Understanding Self-fulfilling Rational Expectations Equilibria in Real Business Cycle Models*

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Abstract

Necessary conditions for indeterminacy in standard RBC models have been extensively studied, but intuitive understanding of the economic mechanism that generates indeterminacy has yet to be fully explored. Following the permanent income theory, this paper provides an alternative framework for understanding and deriving the technical conditions of indeterminacy in RBC models. A virtue of this approach is that in deriving the conditions of indeterminacy, one can see clearly not only how indeterminacy arises but also how robust the indeterminacy is to structural perturbations in preferences, technologies, and market structures.

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1. Introduction

There has been recently a growing interest in models with multiple, self-ful¯lling rational expectations equilibria (or indeterminacy) in the real-business-cycle (RBC) literature.[1] Although technical conditions for indeterminacy have been extensively studied (e.g., Boldrin and Rustichini, 1994, Benhabib and Rustichini, 1994, and Schmitt-Grohe, 1997), intuitive understanding of the mechanisms that generate indeterminacy in RBC models has yet to be fully explored. The standard way to prove the existence of indeterminacy is to derive conditions for indeterminacy by checking the eigenvalues of the linearized dynamic system of a model. This method can straight forwardly demonstrate the stability of the steady state, it is, however, not constructive in giving the reader the opportunity to understand the economic nature of the multiple equilibrium paths emerging under these conditions. Intuitive interpretations of these mathematical conditions are often hard to give because the eigenvalues bear little apparent relation to the underlying technologies and preferences. Consequently, it is di±cult to provide transparent comparisons among different models and to assess the robustness of indeterminacy under structural perturbations (because the eigenvalues have to be re-derived from the very beginning each time after even a slight model modi¯cation). This di±culty was pointed out in particular, for example, by Boldrin and Rustichini (1994).

A popular approach to provide intuitive understanding on the economic mechanisms that generate indeterminacy is the labor-market diagrammatic analysis introduced by Benhabib and Farmer (1994). According to this approach, indeterminacy (or multiple self-ful¯lling rational expectations equilibria) can arise when the aggregate labor demand curve is upward sloping and is steeper than the labor supply curve. In such a case, an increase in the consumption level that shifts the labor supply curve upward can lead to an increase in the equilibrium employment and output, ratifying the initial rise in consumption and rendering expectations self-ful¯lling.[2]

The labor-market interpretation of indeterminacy is very illustrative, but not

[1] An incomplete list of important works include Beaudry and Devereux (1995), Benhabib and Farmer (1994,1996,1997), Benhabib and Rustichini (1994), Boldrin and Rustichini (1994), Christiano and Harrison (1996), Farmer and Guo (1994), Gali (1994), Perli (1998), Rotemberg and Woodford (1995), Schmitt-Grohe (1997) and Wen (1998), as well as many others. This line of work is closely linked to earlier works done by Azariadis (1981), Cass and Shell (1983), Shell (1977), and Woodford (1986a, 1986b, 1991).

[2] A nice and comprehensive discussion on this approach can be found in Aiyagari (1995).

rigorous. It can also be potentially misleading because it may imply that it would be easier for indeterminacy to arise the closer the slope of the labor demand curve is to that of the labor supply curve. In that case, a slight rise in consumption demand can cause a tremendous increase in equilibrium labor and output. The fact, however, is that it is easier for indeterminacy to arise the greater the slope of the labor demand curve is than that of the labor supply curve (implying larger increases in the equilibrium real wage rather than in the equilibrium employment when the consumption level rises). This is only understandable in light of the permanent income approach adopted here.

In dynamic models with rational expectations, consumption decisions are based on the permanent income, not on the current income. Although an initial rise in consumption can lead to higher labor supply and output in equilibrium if the aggregate labor demand curve is upward sloping and is steeper than the labor supply curve, the exact long-term impact of increasing the current labor supply on permanent income, however, is vague because it depends on the whole trajectory of the equilibrium wage-hours locus, which is difficult to trace by the simple curve-shifting exercises done using the labor supply and demand diagram. For instance, even though current equilibrium labor may rise above the steady state as a result of the increase in consumption under an upward sloping labor demand curve, the next period employment may fall below the steady state as a result of an upward shift of the labor demand curve due to increases in the capital stock. The net effect on total output is thus unclear as it depends on the relative magnitude of movement of the labor supply and demand curves over time, as well as on their relative speed of convergence toward the steady state. Hence, whether a higher initial consumption level can constitute a new equilibrium path or not cannot be precisely determined by the labor-market diagram. A good example is that indeterminacy can no longer arise in the Benhabib-Farmer model when the rate of capital depreciation is sufficiently high. The reason is precisely that a higher rate of capital depreciation reduces the permanent income through the interest-rate effect while leaving the labor market equilibrium condition intact (see Section 5.2 and Proposition 5 below).

By applying the permanent income theory, this paper provides an alternative framework for understanding and deriving the conditions of indeterminacy for standard RBC models. This approach puts the Benhabib-Farmer (1994) interpretation of indeterminacy in terms of the aggregate labor supply and demand on a solid base because it provides a precise method to trace the exact changes in the present value of total income caused by changes in the labor supply. To be more specific, starting with a model's multi-period budget constraint and its intertemporal Euler equation, a relationship between the current consumption and the permanent income is derived. Using the labor market equilibrium condition, the permanent income is then linked to the dynamic path of labor, which can be expressed analytically as a function of the state. Analytical conditions for multiple equilibrium paths (indeterminacy) can then be easily found as depending only on elasticities of labor supply and demand. A virtue of this method is that in deriving these conditions, both the economic mechanism giving rise to indeterminacy and the robustness of indeterminacy under structural perturbations become transparent.

