An OCE Analysis of the Effect of Uncertainty on Saving under Risk Preference Independence

LARRY SELDEN
Columbia University in the City of New York

1. INTRODUCTION AND SUMMARY

This paper is concerned with the effects of capital risk on optimal individual savings decisions in a simple two-period setting. We investigate the respective roles played by *risk* and *time* preferences in answering the following related questions:[1]

Q1: Will savings increase, remain constant or decrease in response to an increase in capital risk?

Q2: Is optimal saving in the presence of capital risk greater than, equal to or less than optimal saving in the certainty case where the rate of return equals the mean (uncertain) return?

A key element in our analysis is the OCE (Ordinal Certainty Equivalent) numeric representation of preferences over " certain × uncertain " consumption pairs developed in Selden (1978). This proposed alternative to the standard *two-period* expected utility model is based on a conditional *second-period* expected utility function and a two-period ordinal time preference index. Consistent with most of the literature dealing with Q1 and Q2, we shall assume risk preferences defined on random future consumption to be independent of current consumption. The resulting OCE representation includes the corresponding two-period expected utility paradigm as a limited special case. The latter requires additional axiomatic structure (cf. Rossman and Selden (1978)), which results in a specific, strong interdependence between risk and time preferences. Because the more general OCE representation hypothesis permits risk and time preferences to be prescribed separately, it will enable us to more clearly distinguish their respective roles in answering Q1 and Q2.

Motivated somewhat differently, Kreps and Porteus (1978) have, for the case of sequential decision problems, also developed an alternative to the vector-payoff NM (von Neumann–Morgenstern) axiomatization.

In the next section we review previous work concerning Q1 and Q2 based on additively separable two-period expected utility and on the stationary, constant elasticity of marginal utility special case. The OCE representation hypothesis is summarized in Section 3. The special case of OCE utility based on constant relative risk aversion risk preferences and CES time preferences, where the elasticity of substitution is interpreted as a measure of intertemporal complementarity, is considered in Section 4 and compared to the corresponding two-period expected utility model. Q1 and Q2 are reconsidered in Section 5 using OCE utility. The *strong interdependence between risk and time preferences* implicit in the conventional special forms of two-period expected utility is seen to result in a number of misleading conclusions concerning the effects of capital risk on optimal saving.

2. CAPITAL RISK AND THRIFT: A REVIEW

Let us adopt the following notation:

c_t:	value of real consumption flow in time-period t ($t = 1, 2$).
y_1:	(positive) certain, exogenously endowed income or wealth, to be received in c_1-units at the beginning of time-period one.
$\mathbb{R}$:	real line or set of real numbers.
ω:	state of nature.
Ω:	set of states of nature.
$\tilde{r}: \omega \mapsto \tilde{r}(\omega) \in (-1, l]$:	random variable mapping states of nature into real (net) rates of return on a single asset ($0 < l < \infty$).

Following the terminology of Sandmo (1970), consider a consumer who confronts " capital " (and not " income ") risk in a simple two-period setting. His budget constraint can be expressed

$$\tilde{c}_2(\omega) = (y_1 - c_1)(1 + \tilde{r}(\omega)) \quad \omega \in \Omega. \qquad \text{...(2.1)}$$

Let us suppose that his preferences over " certain × uncertain " consumption pairs are representable in accord with the expected utility hypothesis where the *two-period* NM index is denoted W (and assumed to be C^2 and to possess strictly positive partial first derivatives). Then the capital risk consumption/savings decision problem can be written

$$\max_{c_1 \in [0, y_1]} EW(c_1, [y_1 - c_1][1 + \tilde{r}]). \qquad \text{...(2.2)}$$

Assuming c_1^* to be an interior solution to (2.2), Q1 and Q2 can be restated as follows where optimal saving $s_1^* \equiv y_1 - c_1^*$:

- Q1: In response to an increase in capital risk, will the consumer increase, hold constant, or decrease s_1^*?
- Q2: Is optimal saving with capital risk (s_1^*) greater than, equal to or less than optimal saving in the certainty case (s_1^{**}) where the rate of return equals the mean return $\bar{r}$?

In order to investigate the first question, we adopt the simple Sandmo (1970)–Arrow (1971) notion of a " pure increase in risk ". Write the (net) rate of return on investment as $\gamma\tilde{r} + \theta$. Then in order for a multiplicative shift around zero to keep the mean constant, we must have $d\theta/d\gamma = -E(\tilde{r}) \equiv -\bar{r}$. (For a more general definition of increased risk, see Rothschild and Stiglitz (1970).)

