts the equilibrium condition (18).

Finally, consider now the case of exponential utility $(\sigma = 0)$. The first order conditions for agent θ imply that

$$c_{2}\left(\theta\right)=c_{1}\left(\theta\right)+\alpha\left(\theta\right)\ln\left(\frac{p\left(1-\hat{\delta}\left(\theta,p\right)\right)}{\hat{\delta}\left(\theta,p\right)\left(1-p\right)}\right).$$

This equation, the individual budget constraints, and the market clearing condition for state 1 imply that

$$\sum_{\theta}\mu_{j}\left(\theta;p\right)\left(\omega_{1}\left(\theta\right)-\omega_{2}\left(\theta\right)\right)=\sum_{\theta}\mu_{j}\left(\theta;p\right)\alpha\left(\theta\right)\left[\ln\left(\frac{\hat{\delta}\left(\theta,p\right)}{1-\hat{\delta}\left(\theta,p\right)}\right)-\ln\left(\frac{p}{1-p}\right)\right] \quad \text{for } j=l,h.$$

If $\sigma = 0$, the exposure of a type θ agent is determined by the ratio

$$e_{1}\left(\theta\right)=\frac{\omega_{1}\left(\theta\right)-\omega_{2}\left(\theta\right)}{\alpha\left(\theta\right)}.$$

The previous two equations imply that

$$\ln\left(\frac{p}{1-p}\right)=\sum_{\theta}\mu_j\left(\theta;p\right)\frac{\alpha\left(\theta\right)}{\bar{\alpha}\left(\varepsilon^j\right)}\varphi\left(\hat{\delta}\left(\theta,p\right),p\right)\quad\text{for } j=l,h,$$

where

$$\varphi\left(\delta,p\right)=\ln\left(\frac{\hat{\delta}\left(\theta,p\right)}{1-\hat{\delta}\left(\theta,p\right)}\right)-e_1\left(\theta\right).$$

Assume towards a contradiction that $\delta^h\left(p\right)\leq\delta^l\left(p\right)$. The function φ is strictly increasing in δ and strictly decreasing in e_1, which implies $\varphi\left(s,\bar{e},p\right)<\varphi(s,\underline{e},p)$ and $\varphi\left(\bar{s},e,p\right)\geq\varphi\left(\underline{s},e,p\right)$. Now define $\tilde{\varepsilon}^j$ as

$$\tilde{\varepsilon^j}:=\frac{\varepsilon^j\alpha\left(\bar{e}\right)}{\varepsilon^j\alpha\left(\bar{e}\right)+\left(1-\varepsilon^j\right)\alpha\left(\underline{e}\right)}=\frac{1}{1+\frac{\left(1-\varepsilon^j\right)}{\varepsilon^j}\frac{\alpha\left(\underline{e}\right)}{\alpha\left(\bar{e}\right)}}.$$

For $\alpha\left(\underline{e}\right)$, $\alpha\left(\bar{e}\right)>0$, it is easy to verify that $\tilde{\varepsilon}^j$ is strictly increasing in ε^j.

Therefore,

$$\begin{aligned}
\ln\left(\frac{p}{1-p}\right) &= \tilde{\varepsilon^h}\left[\delta^h\varphi\left(\bar{s},\bar{e},p\right)+\left(1-\delta^h\right)\varphi\left(\underline{s},\bar{e},p\right)\right]+\left(1-\tilde{\varepsilon^h}\right)\left[\delta^h\varphi\left(\bar{s},\underline{e},p\right)+\left(1-\delta^h\right)\varphi\left(\underline{s},\underline{e},p\right)\right]\\
&< \tilde{\varepsilon^l}\left[\delta^h\varphi\left(\bar{s},\bar{e},p\right)+\left(1-\delta^h\right)\varphi\left(\underline{s},\bar{e},p\right)\right]+\left(1-\tilde{\varepsilon^l}\right)\left[\delta^h\varphi\left(\bar{s},\underline{e},p\right)+\left(1-\delta^h\right)\varphi\left(\underline{s},\underline{e},p\right)\right]\\
&\leq \tilde{\varepsilon^l}\left[\delta^l\varphi\left(\bar{s},\bar{e},p\right)+\left(1-\delta^l\right)\varphi\left(\underline{s},\bar{e},p\right)\right]+\left(1-\tilde{\varepsilon^l}\right)\left[\delta^l\varphi\left(\bar{s},\underline{e},p\right)+\left(1-\delta^l\right)\varphi\left(\underline{s},\underline{e},p\right)\right]\\
&= \ln\left(\frac{p}{1-p}\right),
\end{aligned}\qquad(21)$$

where the first inequality holds because $\varphi\left(s,\bar{e},p\right)<\varphi(s,\underline{e},p)$ and $\varepsilon^h>\varepsilon^l$, and where the second inequality holds because $\varphi\left(\bar{s},e,p\right)\geq\varphi\left(\underline{s},e,p\right)$ and $\delta^h\leq\delta^l$.