This paper is not the only nor the first attempt to provide a general framework for the intuitive understanding of indeterminacy in infinite-horizon general equilibrium models. An important caveat is that the approach being taken here should not be viewed as a substitute for the existing approaches, but rather as a complement to them. Schmitt-Grohe (1997) compares four models of aggregate fluctuations due to self-fulfilling expectations using the eigenvalue method. The four models share as a common feature increasing returns to scale production technologies and differ in the behavior of marginal costs and markups. Benhabib and Farmer (1994) offer a framework based on an interpretation of the intertemporal Euler equation.[3] According to this framework, indeterminacy arises when the rate of return of an asset and its quantity are able to co-move together. This is possible under individual rationality only when there exist external forces that act to reinforce the decisions of individual agents. This framework, however, does not allow for direct derivations of the analytical conditions for indeterminacy. In contrast, the permanent income approach presented in this paper allows for direct derivations for the analytical conditions of indeterminacy and, at the same time, is more intuitive than the eigenvalue method adopted by Schmitt-Grohe (1994).

The rest of the paper is organized as follows. The permanent income framework is presented in Sections 2-4. Some interesting applications of this methodology are presented in Section 5. Section 6 concludes the paper.

[3] Also see Christiano (1995).

2. The Rise of Indeterminacy

Define the expected life-time utilities of a representative agent as

$$V = E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - n_t), \tag{1}$$

where U is strictly concave in consumption c_t and leisure time $(1 - n_t)$, and $\beta \in (0, 1)$ is the time discount factor. Define the stochastic path of consumption and labor supply as $\{c_t\}_{t=0}^{\infty}$ and $\{n_t\}_{t=0}^{\infty}$; and the intertemporal budget constraint as

$$c_t + E_t \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} R_{t+i}^{-1} \right) c_{t+j} = R_t k_t + w_t n_t + E_t \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} R_{t+i}^{-1} \right) w_{t+j} n_{t+j}, \tag{2}$$

where w_t is the real wage rate (hence $w_t n_t$ is the labor income), R_t is the gross rate of return on capital and is defined as $R_t = r_t + 1 - \delta_t$, with r_t being the marginal product of capital and $\delta_t \in (0, 1]$ the rate of capital depreciation. The right-hand side of the budget constraint defines an agent's permanent income $Y_t^p \equiv R_t k_t + w_t n_t + E_t \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} R_{t+i}^{-1} \right) w_{t+j} n_{t+j}$.

In standard RBC models without market failures, the solution path for consumption (as well as labor) is unique in the sense that V is strictly concave over the domain of $\{c_t\}_{t=0}^{\infty}$ and $\{n_t\}_{t=0}^{\infty}$. This is the so-called saddle-path property. With market failures, such as productive externalities, V may no longer be strictly concave and indeterminacy may arise in the sense that there may exist many sequences of $\{c_t\}_{t=0}^{\infty}$ and $\{n_t\}_{t=0}^{\infty}$ that converge to the same steady state (c^*, n^*).

Since U is strictly concave, multiple equilibria for consumption are possible only if there are external effects on individual's resource-allocation decisions (such as imperfect competition and production externalities) so that different equilibrium consumption paths can be ratified by the expected changes in the permanent income resulting from different labor supply schedules.[4] In other words, multiple equilibrium consumption paths are possible only if the permanent income Y_t^p is indeterminate in equilibrium such that any deviation of c_0 away from c^* is equally optimal due to the

[4] Boldrin and Rustichini (1994) have shown that multiple equilibria are not possible in a standard one-sector RBC model if the labor supply is exogenously fixed.

corresponding changes in the expected permanent income. Therefore, indeterminacy in $\{c_t\}_{t=0}^{\infty}$ is linked to indeterminacy in the permanent income.

3. The Permanent Income and the Labor Market

To illustrate, consider the case where there is no uncertainty and the utility function U takes the standard form[5]:

$$U(c_t, n_t) = \log(c_t) - \psi(n_t); \quad \psi'(n) > 0, \psi'' \geq 0. \tag{3.1}$$

The intertemporal optimality condition for consumption (the Euler equation) is:

$$c_{t+1} = (\beta R_{t+1})\, c_t, \tag{3}$$

which can also be written as:

$$c_{t+j} = \beta^j \left(\prod_{i=1}^{j} R_{t+i} \right) c_t. \tag{4}$$

Inserting (4) into the budget constraint (2) immediately gives:

$$c_t = (1-\beta)\left(R_t k_t + w_t n_t + \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} R_{t+i}^{-1} \right) w_{t+j} n_{t+j} \right), \tag{5}$$

indicating that the representative agent always consumes a constant fraction $(1-\beta)$ of his/her permanent income. Equation (5) is valid for any sequences $\{R_t\}_{t=0}^{\infty}$ and $\{w_t n_t\}_{t=0}^{\infty}$ such that the right side converges. It shows more clearly that indeterminacy in c_t implies indeterminacy in the permanent income.

In order for a higher initial consumption level, c_0, for instance, to constitute another equilibrium path given k_0, the agent needs to be optimistic that the permanent income will also be higher. A higher permanent income may be possible in equilibrium given the same level of initial capital stock if there exists external increasing returns to work. Thus expectations of higher permanent income may be self-fulfilling when a rise in consumption demand that shifts the labor supply curve upward leads to increases in the present value of labor income.

[5] The log utility function is used through out the text for analytical convenience. More general utility functions require linearizations and are therefore not pursued in the paper.