It is well known that, in general, neither Q1 nor Q2 can be answered unambiguously. In order to provide behavioural interpretation for the possible variations in consumer behaviour, the standard approach has been to restrict the form of the two-period NM utility W. Mirman (1971) and Diamond and Stiglitz (1974, p. 354), for instance, assume the following *additively separable* form:

$$W(c_1, c_2) = w_1(c_1) + w_2(c_2). \qquad \text{...(2.3)}$$

The far stronger assumption of a stationary, constant elasticity of marginal utility form will be referred to as the PLS (Phelps–Levhari–Srinivasan) utility hypothesis:[2]

$$W(c_1, c_2) = -\alpha c_1^{-\delta}/\delta - (1-\alpha)c_2^{-\delta}/\delta \qquad \text{...(2.4)}$$

where $0 < \alpha < 1$ and $-1 < \delta < \infty$. For these two special forms of utility, the *two-period* relative risk aversion function defined by (in the spirit of Sandmo (1970) and Drèze and Modigliani (1972)) $\tau_R(c_1, c_2) =_{\text{def}} -c_2 W_{22}(c_1, c_2)/W_2(c_1, c_2)$ simplifies to, respectively,

$$\tau_R(c_2) = -c_2 w_2''(c_2)/w_2'(c_2) \qquad \text{...(2.5)}$$

and

$$\tau_R = \delta + 1. \qquad \ldots(2.6)$$

If one assumes the two-period NM utility W to be additively separable, then Q1 can be answered as follows:

Result A (adapted from Sandmo (1970)):

Given the Sandmo–Arrow definition of a (multiplicative, mean preserving) increase in capital risk, then

$$\left.\frac{\partial s_1^*}{\partial \gamma}\right|_{\frac{d\theta}{d\gamma} = -\bar{r}} \gtreqless 0 \quad \text{as} \quad E\left[(\tilde{r}-\bar{r})w_2'(\tilde{c}_2)\left\{\frac{\tilde{c}_2 w_2''(\tilde{c}_2)}{w_2'(\tilde{c}_2)}+1\right\}\right] \gtreqless 0$$

$$\text{or as} \quad E[(\tilde{r}-\bar{r})w_2'(\ _2)\{1-\tau_R(\tilde{c}_2)\}] \gtreqless 0.$$

Thus the qualitative effect of increased capital risk on thrift seems to involve the consumer's risk preferences in general and the relative risk aversion measure $\tau_R(c_2)$ in particular.

By assuming W to take the PLS form (2.4), one can obtain extremely simple answers to both Q1 and Q2 which seem to depend only on the (constant) measure of relative risk aversion τ_R. Note first that under this stronger utility hypothesis, the consumption/savings problem (2.2) possesses the unique solution

$$c_1^* = \left(\frac{Z}{1+Z}\right)y_1 \quad s_1^* = \left(\frac{1}{1+Z}\right)y_1 \qquad \ldots(2.7)$$

where

$$Z =_{\text{def}} \left[\frac{(1-\alpha)}{\alpha} E([1+\tilde{r}]^{-\delta})\right]^{-1/(\delta+1)}$$

Then using (2.6), we have

Result B:

$$\left.\frac{\partial s_1^*}{\partial \gamma}\right|_{\frac{d\theta}{d\gamma} = -\bar{r}} \gtreqless 0 \quad \textit{and} \quad s_1^* \gtreqless s_1^{**}$$

$$\textit{as } \delta \gtreqless 0 \quad \textit{or as} \quad \tau_R \gtreqless 1,$$

where s_1^{**} denotes the optimal saving for (2.2) when $\tilde{r}(\omega)$ is replaced by its mean $\bar{r}$.

Note that if $\tau_R = 1$, the PLS W is log-additive. It is thus easy to see why Samuelson (1969), Merton (1969) and Rothschild and Stiglitz (1971) assert that whether saving increases, remains constant or decreases depends on *whether the consumer's degree of (relative) risk aversion exceeds, equals or is less than that of the so-called " watershed " log-additive (NM index) case.* (Also see the comments of Mirrlees (1974, p. 43) and Sandmo (1974, p. 28).) Similarly, Result B suggests that, depending on the consumer's degree of relative risk aversion, optimal saving under capital risk exceeds, equals or is less than that of the corresponding certainty problem. However since W also represents the consumer's preferences over certain consumption pairs, one is led to ask why the parameter δ cannot alternatively be interpreted in terms of his " time preferences ".