■

Proof of Proposition 4.2.

Using analogous notation as for the REE price function, we define $P_{FI}^j\left(\delta\right):=P_{FI}\left(\delta,\varepsilon^j\right)$. By (7) in the proof of Proposition 4.1, the functions P_{FI}^j are strictly increasing and thus have well-defined inverse functions $\delta_{FI}^j\left(p\right)=\left(P_{FI}^j\right)^{-1}\left(p\right)$.

We now establish that for all $p \in (0,1)$,

$$\delta^h(p) > \delta^h_{FI}(p) > \delta^l_{FI}(p) > \delta_l(p).$$

Since both $\tilde{P}$ and $\tilde{P}_{FI}$ are strictly increasing and continuous in δ, this proves part 3.

In the full information case, we can follow the same algebra as in the proof of Proposition 4.1. above to arrive at equation (18). If all beliefs are equal at $\hat{\delta}(\theta, p) = \delta^j(p)$, that equation simplifies to

$$\frac{p}{1-p} = \frac{\psi\left(\delta^j_{FI}(p), p\right)}{1-\psi\left(\delta^j_{FI}(p), p\right)} \frac{\bar{\alpha}\left(\varepsilon^j\right) + \sigma\Omega_2\left(\varepsilon^j\right)}{\bar{\alpha}\left(\varepsilon^j\right) + \sigma\Omega_1\left(\varepsilon^j\right)} \quad \text{for } j = h, l. \tag{22}$$

To establish $\delta^h(p) > \delta^h_{FI}(p)$, start again with the case $\sigma > 0$. Since all the $\bar{\psi}_i\left(\varepsilon^j\right)$ are averages of the $\psi\left(\hat{\delta}(\theta, p), p\right)$, and moreover the $\hat{\delta}(\theta, p)$ are averages of δ^h and δ^l, we have $\bar{\psi}_i\left(\varepsilon^h\right) < \max_\theta \psi\left(\hat{\delta}(\theta, p), p\right) < \psi\left(\delta^h, p\right)$ for $i = 1, 2$. Therefore

$$\frac{p}{1-p} < \frac{\psi\left(\delta^h, p\right)}{1-\psi\left(\delta^h, p\right)} \frac{\bar{\alpha}\left(\varepsilon^h\right) + \sigma\Omega_2\left(\varepsilon^h\right)}{\bar{\alpha}\left(\varepsilon^h\right) + \sigma\Omega_1\left(\varepsilon^h\right)}. \tag{23}$$

The definition of $\hat{\delta}^h$ in (22) together with the fact that ψ is strictly increasing thus implies $\hat{\delta}^h < \delta^h$. The argument for $\hat{\delta}^l > \delta^l$ follows analogously from the fact that $\bar{\delta}_i\left(\varepsilon^l\right) > \delta^l$ for $i = 1, 2$.

Now if $\sigma < 0$, the ψs are decreasing in δ, so that

$$\bar{\psi}_i\left(\varepsilon^h\right) > \min_\theta \psi\left(\hat{\delta}(\theta, p), p\right) > \psi\left(\delta^h, p\right)$$

and the inequality (23) is reversed. However, The definition of $\hat{\delta}^h$ in (22) together with the fact that ψ is strictly decreasing once more implies $\hat{\delta}^h < \delta^h$. Again, the argument for $\hat{\delta}^l > \delta^l$ follows analogously from the fact that $\bar{\delta}_i\left(\varepsilon^l\right) > \delta^l$ for $i = 1, 2$.

case $\sigma = 0$**:** to be written

■

Proof of Proposition 4.3. We want to show

$$\eta^h_p \hat{\delta} + (1-\eta_p)\,\hat{\delta}^l < \eta_p \delta^h + (1-\eta_p)\,\delta^l.$$

or equivalently

$$\eta_p \left(\hat{\delta}^h - \delta^h\right) + (1-\eta_p)\left(\hat{\delta}^l - \delta^l\right) < 0$$