The exact long-term impact of increasing the current labor supply on permanent income, however, is vague because it depends on the whole trajectory of the equilibrium wage-hours locus, which is difficult to trace by simple curve-shifting exercises done using the labor supply and demand diagram (e.g., see Aiyagari, 1995). This difficulty is clearly indicated by Equation (5) where the direction of changes in the permanent income is not easy to be traced as it depends both on the path of the real interest rate $\{R_t\}_{t=0}^{\infty}$ and on the path of the real wage $\{w_t\}_{t=0}^{\infty}$.

4. More Rigorous Treatment

In order to trace the net changes in the present value of total income, we need to go beyond Equation (5). This can be achieved by relating the wage income to the consumption level using the labor market efficiency condition:

$$w_t n_t = c_t \psi'(n_t) n_t. \tag{6}$$

Substituting out the current and future labor incomes in Equation (5) using Equation (6) gives:

$$c_t = (1-\beta)\left(R_t k_t + c_t \psi'(n_t) n_t + \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} R_{t+i}^{-1} \right) c_{t+j} \psi'(n_{t+j}) n_{t+j} \right). \tag{7}$$

Applying Euler's equation (4) to get rid of c_{t+j} then gives:

$$c_t = (1-\beta)\left(R_t k_t + c_t \psi'(n_t) n_t + c_t \sum_{j=1}^{\infty} \beta^j \psi'(n_{t+j}) n_{t+j} \right). \tag{8}$$

After rearranging terms, we get:

$$c_t = \frac{(1-\beta)}{1-(1-\beta)\sum_{j=0}^{\infty} \beta^j \psi'(n_{t+j}) n_{t+j}} R_t k_t. \tag{9}$$

Denote $\tilde{\beta}_t \equiv (1-\beta)\sum_{j=0}^{\infty} \beta^j \psi'(n_{t+j}) n_{t+j}$,[6] Equation (9) can also be expressed as $c_t = (1-\beta)\frac{R_t k_t}{1-\tilde{\beta}_t} = (1-\beta)\sum_{j=0}^{\infty} \tilde{\beta}_t^j R_t k_t$. So the permanent income has an alternative expression: $Y_t^p = \sum_{j=0}^{\infty} \tilde{\beta}_t^j R_t k_t$, which is just the present value of wealth-induced

[6] It can be shown that $0 < \tilde{\beta} < 1$ in the steady state (see e.g., Proposition 2).

income discounted by a dynamic discounting factor that depends on the trajectory of labor supply.

Hence, the permanent income Y_t^p is seen now depending solely on the dynamic path of equilibrium labor supply $\{n\}_{t+j}^{\infty}$ $(j \geq 0)$ and the initial wealth k_t (notice that the gross interest rate R_t depends only on k_t and n_t). Equation (9) also gives the optimal consumption level that becomes proportional to the wealth $R_t k_t$ when the optimal path of labor supply is constant (such as in the case of 100 percent depreciation for the capital stock). When the optimal path of labor supply is not constant, the marginal propensity to consume out of wealth, $\frac{1-\beta}{1-(1-\beta)\sum_{j=0}^{\infty}\beta^j\psi'(n_{t+j})n_{t+j}}$, is a variable depending on the dynamic path of labor. Since the marginal propensity to consume can be rewritten as $\frac{1-\beta}{1-(1-\beta)\sum_{j=0}^{\infty}\beta^j\psi'(n_{t+j})n_{t+j}} = \frac{1}{\frac{1}{1-\beta}-\sum_{j=0}^{\infty}\beta^j\psi'(n_{t+j})n_{t+j}}$ $= \frac{1}{\sum_{j=0}^{\infty}\beta^j[1-\psi'(n_{t+j})n_{t+j}]}$; Equation (9) can be rewritten as:

$$\sum_{j=0}^{\infty}\beta^j\left[1-\psi'(n_{t+j})n_{t+j}\right] = R(k_t,n_t)k_t\frac{\psi'(n_t)n_t}{w(k_t,n_t)n_t}, \tag{10}$$

where the left-hand side is the present value of leisure and the right-hand side is the value of wealth in utility terms (since $u'(c_t) = \frac{1}{c_t} = \frac{\psi'(n_t)n_t}{w_t n_t}$).

Notice that Equation (10) describes the equilibrium path of labor supply as an implicit function of the state, k_t. Once the equilibrium path of labor is determined, the equilibrium paths for consumption and capital stock are also determined in the model. The task of studying the indeterminacy of the model is therefore reduced to the examination of the indeterminacy of the equilibrium path of labor supply.

Proposition 1. If the equilibrium transition path for n_t around the steady state follows a stationary AR(1) process, then the equilibrium is locally unique. If the equilibrium path is not unique, then it must follow a stationary AR(2) process around the steady state.

Proof. By log-linearizing Equation (10) around the steady state (k^*, n^*), we get

$$\sum_{j=0}^{\infty}\varphi_j\hat{n}_{t+j} = \theta\hat{k}_t, \tag{11}$$

where hat variable $\hat{x}_t$ denotes $\log(x_t) - \log(x^*)$, and ϕ_j and θ are elasticity parameters. First, we show that if $\hat{n}_t$ is AR(1), then the equilibrium must be unique. Let

$$\hat{n}_{t+1} = \rho \hat{n}_t. \tag{12}$$

Under repeated iteration, Equation (11) then implies

$$\hat{n}_t = \hat{\theta} \hat{k}_t, \tag{13}$$

where $\hat{\theta}$ is a new constant. This function is just the (linearized) equilibrium decision rule for labor. Thus, given any initial capital stock, there exists a unique value of labor to solve Equation (10) around the steady state.