3. OCE UTILITY

For choices over " certain × uncertain " consumption pairs, an alternative to the standard *two-period* expected utility hypothesis is developed in Selden (1978). Suppose, as in the previous section, c_1 and c_2 denote real consumption in time-periods one and two. Let F and G be c.d.f.'s (cumulative distribution functions) defined on second-period consumption. Define S to be some (suitable) space of (c_1, F)-pairs. Assume that a consumer possesses a complete preordering over S, denoted $\preccurlyeq^S$. Further, let this ordering be representable by a continuous " Bernoulli index " $\Psi: S \to \mathbb{R}$, i.e.

$$(c_1', F) \preccurlyeq^S (c_1'', G) \Leftrightarrow \Psi(c_1', F) \leqq \Psi(c_1'', G)$$

for any values of period-one consumption c_1' and c_1'' and c_2-c.d.f.'s F and G. (We use the term " Bernoulli index " to refer to any real-valued, order-preserving representation where the completely preordered space is at least partially stochastic.)

Given a complete preordering on S, each " cross-section " thereof S_{c_1} (defined by a value c_1 of first-period consumption) will possess an ordering denoted $\preccurlyeq^{S_{c_1}}$. The set of these orderings, $\{\preccurlyeq^{S_{c_1}}\}$, will be referred to as the consumer's " conditional risk preferences ". These preferences over c.d.f.'s on second-period consumption are assumed to be unaffected by the level of consumption in period one—identified as the " risk preference independence " postulate. This assumption implies that each of the conditional orderings are identical. (Note that the two-period expected utility representations defined by the additively separable form (2.3) and by the PLS form (2.4) both exhibit risk preference independence.) Let us further suppose that the (common) conditional ordering, $\preccurlyeq^{S_{c_1}}$, can be represented by a (single-attribute) *second* period expected utility function, i.e. there exists a continuous (strictly monotonically increasing) period-two NM index V such that for any pair of c_2-c.d.f.'s F and G

$$F \preccurlyeq^{S_{c_1}} G \Leftrightarrow \int V(c_2)dF(c_2) \leqq \int V(c_2)dG(c_2).$$

Then the intertemporal choice between pairs such as (c_1', F) and (c_1'', G) can be decomposed into two steps. First, these pairs can be converted into the certain first-period, certainty equivalent second-period consumption pairs $(c_1', \hat{c}_2^F)$ and $(c_1'', \hat{c}_2^G)$ by using the consumer's second-period expected utility function,

$$\hat{c}_2^F =_{\text{def}} V^{-1} \int V(c_2)dF(c_2) \quad \text{and} \quad \hat{c}_2^G =_{\text{def}} V^{-1} \int V(c_2)dG(c_2).$$

Then the latter pairs can be ordered by a (continuous) ordinal time preference function U defined on certain consumption plans (which, as is shown in the proof of Theorem 1 in Selden (1978), in essence corresponds to Ψ restricted to the set $\{(c_1, F^*)\}$, where F^* denotes a degenerate or one-point c.d.f.). Together these two steps are order-preserving in the sense that (for any c_1', c_1'', F and G)

$$(c_1', F) \preccurlyeq^S (c_1'', G) \Leftrightarrow \Psi(c_1', F) = U(c_1', \hat{c}_2^F) \leqq U(c_1'', \hat{c}_2^G) = \Psi(c_1'', G). \qquad \ldots(3.1)$$

This procedure will be referred to as the OCE (Ordinal Certainty Equivalent) representation.[3] Thus assuming risk preference independence, our proposed alternative to the two-period (multi-attribute) expected utility model is based on a second-period (single-attribute) expected utility function and a two-period ordinal index.