Market clearing at the price p is

$$\sum_\theta \mu^j(\theta)\,\psi\left(\hat{\delta}(\theta,p),p\right) w(\theta,p) = p\sum_\theta \mu^j(\theta)\,\omega_g(\theta), \tag{24}$$

where

$$\begin{aligned}
\hat{\delta}(\theta,p) &= \hat{\eta}(\theta,p)\,\delta^h(p) + \left(1-\hat{\eta}(\theta,p)\right)\delta^l(p) \\
\hat{\eta}(\theta,p) &= \frac{\eta_p\mu_h(\theta)}{\eta_p\mu_h(\theta) + (1-\eta_p)\mu_l(\theta)} \\
\eta_p &= \frac{\eta\delta^{h\prime} f(\delta_h)}{\eta\delta^{h\prime} f(\delta_h) + (1-\eta)\,\delta^{l\prime} f(v_l)}
\end{aligned}$$

Multiply the market clearing equation for state h by η_p, multiply that for state l by $(1-\eta_p)$ and add the two equations to get

$$\begin{aligned}
&\eta_p \sum_\theta \mu^h(\theta)\,\frac{\eta_p\mu_h(\theta)\,\delta^h + (1-\eta_p)\,\mu_l(\theta)\,\delta^l}{\eta_p\mu_h(\theta) + (1-\eta_p)\mu_l(\theta)} w(\theta,p) \\
&+ (1-\eta_p)\sum_\theta \mu^l(\theta)\,\frac{\eta_p\mu_h(\theta)\,\delta^h + (1-\eta_p)\,\mu_l(\theta)\,\delta^l}{\eta_p\mu_h(\theta) + (1-\eta_p)\mu_l(\theta)} w(\theta,p) \\
=\;& p\left(\eta_p\sum_\theta \mu^l(\theta)\,\omega_g(\theta) + (1-\eta_p)\sum_\theta \mu^l(\theta)\,\omega_g(\theta)\right)
\end{aligned}$$

Rearranging terms we get

$$\eta_p\delta^h W\left(\varepsilon^h,p\right) + (1-\eta_p)\,\delta^l W\left(\varepsilon^l,p\right) = p\left(\eta_p\Omega_g\left(\varepsilon^h\right) + (1-\eta_p)\,\Omega_g\left(\varepsilon^l\right)\right), \tag{25}$$

where $W(\varepsilon^j,p)$ is aggregate wealth in state j.

Now the subjective beliefs fit by the econometrician satisfy

$$\hat{\delta}^j W\left(\varepsilon^j,p\right) = p\Omega_g\left(\varepsilon^j\right), \qquad j = h,l.$$

We can again multiply the equations for h and l by η_p and $1-\eta_p$, respectively. We get that

$$\eta_p^h\hat{\delta} W\left(\varepsilon^h,p\right) + (1-\eta_p)\,\hat{\delta}^l W\left(\varepsilon^l,p\right) = p\left(\eta_p\Omega_g\left(\varepsilon^h\right) + (1-\eta_p)\,\Omega_g\left(\varepsilon^l\right)\right), \tag{26}$$

Combining (25) and (26), we have

$$\eta_p \left(\hat{\delta}^h - \delta^h\right) W\left(\varepsilon^h, p\right) + (1-\eta_p)\left(\hat{\delta}^l - \delta^l\right) W\left(\varepsilon^l, p\right) = 0$$

But then

$$\begin{aligned} \eta_p \left(\hat{\delta}^h - \delta^h\right) + (1-\eta_p)\left(\hat{\delta}^l - \delta^l\right) &= \eta_p \left(\hat{\delta}^h - \delta^h\right) - (1-\eta_p)\frac{\eta_p \left(\hat{\delta}^h - \delta^h\right) W\left(\varepsilon^h, p\right)}{(1-\eta_p) W\left(\varepsilon^l, p\right)} \\ &= \frac{\eta_p}{W\left(\varepsilon^l, p\right)} \left(\hat{\delta}^h - \delta^h\right)\left(W\left(\varepsilon^l, p\right) - W\left(\varepsilon^h, p\right)\right) \end{aligned}$$

Since $\hat{\delta}^h < \delta^h$ from Proposition 4.2., the condition follows.■