This means that under indeterminacy, $\hat{n}_t$ must follow an autoregressive process with an order higher than one. Consider the stationary AR(2) process

$$\hat{n}_{t+2} = \rho_1 \hat{n}_{t+1} + \rho_2 \hat{n}_t. \tag{14}$$

Under repeated iteration, Equation (11) then implies

$$\hat{\rho}_1 \hat{n}_{t+1} + \hat{\rho}_2 \hat{n}_t = \hat{k}_t, \tag{15}$$

where $\hat{\rho}_1$ and $\hat{\rho}_2$ are nonzero constants that are functions of the two stable roots in (14). Given $\hat{k}_t$, $\hat{n}_t$ is indeterminate unless $\hat{n}_{t+1}$ is known. Since $\hat{n}_{t+1}$ cannot be solved by backward iteration using (14), it is unknown at time t. The equilibrium is therefore not unique. Another way to see this is to solve for $\hat{n}_t$ explicitly using (14):

$$\hat{n}_t = a(\hat{k}_0, \hat{n}_0)\lambda_1^t + b(\hat{k}_0, \hat{n}_0)\lambda_2^t \tag{16}$$

where a and b are constants depending on the initial values of k_0 and n_0. Since λ_1 and λ_2 are the two stable roots of (14), the initial value $\hat{n}_0$ is clearly indeterminate.

Notice that $\hat{n}_t$ cannot follow a higher order autoregressive process than AR(2) in the model because the state space of the model has the dimension of at most two, including one state variable and one co-state variable.[7] Q.E.D.

Proposition 2. The labor elasticity of the present value of leisure on the left-hand side of (10) is negative.

[7] The forecasting errors in the model cannot depend on lagged state variables because they must be i.i.d as specified by the intertemporal Euler equation.

Proof. Log-linearizing the left-hand side of (10) around the steady state gives:

$$(1-\beta)(1+\xi_a)\frac{-\psi'(n^*)n^*}{1-\psi'(n^*)n^*}\left(\hat{n}_t+\beta\hat{n}_{t+1}+\beta^2\hat{n}_{t+2}+\dots\right), \tag{17}$$

where $\xi_a \geq 0$ is the inverse of the labor supply elasticity. The steady-state value $\psi'(n^*)n^*$ is less than one because the labor market equilibrium condition implies

$$\psi'(n^*)n^* = \frac{w^*n^*}{c^*} = \frac{(1-a)}{(1-s)} < 1, \tag{18}$$

where $(1-a)$ is the steady-state labor's share of national income and $(1-s)$ is the steady-state consumption to output ratio. The last inequality in (18) comes from the fact that the steady state savings to output ratio, s, is given by $s = \delta\frac{k^*}{y^*} = \frac{\delta a\beta}{1-\beta(1-\delta)}$, where a is the steady-state capital's share of national income and δ is the steady state rate of capital depreciation. It is easy to show that $s < a$. Hence, the sign of $\frac{-\psi'(n^*)n^*}{1-\psi'(n^*)n^*}$ is negative.

Finally, it remains to be shown that

$$\text{sign}\{\hat{n}_t+\beta\hat{n}_{t+1}+\beta^2\hat{n}_{t+2}+\dots\} = \text{sign}\{\hat{n}_t\}. \tag{20}$$

Notice that the equilibrium path $\{\hat{n}_t\}_{t=0}^{\infty}$ is a convergent sequence with an autoregressive order of at most 2 (Proposition 1), namely,

$$\hat{n}_{t+1} = \rho_1\hat{n}_t + \rho_2\hat{n}_{t-1}, \tag{21}$$

where ρ_1 and ρ_2 must satisfy $1-\rho_1-\rho_2 > 0$ to guarantee stationarity (the case of AR(1) is just a special case with $\rho_2 = 0$). Define the two eigenvalues of this second-order system as λ_1 and λ_2, which satisfy $\lambda_1+\lambda_2 = \rho_1$, $\lambda_1\lambda_2 = -\rho_2$. It is easy to show that

$$\hat{n}_{t+1+j} = \frac{\lambda_1^{j+2}-\lambda_2^{j+2}}{\lambda_1-\lambda_2}\hat{n}_t + \frac{\lambda_1\lambda_2^{j+2}-\lambda_2\lambda_1^{j+2}}{\lambda_1-\lambda_2}\hat{n}_{t-1}, \quad j \geq 0. \tag{22}$$

Successive iteration then gives:

$$\begin{aligned} &\{\hat{n}_t+\beta\hat{n}_{t+1}+\beta^2\hat{n}_{t+2}+\dots\} \qquad (23)\\ =\ & \frac{1}{1-\beta(\lambda_1+\lambda_2)+\beta^2\lambda_1\lambda_2}\hat{n}_t - \frac{\beta\lambda_1\lambda_2}{1-\beta(\lambda_1+\lambda_2)+\beta^2\lambda_1\lambda_2}\hat{n}_{t-1}\\ =\ & \frac{1}{1-\beta\rho_1-\beta^2\rho_2}\hat{n}_t + \frac{\beta\rho_2}{1-\beta\rho_1-\beta^2\rho_2}\hat{n}_{t-1}. \end{aligned}$$

Since $1-\rho_1-\rho_2>0$ implies $1-\beta\rho_1-\beta^2\rho_2>0$, the sign of $\{\hat{n}_t+\beta\hat{n}_{t+1}+\beta^2\hat{n}_{t+2}+...\}$ is thus determined by the sign of $\hat{n}_t$ (this obviously applies to the case of AR(1) where $\rho_2=0$). Notice that the sign of $\hat{n}_{t-1}$ is irrelevant because one can always choose to start from the steady state where $\hat{n}_{t-1}=0$. Hence, the claim that the elasticity of labor on the left-hand side of (10) is negative is proved. Q.E.D.