Let us next consider exactly how the OCE and two-period expected utility models are related. First of all, the preference ordering over *all* of S (not just each S_{c_1}) will be representable in accord with the expected utility principle if and only if there exists a (continuous) two-period NM index W such that (for all c_1', c_1'', F and G)

$$(c_1', F) \preccurlyeq^S (c_1'', G) \Leftrightarrow h[\Psi(c_1', F)] = \int W(c_1', c_2)dF(c_2) \leqq \int W(c_1'', c_2)dG(c_2)$$

$$= h[\Psi(c_1'', G)] \quad h' > 0. \qquad \ldots(3.2)$$

Further assuming $\preccurlyeq^S$ to exhibit risk preference independence implies that (Pollak (1967) and Keeney (1972))

$$W(c_1, c_2) = \alpha(c_1)+\beta(c_1)V(c_2) \quad \beta(c_1)>0. \qquad \text{...(3.3)}$$

Now it is shown in Selden (1978, Theorem 2) that under comparable assumptions, the OCE representation hypothesis includes the two-period expected utility paradigm as a quite limited special case. Thus, together the existence of a (continuous) ordinal time preference function U and the NM representability of the consumer's conditional risk preferences are sufficient for there to exist an OCE representation of $\preccurlyeq^S$, but are *not enough* for it necessarily to be linear in the probabilities as is required to have a two-period expected utility function (cf., Equation (3.2)). In order to obtain the latter representation, Rossman and Selden (1978) have shown that an additional axiom, referred to as " coherence ", is required.

In adding that extra axiomatic structure required for $\preccurlyeq^S$ to be NM representable according to (3.2), one however produces a specific strong interdependence between risk and time preferences. An individual's time preference representation U and his (two-period) NM index W, both defined on certain first- and second-period consumption pairs, are closely related; since they define the same indifference classes, each is an increasing monotonic transformation of the other (Pollak (1967)). Thus, if $\preccurlyeq^S$ is NM representable and exhibits risk preference independence,

$$U(c_1, c_2) = T[\alpha(c_1)+\beta(c_1)V(c_2)] \quad T'>0 \qquad \text{...(3.4)}$$

(which clearly includes the two-period additively separable (2.3) as a special case). In contrast, the more general OCE representation permits one to *prescribe* risk preferences (V) and time preferences (U) *separately*—thereby making possible an explicit modelling of their interrelationship (including the possible cases of complete independence and the (two-period) expected utility-dependence (3.4)).

As we shall see next, the PLS two-period NM utility hypothesis (2.4) involves an even more extreme degree of interdependence between risk and time preferences.

4. CES TIME PREFERENCES AND CONSTANT RELATIVE RISK AVERSION RISK PREFERENCES

The PLS two-period NM utility (2.4) is characterized by CES (constant elasticity of substitution) time preferences and constant relative risk aversion risk preferences. Maintaining these same properties, the natural OCE generalization is defined by

$$U(c_1, c_2) = -\alpha c_1^{-\delta_1}/\delta_1-(1-\alpha)c_2^{-\delta_1}/\delta_1 \qquad \text{...(4.1)}$$

$$V(c_2) = -c_2^{-\delta_2}/\delta_2 \qquad \text{...(4.2)}$$

where $0<\alpha<1$ and $-1<\delta_1, \delta_2<\infty$. Then following equation (3.1), the OCE representation of $\preccurlyeq^S$ can be expressed (for any c_1 and F)

$$\Psi(c_1, F) = -\alpha c_1^{-\delta_1}/\delta_1-\frac{(1-\alpha)}{\delta_1}\left[\int c_2^{-\delta_2}dF(c_2)\right]^{\delta_1/\delta_2} = U(c_1, \hat{c}_2^F). \qquad \text{...(4.3)}$$

Note that the " Bernoulli index " Ψ is *not* in general an expected utility functional (i.e. it is not " linear in the probabilities "), although there exists an exceedingly simple condition for it to be one:

$$\delta_1 = \delta_2. \qquad \text{...(4.4)}$$

The parameters δ_1 and δ_2 each readily admit interesting economic interpretations. The latter bears the following simple relationship to the classic Pratt (1964)–Arrow (1971) univariate measure of relative risk aversion,

$$\rho_R(c_2) =_{\text{def}} -c_2V''(c_2)/V'(c_2) = \delta_2+1. \qquad \text{...(4.5)}$$

Letting η denote an intertemporal elasticity of substitution (rather like the elasticity of substitution of a production function), straightforward computation for the time preference index (4.1) yields

$$\eta = \frac{1}{\delta_1+1}. \qquad \ldots(4.6)$$

Following Katzner (1970, p. 147), the constant elasticity η can be interpreted as a measure of intuitive intertemporal complementarity.[4] We shall say that c_1 and c_2 are intertemporal {substitutes, independents, complements} iff. $\eta\{>, =, <\}1$ (or $\delta_1\{<, =, >\}0$).