Proposition 3. In the absence of any market distortions such as externalities, the equilibrium of the model is locally unique.

Proof. It is sufficient to show that given the steady state level of capital, k^*, there is no n_t other than the steady-state n^* that satisfies (10). We prove this by contradiction. Suppose that there exists a $n_t>n^*$ such that (10) is satisfied, then:

$$\sum_{j=0}\beta^j\left[1-\psi'(n_{t+j})n_{t+j}\right]=R(k^*;n_t)k^*\frac{\psi'(n_t)n_t}{w(k^*;n_t)n_t}. \tag{24}$$

By Proposition 2, the left-hand side must satisfy the following inequality:

$$\sum_{j=0}\beta^j\left[1-\psi'(n_{t+j})n_{t+j}\right]<\sum_{j=0}\beta^j\left[1-\psi'(n^*)n^*\right]. \tag{25}$$

In the absence of market distortions, the gross interest rate R_t (as a function of the marginal product of capital) is increasing in n_t and the real wage w_t is decreasing in n_t, therefore we also have the following inequality for the right-hand side:

$$R(k^*;n_t)k^*\frac{\psi'(n_t)n_t}{w(k^*;n_t)n_t}>R^*k^*\frac{\psi'(n^*)n^*}{w^*n^*}, \tag{26}$$

implying

$$\begin{aligned}\sum_{j=0}\beta^j\left[1-\psi'(n_{t+j})n_{t+j}\right] &< \sum_{j=0}\beta^j\left[1-\psi'(n^*)n^*\right] \\ &= R^*k^*\frac{\psi'(n^*)n^*}{w^*n^*}<R(k^*;n_t)k^*\frac{\psi'(n_t)n_t}{w(k^*;n_t)n_t}.\end{aligned} \tag{27}$$

This is a contradiction. Q.E.D.

Proposition 4. A necessary and sufficient condition for indeterminacy is that the labor elasticities on both sides of Equation (10) are the same.

Proof. Starting from an equilibrium path, a small deviation of n_t away from this path will not violate Equation (10) if and only if the changes on both sides of (10) brought about by the deviation in n_t exactly offset each other. In such a case, n_t is indeterminate given the state k_t. This is the same as saying that the labor elasticities on both sides of (10) are the same. Q.E.D.

Remark 1. The set of equilibria under indeterminacy is a continuum. For example, if $\{n_t\}_{t=0}^{\infty}$ is another equilibrium path in addition to the steady-state path, $\{n^*\}_{t=0}^{\infty}$; given the steady-state capital stock k^* as an initial point, then any convex combination between this path and the steady state path is also an equilibrium path.

Proposition 5. A necessary condition for indeterminacy is

$$\xi_w > \xi_a + \xi_R; \tag{28}$$

where ξ_a is the inverse labor-supply elasticity, ξ_R is the labor elasticity of gross interest, and ξ_w is the labor elasticity of real wages. That is, in order for indeterminacy to arise, the labor elasticity of wages must be greater than the labor elasticity of interest plus the inverse labor-supply elasticity.

Proof. Since the labor elasticity of the present value of leisure on the left-hand side of Equation (10) is negative (Proposition 2), then a necessary condition for indeterminacy is obviously that the labor elasticity of the value of wealth on the right-hand side of (10) is also negative. This elasticity term is:

$$(1 + \xi_a) + \xi_R - (1 + \xi_w); \tag{29}$$

which would be negative if and only if $\xi_w > \xi_a + \xi_R$. Q.E.D.

Remark 2. Since ξ_w can be interpreted as the slope of the aggregate labor demand curve and ξ_a the slope of the labor supply curve in the Benhabib-Farmer (1994) model, this necessary condition says that indeterminacy requires the labor demand curve to cut through the labor supply curve from below. But what this condition implies in general is that the equilibrium wage-hours loci need to have a slope that is greater than that of the labor supply curve taking into account the interest rate effect. Hence, it is not how much the equilibrium labor can increase, but how much the equilibrium real wage can increase as a result of an upward shift of the labor supply curve that is crucial for the rise of indeterminacy. This is why in the Benhabib-Farmer model, the slope of the labor demand curve is required to be sufficiently steeper than that of the labor supply curve, because only then can there exist a sufficiently positive labor elasticity of wages. Notice that indeterminacy may also arise in models with a downward sloping aggregate labor demand curve, because $\xi_w > 0$ does not have to imply that the slope of the aggregate labor demand curve is positive (ξ_w is simply the slope of the equilibrium wage-hour loci). In addition, ξ_w does not even have to be positive if $\xi_a + \xi_R < 0$ (e.g., if the aggregate labor supply curve is downward sloping or if ξ_R is sufficiently negative).

5. Some Interesting Examples

This section demonstrates the usefulness of the permanent income approach to indeterminacy by applying it to several representative models in the literature. Since the necessary condition given by (28) in Proposition 5 is easier to analyze and to interpret than the sufficient condition given in Proposition 4, we shall consider only the necessary condition of indeterminacy in these models.[8]

5.1. The Standard RBC Model

Consider the standard RBC model with preferences and technology specified as: $U(c_t; n_t) = \log(c_t) - \alpha\frac{n_t^{1+\gamma}}{1+\gamma}$; $\gamma \geq 0$; and $y_t = k_t^a n_t^{1-a}$: Since the wage rate $w_t = (1-a)k_t^a n_t^{-a}$; the interest rate $r_t = a k_t^{a-1} n_t^{1-a}$ and the gross interest rate $R = (1-\delta+r) = \frac{1}{\beta}$ in the steady state, we have: $\xi_r = (1-a)$; $\xi_w = -a$; $\xi_R = (1-\beta(1-\delta))\xi_r$; and $\xi_a = \gamma$: Clearly, this model economy has a unique equilibrium because the necessary condition (28) for multiple equilibria is not satisfied, namely:

$$\xi_w - \xi_a - \xi_R = -a - \gamma - [1-\beta(1-\delta)](1-a) < 0: \tag{30}$$

[8] Discussing the sufficient condition for indeterminacy in these specific RBC models requires analyses of the roots of a second-order polynomial, which is quite involved and does not add much new insight.

5.2. The Benhabib-Farmer-Guo Model

Benhabib and Farmer (1994) and Farmer and Guo (1994) considered a RBC model with externalities in the production technology:

$$y_t = k_t^a n_t^{1-a} (\bar{k}_t^a \bar{n}_t^{1-a})^x; \tag{31}$$

where $\bar{k}$ and $\bar{n}$ are the average economy-wide levels of capital stock and labor, which are taken as parametric by individual agents, and $x \geq 0$ measures the degree of externality. Since $r_t = ak_t^{a(1+x)-1} n_t^{(1-a)(1+x)}$ and $w_t = (1-a)k_t^{a(1+x)} n_t^{(1-a)(1+x)-1}$ in this model, we therefore have: $\xi_r = (1-a)(1+x)$; $\xi_w = (1-a)(1+x)-1 = \xi_r - 1$; $\xi_R = [1-\beta(1-\delta)]\xi_r$; and $\xi_a = \gamma$: Condition (28) implies

$$\beta(1-\delta)\xi_r > 1 + \xi_a; \tag{32}$$

or

$$(1-a)(1+x) > \frac{1+\gamma}{\beta(1-\delta)}: \tag{33}$$

This condition can be satisfied only if the externality x is large enough. Let $\beta \to 1; \delta \to 0$; (33) is exactly the condition for indeterminacy derived by Benhabib and Farmer (1994) using a continuous-time model. Namely, the slope of the aggregate labor demand curve, $(1-a)(1+x)-1$; must be positive and be greater than that of the labor supply curve, γ.

As a numerical example, suppose that the capital's share $a = 0{:}4$, the time discount factor $\beta = 0{:}99$, the labor supply elasticity parameter $\gamma = 0$, and the depreciation rate $\delta = 0{:}025$ (these parameter values are standard in the RBC literature), then the production externality x must be at least as large as 0:73 to generate indeterminacy in the current model (the implied slope for the aggregate labor demand curve is 0:038). An externality of this magnitude, of course, is empirically implausible.[9]

5.3. A Variable Capacity Utilization Model

Wen (1998_b) studied a model in which the production function is given by:

$$y_t = (e_t k_t)^a n_t^{1-a} \left((\bar{e}_t \bar{k}_t)^a \bar{n}_t^{1-a} \right)^x; \tag{34}$$

[9] Notice that for any finite value of the externality x, indeterminacy is not possible in this model if the rate of capital depreciation is one hundred percent. This is the case because when the capital stock depreciates very fast, the permanent effect of increasing the current labor supply on future income becomes too small.

where $e_t \in [0, 1]$ is the rate of capital utilization. It is assumed that increasing the intensity of capital utilization accelerates the rate of capital depreciation (Greenwood et al., 1988): $\delta_t = \frac{1}{h} e_t^h$ $(h > 1)$; so an interior solution exists for the optimal rate of capital utilization in the model. Wen (1998b) shows that at the optimal rate of capacity utilization, a reduced form production function can be found as

$$y_t = Ak_t^{a(1+x)\tau_k} n_t^{(1-a)(1+x)\tau_n}, \tag{35}$$

where τ_k and τ_n are defined as $\tau_k \equiv \frac{h-1}{h-a(1+x)}$, $\tau_n \equiv \frac{h}{h-a(1+x)}$. Notice that $\tau_k < 1$ and $\tau_n > 1$ for small externality x. The real wage and real interest rate are then given by

$$r_t = a\frac{y_t}{k_t} = aAk_t^{a(1+x)\tau_k - 1} n_t^{(1-a)(1+x)\tau_n}, \tag{36}$$

$$w_t = (1-a)\frac{y_t}{n_t} = (1-a)Ak_t^{a(1+x)\tau_k} n_t^{(1-a)(1+x)\tau_n - 1}, \tag{37}$$

and the gross interest rate is given by $R_t = 1 - \delta_t + r_t$. Therefore, we have $\eta_w = (1-a)(1+x)\tau_n - 1$, $\eta_r = (1-a)(1+x)\tau_n$, and $\eta_R = [1 - \beta(1-\delta^*)]\eta_r - \beta\delta^*\eta_\delta$, where δ^* is the steady-state rate of capital depreciation and η_δ is the equilibrium labor elasticity of depreciation. Condition (28) then implies

$$\beta(1-\delta^*)\eta_r > 1 + \eta_a - \beta\delta^*\eta_\delta. \tag{38}$$

A comparison of equation (38) with (32) in the Benhabib-Farmer-Guo model indicates that condition (38) is much less demanding than condition (32) for two reasons: one is that η_r in the current model is larger due to the effect of capacity utilization on labor-output elasticity ($\tau_n > 1$); another is the interest effect of variable capital depreciation ($\eta_\delta > 0$). The exact condition for indeterminacy is therefore:

$$(1+x) > \frac{(1+\gamma)h}{\beta(1-a)h + a(1+\gamma)}. \tag{39}$$

As a numerical example, suppose that $a = 0.4$, $\beta = 0.99$, the steady state rate of capital depreciation $\delta^* = 0.025$ (which implies $h = 1.4$), and $\gamma = 0$; then the minimum degree of externality necessary for generating indeterminacy is 0.14 (which is a much smaller number compared with 0.73 in the Benhabib-Farmer-Guo model). The corresponding slope of the aggregate labor demand curve is -0.3.[10]