Hence the requisite condition for the OCE representation (4.3) to be an expected utility functional as well, (4.4), can be expressed as

$$\eta = 1/\rho_R. \qquad \ldots(4.7)$$

That is, given risk preferences exhibiting constant relative risk aversion and CES time preferences, $\preccurlyeq^S$ will be representable by a two-period expected utility functional only if the individual's measure of intertemporal complementarity equals the reciprocal of his measure of relative risk aversion.

Finally, two interesting special cases of the OCE representation (4.3) can be noted.[5] One is extreme risk aversion ($\rho_R \to \infty$), when we have

$$U(c_1, \hat{c}_2^F) = -\alpha c_1^{-\delta_1}/\delta_1 - \frac{(1-\alpha)}{\delta_1}\{\min(\tilde{c}_2(\omega))\}^{-\delta_1},$$

where $\min(\tilde{c}_2(\omega))$ is over the domain of the c.d.f. F (assuming a finite number of states). The second is extreme intertemporal complementarity ($\eta \to 0$):

$$U(c_1, \hat{c}_2^F) = \min\left\{c_1, \left(\int c_2^{-\delta_2} dF(c_2)\right)^{-1/\delta_2}\right\}.$$

5. CAPITAL RISK AND THRIFT: AN OCE REFORMULATION

Given the OCE representation of $\preccurlyeq^S$ described in Section 3, the "capital risk" consumption/savings decision problem can be expressed as

$$\max_{c_1 \in [0, y_1], \hat{c}_2} U(c_1, \hat{c}_2)$$

subject to

$$\hat{c}_2 - V^{-1}\{EV([y_1 - c_1][1+\tilde{r}])\} = 0, \qquad \ldots(5.1)$$

where U is assumed to be strictly quasi-concave and V strictly concave. Assuming we have a regular, interior maximum (c_1^*), it will satisfy the first-order condition

$$U_1(c_1, \hat{c}_2)/U_2(c_1, \hat{c}_2) = \frac{E[V'(\tilde{c}_2)(1+\tilde{r})]}{V'(\hat{c}_2)}. \qquad \ldots(5.2)$$

We see that the consumer will equate his marginal rate of time preference and his "risk preference adjusted expected (net) rate of return".

5.1. *Ordinally Additively Separable Time Preferences*

Paralleling the two-period additively separable NM utility (2.3), let time preferences be represented according to

$$U(c_1, c_2) = T[u_1(c_1) + u_2(c_2)] \quad T' > 0.$$

Then assuming an OCE representation, we have

$$\Psi(c_1, F) = T\left[u_1(c_1)+u_2\left(V^{-1}\int V(c_2)dF(c_2)\right)\right], \qquad \ldots(5.3)$$

with which Q1 can be answered as follows:

Result C:

Given the Sandmo–Arrow definition of a (multiplicative, mean preserving) increase in capital risk, then

$$\left.\frac{\partial s_1^*}{\partial \gamma}\right|_{\frac{d\theta}{d\gamma} = -\bar{r}} \gtreqless 0 \quad \text{as} \quad E[(\tilde{r}-\bar{r})V'(\tilde{c}_2)\{1-\rho_R(\tilde{c}_2)\}]+\left[\frac{u_1'(c_1)}{u_2'(\hat{c}_2)} \cdot\left\{\frac{u_2''(\hat{c}_2)}{u_2'(\hat{c}_2)}-\frac{V''(\hat{c}_2)}{V'(\hat{c}_2)}\right\}E[(\hat{c}_2-\tilde{c}_2)V'(\tilde{c}_2)]\right] \gtreqless 0$$

where $\rho_R(c_2)$ is defined by Equation (4.5).

The effect of an increase in capital risk on thrift depends on *two* (bracketed) terms, the first of which (explicitly) involves only risk preferences and the second both risk and time preferences. Comparing Results A and C, we see that the second term is not present in A. The reason is quite simple: For the OCE representation (5.3) also to be a two-period expected utility function, risk and time preferences must exhibit the strong interdependence $V(c_2) = u_2(c_2)$, but this in turn results in the second bracket in Result C identically equalling zero. (Note however that the remaining first bracket involves derivatives of a function which is simultaneously the period-two NM utility and the period-two time preference index—cf. Result D.)

5.2. CES *Time Preferences and Constant Relative Risk Aversion Risk Preferences*

If $\preccurlyeq^S$ is representable by the OCE expression (4.3), then the consumption/savings problem can be expressed

$$\max_{c_1\in[0,y_1],\,\hat{c}_2} -\alpha c_1^{-\delta_1}/\delta_1-(1-\alpha)\hat{c}_2^{-\delta_1}/\delta_1$$

subject to $\qquad\ldots(5.4)$

$$\hat{c}_2-(y_1-c_1)(1+\hat{r}) = 0,$$

where $(1+\hat{r}) =_{\text{def}} [E([1+\tilde{r}]^{-\delta_2})]^{-1/\delta_2}$. As can be seen from Figure (1), this OCE formulation permits us to generalize the classic two-period Fisherian diagrammatic analysis of the (deterministic) consumption/savings problem to the capital risk case.[6]

The problem (5.4) possesses the unique solution

$$c_1^* = \left(\frac{Q}{1+Q}\right)y_1, \quad s_1^* = \left(\frac{1}{1+Q}\right)y_1 \qquad \ldots(5.5)$$

where

$$Q =_{\text{def}}\left[\frac{(1-\alpha)}{\alpha}(1+\hat{r})^{-\delta_1}\right]^{-1/(\delta_1+1)}.$$

Not surprisingly, this OCE result includes the PLS, two-period expected utility solution (2.7) as a special case where $\delta_1 = \delta_2$ (or $\eta = 1/\rho_R$).

Now clearly, following Rothschild and Stiglitz (1970), any mean preserving spread in the distribution of the random return will decrease $\hat{r}$. But then from Figure 1, we see that the effect of increased risk on saving is exactly the same as the effect of decreased certain return on saving. Thus, decreased $\hat{r}$ will, for instance, decrease Q and increase s_1^* if

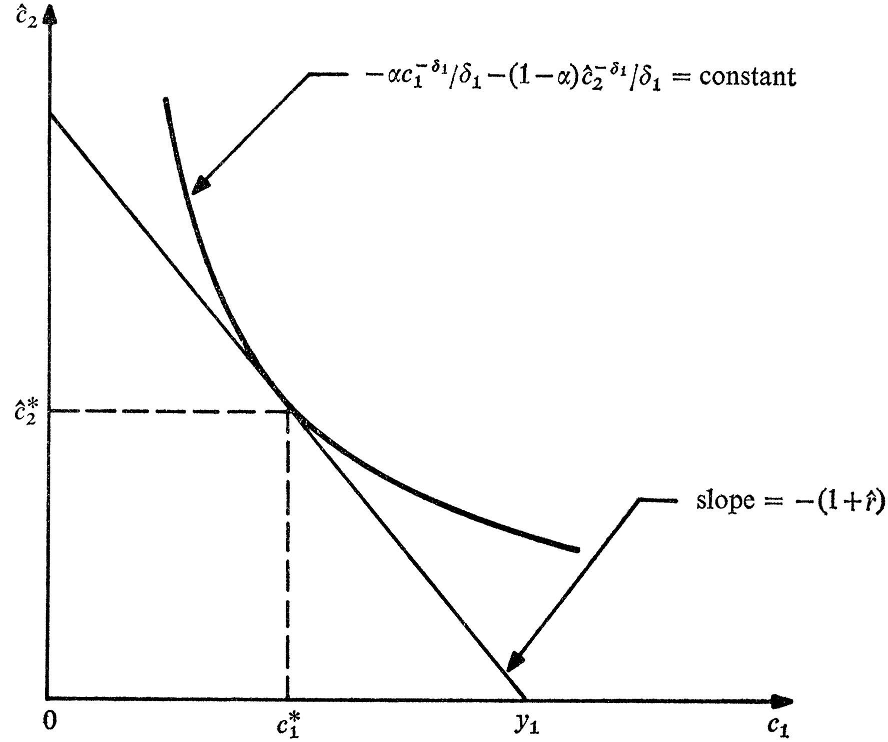

FIGURE 1

$\eta<1$ (or $\delta_1>0$). Paralleling our earlier results, this can be expressed in terms of the Sandmo–Arrow notion of increased risk as follows:

Result D:

$$\left.\frac{\partial s_1^*}{\partial \gamma}\right|_{\left.\frac{d\theta}{d\gamma}\right| = -\bar{r}} \gtreqless 0 \quad \textit{and} \quad s_1^* \gtreqless s_1^{**} \quad \textit{as} \quad \eta \lesseqgtr 1,$$

where s_1^{**} *denotes the optimal saving for* (5.1) *when* $\tilde{r}(\omega)$ *is replaced by* $\bar{r}$.