5.4. A Model of Tax Distortions

Schmitt-Grohe and Uribe (1997b) showed that a distortionary labor-income tax under the balanced-budget fiscal policy rule can result in indeterminacy even with constant returns-to-scale production technologies. Their results can be easily nested into the current framework. Let G denote the constant government purchases and τ_t the labor income tax rate. The government budget constraint is then given by $G = \tau_t w_t n_t$, where $w_t n_t$ is the labor income. The aggregate budget constraint in their model is given by $c_t + k_{t+1} - (1-\delta)k_t = y_t - G$. Since the income distribution satisfies $y_t = r_t k_t + w_t n_t$, the aggregate budget constraint can then be rewritten as

$$k_{t+1} = R_t k_t + (1-\tau_t) w_t n_t - c_t. \tag{40}$$

The consumption Euler equation is the same as (3) and the optimal condition for labor is given by

$$\psi'(n_t) n_t = \frac{1}{c_t}(1-\tau_t) w_t n_t. \tag{41}$$

Substituting (41) into (40) gives

$$k_{t+1} = R_t k_t - (1 - \psi'(n_t) n_t) c_t, \tag{42}$$

which immediately gives us the following permanent-income relation under forward iteration:

$$\sum_{j=0}^{\infty} \beta^j \left[1 - \psi'(n_{t+j}) n_{t+j} \right] = R_t k_t \frac{1}{c_t} = R_t k_t \frac{\psi'(n_t) n_t}{(1-\tau_t) w_t n_t}. \tag{43}$$

This is a slightly modified version of Equation (10). The elasticity of labor on the right-hand side of (43) is given by

$$(1+\xi_{\Psi}) + \xi_R - \frac{1}{1-\bar{\tau}}(1+\xi_w), \tag{44}$$

where $\bar{\tau}$ is the steady-state tax rate, ξ_R, ξ_{Ψ} and ξ_w are elasticity terms defined in Proposition 5. A necessary condition for indeterminacy is that the expression in (44)

[10] Notice that indeterminacy can arise in the capacity utilization model with a downward sloping aggregate labor demand curve.

be negative, which can be true if τ is sufficiently close to one. For example, given the Cobb-Douglas production function, $k^{a}n^{(1-a)}$; and the utility function, $\log(c) - \alpha\frac{n^{1+\gamma}}{1+\gamma}$; this necessary condition implies

$$\tau > \frac{\xi_R + \xi_a - \xi_w}{1 + \xi_R + \xi_a} = \frac{1 - \beta(1-\delta)(1-a)}{1 - \beta(1-\delta)(1-a) + (1-a)}. \tag{45}$$

Let $\beta \to 1$ and $\delta \to 0$; we get exactly the same condition for indeterminacy, $\tau > a$, as that in a continuous time model of Schmitt-Grohe and Uribe (1997_b).[11]

5.5. A Two-Sector Model

Benhabib and Farmer (1996) showed that indeterminacy can also emerge in multi-sector models with mild externalities. Their results can be easily put into the current framework. Let μ_t denote the fraction of aggregate resources used in producing consumption goods (c_t) and $(1-\mu_t)$ denote the fraction of those used in producing investment goods (i_t). The production functions of the consumption sector and the investment sector are given by

$$c_t = (\mu_t k_t)^{a(1+x)} (\mu_t n_t)^{(1-a)(1+x)}; \tag{46}$$

$$i_t = k_{t+1} - (1-\delta)k_t = ((1-\mu_t)k_t)^{a(1+x)} ((1-\mu_t)n_t)^{(1-a)(1+x)}; \tag{47}$$

where x is a sectorial externality taken as given by individual agents in the economy. The relative price of investment goods is $p_t = \mu_t^x/(1-\mu_t)^x$. In equilibrium, the value of the marginal product of labor must equalize across sectors so that $(p_t i_t)/(1-\mu_t) = c_t/\mu_t$. By redefining variables: $I_t \equiv \left(\frac{\mu_t}{1-\mu_t}\right) i_t$; $r_t \equiv \frac{aI_t}{\mu_t k_t}$; the production function of the investment sector and the intertemporal Euler equation (3) can be expressed as:

$$k_{t+1} = (r_t + 1 - \delta)k_t - \left(1 - \frac{1-a}{\mu_t}\right) I_t; \tag{48}$$

$$I_{t+1} = \beta(r_{t+1} + 1 - \delta)I_t; \tag{49}$$

[11] The reason that we can get indeterminacy in this model without productive externalities is that government purchases function as a source of externality (market distortion) for aggregate labor demand. The labor-income tax can be re-interpreted as a tax on the marginal product of labor. Given the real wage, increases in employment reduce the marginal cost of labor because of the corresponding reduction in the tax rate, $\tau_t = \frac{G}{w_t n_t}$. This gives rise to short-run increasing returns to labor as in the Benhabib-Farmer-Guo model.