(It is worth noting that given (4.1) and (4.2), the two-term condition in Result C becomes

$$-\delta_2 E[(\tilde{r}-\bar{r})V'(\tilde{c}_2)]+[\delta_2 E[(\tilde{r}-\bar{r})V'(\tilde{c}_2)]-\delta_1 E[(\tilde{r}-\bar{r})V'(\tilde{c}_2)]] \gtreqless 0,$$

from which the inequality $\eta \lesseqgtr 1$ readily follows.)

Thus for this natural generalization of the PLS two-period expected utility hypothesis (based on CES time preferences and constant relative risk aversion risk preferences), the consumer's degree of relative risk aversion is, contrary to Result B, *irrelevant to both* the qualitative effect on thrift of an increase in capital risk and whether optimal saving is larger in the presence or absence of capital risk. What matters is the degree to which he insists first- and second-period consumption " go together " (i.e. are complements). If he views them as complements, then on the one hand increased capital risk will cause him to increase his saving and, on the other, he will save more in an uncertain setting than in a certain one.[7]

First version received March 1977; *final version accepted May* 1978 (*Eds.*).

This paper was in part supported by a grant from the Institute for Quantitative Research in Finance. Helpful discussions with Karl Shell are gratefully acknowledged. I also wish to thank Agnar Sandmo and members of the Micro-economics Workshop at Columbia University for providing useful comments. Finally, I am indebted to Robert Willig, an anonymous referee and an editor for detailed criticisms which have led to significant improvement. Of course, responsibility for error remains with the author.

NOTES

1. Various different versions of these questions have been considered extensively in the literature: Phelps (1962), Levhari and Srinivasan (1969), Merton (1969), Samuelson (1969), Hahn (1970), Sandmo (1970), Mirman (1971), Rothschild and Stiglitz (1971), Drèze and Modigliani (1972), Diamond and Stiglitz (1974) and Mirrlees (1974).

2. In their respective analyses of optimal savings behaviour under uncertainty, Phelps (1962) and Levhari and Srinivasan (1969) employ a maximand of the form $E[\sum_{t=1}^{T} \beta^t u(c_t)]$ where u is strictly concave and non-decreasing. Phelps assumes T to be finite while Levhari and Srinivasan consider an infinite horizon. In order to discuss the effect of increased risk on optimal saving, the authors of both papers specialize the form of u to $-c^{-\delta}/\delta$, $-1 < \delta \leqq \infty$, (or a positive affine transform thereof).

Also in this same context see Merton (1969), Samuelson (1969), Hahn (1970), Rothschild and Stiglitz (1971) and Mirrlees (1974).

3. Subject to minor changes, essentially the same argument can be used to prove the two-period OCE representation theorem when risk preference independence is not assumed—cf. Selden (1978).

The procedure described in the text is referred to as the " Ordinal Certainty Equivalent " representation because of (i) the invariance of U under increasing monotonic transforms and (ii) the important role played by certainty equivalents. This terminology is not meant to imply that the certainty equivalents are ordinal; rather it seeks to emphasize the ordinality of U.

4. Although differing somewhat in their use of the term complementarity, Hicks (1965) and Koopmans (1967) have argued, to use the words of the former, that there exists " . . . complementarity between the consumption that is planned for . . . [a] . . . particular period and that which is planned for its neighbour " (1965, p. 261).

5. I am grateful to a referee for pointing out these interesting limiting cases.

6. Essentially the same graphical analysis can be used for the more general formulation (5.1). However, the $c_1\hat{c}_2$ constraint curve will not in general be linear as portrayed in Figure 1 (where the NM index V exhibits constant relative risk aversion). Cf. Tobin (1968).

7. Presuming the existence of the appropriate multivariate " NM index ", Kihlstrom and Mirman (1974) propose a definition of multivariate risk aversion. Suppose preferences over certain consumption plans are representable by a constant elasticity of substitution (ordinal) utility function. They then show that increasing risk aversion (in their sense) $\Rightarrow$ increased (decreased) savings as $\eta < (>) 1$ *or* as $\partial c_1/\partial r_F > (<) 0$, where r_F denotes the (net) risk-free rate of interest. A similar conclusion can be drawn in our setting using a less controversial measure of risk aversion and a more general representation of preferences. To wit, note that in the case of (5.4) increasing ρ_R (increasing δ_2) results in a less steep $c_1\hat{c}_2$ constraint curve. But just as in the case of a mean preserving increase in risk, whether a downward rotation of the constraint curve results in increased or decreased saving depends on whether η is less than or greater than unity.

REFERENCES

ARROW, K. (1971) *Essays in the Theory of Risk Bearing* (Chicago, Ill.: Markham Publishing Co.).

DIAMOND, P. and STIGLITZ, J. (1974), " Increases in Risk and Risk Aversion ", *Journal of Economic Theory*, **8**, 337-360.

DRÈZE, J. H. and MODIGLIANI, F. (1972), " Consumption Decisions Under Uncertainty ", *Journal of Economic Theory*, **5**, 308-335.

HAHN, F. H. (1970), " Savings and Uncertainty ", *Review of Economic Studies*, **37**, 21-24.

HICKS, J. R. (1965) *Capital and Growth* (London: Oxford University Press).

KATZNER, D. W. (1970) *Static Demand Theory* (Toronto: Macmillan).

KEENEY, R. L. (1972), " Utility Functions for Multi-attributed Consequences ", *Management Science*, **18**, 276-287.

KIHLSTROM, R. and MIRMAN, L. (1974), " Risk Aversion with Many Commodities ", *Journal of Economic Theory*, **8**, 361-388.

KOOPMANS, T. C. (1967), " Intertemporal Distribution and ' Optimal ' Aggregate Economic Growth ", in Fellner, W. *et al.* (eds.) *Ten Economic Studies in the Tradition of Irving Fisher* (New York: John Wiley).

KREPS, D. M. and PORTEUS, E. L. (1978), " Temporal Resolution of Uncertainty and Dynamic Choice Theory ", *Econometrica*, **46**, 185-200.

LEVHARI, D. and SRINIVASAN, T. N. (1969), " Optimal Savings Under Uncertainty ", *Review of Economic Studies*, **36**, 153-163.

MERTON, R. C. (1969), " Lifetime Portfolio Selection Under Uncertainty: the Continuous-Time Case ", *Review of Economics and Statistics*, **51**, 247-257.

MIRMAN, L. J. (1971), " Uncertainty and Optimal Consumption Decisions ", *Econometrica*, **39**, 179-185.

MIRRLEES, J. A. (1974), " Optimum Accumulation Under Uncertainty: the Case of Stationary Returns to Investment ", in Drèze, J. H. (ed.) *Allocation Under Uncertainty: Equilibrium and Optimality* (New York: John Wiley and Sons).

PHELPS, E. S. (1962), " The Accumulation of Risky Capital: a Sequential Utility Analysis ", *Econometrica*, **30**, 729-743.

POLLAK, R. A. (1967), " Additive von Neumann–Morgenstern Utility Functions ", *Econometrica*, **35**, 484-494.

PRATT, J. W. (1964), " Risk Aversion in the Small and in the Large ", *Econometrica*, **32**, 122-136.

ROSSMAN, M. and SELDEN, L. (1978), " Time Preferences, Conditional Risk Preferences and Two-Period Cardinal Utility ", *Journal of Economic Theory*.

ROTHSCHILD, M. and STIGLITZ, J. E. (1970), " Increasing Risk I: A Definition ", *Journal of Economic Theory*, **2**, 225-243.

ROTHSCHILD, M. and STIGLITZ, J. E. (1971), " Increasing Risk II: Its Economic Consequences ", *Journal of Economic Theory*, **3**, 66-84.

SAMUELSON, P. A. (1969), " Lifetime Portfolio Selection by Dynamic Stochastic Programming ", *Review of Economics and Statistics*, **51**, 239-246.

SANDMO, A. (1970), " The Effect of Uncertainty on Saving Decisions ", *Review of Economic Studies*, **37**, 353-360.

SANDMO, A. (1974), " Two-Period Models of Consumption Decisions Under Uncertainty: a Survey ", in Drèze, J. H. (ed.) *Allocation Under Uncertainty: Equilibrium and Optimality* (New York: John Wiley and Sons).

SELDEN, L. (1978), " A New Representation of Preferences Over ' Certain × Uncertain ' Consumption Pairs: the ' Ordinal Certainty Equivalent ' Hypothesis ", *Econometrica*, **46**, 1045-1060.

TOBIN, J. (1968), " Notes on Optimal Monetary Growth ", *Journal of Political Economy*, **76**, 833-859.