and the labor market efficiency condition is given by

$$\alpha n_t^{1+\gamma} = \frac{1-a}{\mu_t}. \tag{50}$$

Using (51) and (52) to form permanent income, one can derive an equation analogous to (10):

$$\sum_{j=0}^{\infty} \beta^j \left[1 - \frac{1-a}{\mu_{t+j}} \right] = \frac{(r_t + 1 - \delta)k_t}{I_t}, \tag{51}$$

which can be re-written as (after substituting out $\frac{1-a}{\mu_t}$ using (53) and I_t and r_t using their definitions):

$$\sum_{j=0}^{\infty} \beta^j \left[1 - \alpha n_{t+j}^{1+\gamma} \right] = \alpha n_t^{1+\gamma} \left(\frac{a}{1-a} + \frac{(1-\delta)k_t}{(1-a)k_t^{a(1+x)} n_t^{(1-a)(1+x)}} \frac{1}{(1-\mu_t)^x} \right). \tag{52}$$

On the other hand, if we apply Equation (10) to the Benhabib-Farmer model by substituting out w_t and R_t using $R_t = 1 - \delta + a k_t^{a(1+x)-1} n_t^{(1-a)(1+x)}$ and $w_t = (1-a)k_t^{a(1+x)} n_t^{(1-a)(1+x)-1}$, we get

$$\sum_{j=0}^{\infty} \beta^j \left[1 - \alpha n_{t+j}^{1+\gamma} \right] = \alpha n_t^{1+\gamma} \left(\frac{a}{1-a} + \frac{(1-\delta)k_t}{(1-a)k_t^{a(1+x)} n_t^{(1-a)(1+x)}} \right). \tag{53}$$

Comparison of (55) and (56) shows that this two-sector model differs from the one-sector model by only an extra term $(1-\mu_t)^x$.[12] Since this extra term co-moves with labor when $x > 0$ (by the labor market efficiency condition 50), the requirement on the externality x for inducing indeterminacy is weaker in the two-sector model than in the one sector model. Applying (28), the necessary condition for indeterminacy in this two-sector model is

$$1 + x \geq \frac{(1+\gamma)(1+\frac{a\delta}{1-\delta})}{(1-a) + \mu(a+\gamma)}, \tag{54}$$

where μ is the steady-state value of μ_t and is determined by $\mu = 1 - \frac{a\beta\delta}{1-\beta(1-\delta)}$.

[12] The two-sector model is in fact the one-sector model being presented in a different way when externalities are absent. It is the sectorial externality that makes the two-sector model differ from the one-sector model.

5.6. Robustness of Indeterminacy

Proposition 5 is very useful for studying the robustness of indeterminacy under parametric and structural modifications. According to Proposition 5, the necessary condition for indeterminacy is $\eta_w > \eta_R + \eta_{\gamma}$; where η_w is the labor elasticity of wages, η_R is the labor elasticity of the gross interest rate, and η_{γ} is the inverse labor supply elasticity. Parametric or structural changes that affect either of the three terms will change the status of indeterminacy, either making it easier to arise or making it more difficult to emerge. In Section 4, we have seen cases where indeterminacy becomes easier to arise compared to the bench-mark model of Benhabib-Farmer-Guo under structural modifications. All of these structural modifications amount essentially to an larger η_w than that in the Benhabib-Farmer-Guo model. The intuition is that for any given level of n_t, a larger η_w implies a larger permanent income.

On the contrary, labor adjustment costs can prevent or eliminate indeterminacy because they work to decrease η_w. This can be easily shown using the following example. Consider a quadratic adjustment cost function, $g(n)$, that has only a second-order effect on output but a first-order effect on the real wage:

$$w_t = f_n(k_t; \bar{k}_t; n_t; \bar{n}_t) - g'(n_t). \tag{55}$$

Due to the presence of adjustment costs, the labor elasticity of wages becomes smaller. Namely, the slope of the aggregate labor demand curve is reduced. Thus, given any degree of externalities, indeterminacy can always be eliminated if the cost of adjusting employment is sufficiently large.[13]

Similarly, indeterminacy can be made difficult to arise if the utility cost of adjusting labor supply (η_{γ}) is large. For example, in the model of Wen (1998) with habit formation on leisure, indeterminacy cannot arise even in the presence of strong productive externalities because rational habit formation substantially reduces the labor supply elasticity. Any structural modifications that have no effect on the three elasticity terms in (64), obviously, will have little effect on indeterminacy.

[13] Adjustment costs in investment can also help eliminate indeterminacy because they decrease the permanent income per unit of labor by decreasing the marginal return of investment to capital income. See Wen (1998_c) for detailed analyses.

6. Conclusion

The concept of self-fulfilling rational expectations equilibria (or indeterminacy) in an infinite-horizon model is a dynamic one in nature, yet intuitive understanding for indeterminacy are often expressed in static forms (e.g., the labor-market diagrammatic analyses). This paper established an analytical link between indeterminacy and labor's elasticities in the labor market via the perspective of permanent income, hence making the economic mechanisms that generate indeterminacy not only precise but also transparent. As a result, analytical conditions for indeterminacy for many standard RBC models can be easily derived without resorting to the eigenvalue method. And the proposed methodology also makes it easier to understand why certain structural modifications have or do not have impact on the conditions of indeterminacy. Nevertheless, the analytical expression being adopted in the permanent-income approach limits its application in certain ways, e.g., models with non-separable utility function and endogenous growth. Hence, it should not be viewed as a substitute for the existing approaches, but rather as a complement to them.

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