Heterogeneous Beliefs, Learning, and Speculative Trade in a Hidden Markov Environment with Short Sale Constraints

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Abstract The paper develops general tools to analyze the prices in dynamic equilibria with heterogeneous prior beliefs and learning in an environment with risk–neutral agents, short–selling constraints and the dividend process following a hidden Markov process. Particularly it analyzes conditions for speculative bubbles to arise in such an environment. I introduce the concept of recursive equilibrium, which uses current beliefs as state variables. I give the precise definition of the speculative bubble as a difference between the price and the market fundamental, with market fundamental being the maximum across the agents of buy-and-hold forever individual valuation of the asset. Then I show that any sequential market equilibrium price must be at least of the magnitude of that of the recursive equilibrium. Since in this environment the fundamental value of the asset is not affected by the particular equilibrium outcome, then this implies that the magnitude of the speculative bubble in equilibrium is also bounded below by that of the recursive equilibrium. The main result is to prove that a recursive equilibrium generates a sequential market equilibrium (Theorem 1) and to propose a simple operator, which characterizes the recursive equilibrium price as its fixed point. This operator is shown to be a monotone contraction, which gives not only uniqueness but also provides a simple criterion for the existence of a speculative bubble in equilibrium. The paper is closed by two illustrative examples.

Keywords: General equilibrium, Short-selling constraints, Heterogeneous priors, Bayesian learning, Speculation, Speculative bubbles, Recursive equilibrium.

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1 Introduction

This paper provides a theoretical insight into the possibility of speculation in asset markets, driven purely by heterogeneous prior beliefs about the relevant dividend process.

The question of speculative trade is inevitably linked to so called *market bubbles*. Historically people give that name to a particular behavior of the price of a commodity or an asset, which was characterized by a significant growth of the price without any apparent fundamental reason, often followed by a collapse. Formally, a bubble is typically understood in the economic literature as the departure of an asset price from its fundamental value. What is problematic with this definition, however, is that there is no broadly accepted definition of the fundamental value. In this work I provide one definition without claiming any superiority over the others. In my opinion it naturally matches the environment of this particular model but would not work in many other environments.

The most famous and the most cited historical example was the Dutch tulip mania in 1636-37. Since then there have been many other notable examples of bubbles, with the most recent ones including the Japanese asset pricing bubble and the dot-com bubble.

All of these bubbles were called speculative bubbles, following the folk belief that they were generated by some traders buying an asset or commodity in order to profitably resell it in the future, without actually believing it has any significant intrinsic value.

In the context of rational expectation models, having rational traders behaving in this way would require either some sort of market imperfection (some agents who will be buying the asset in the future are unable to participate in the market now) or agreement to disagree type of behavior (some agents expect other agents to become over-optimistic in the future). In this paper I will focus on the disagreement based speculative trade.

I will be analyzing both speculative trade and the bubbles in a context of a model with risk neutral agents having heterogeneous beliefs, and facing short selling constraints. Both speculation and the bubble, or the fundamental value, will be given a precise meaning and it will be clear that both phenomena are very closely related.

By speculative trade I will understand a dynamic general equilibrium outcome, in which there is a contingency where an agent who currently holds the asset is paying a price that is higher then his subjective net present value of holding it forever. This means this particular trader buys the asset with the intention to re-sell it at some future date.

The bubble will be defined as an equilibrium outcome, in which the current price of an asset is higher than the fundamental value, which is defined as the maximum of any agent's subjective present value of holding it forever.

Here, I will be solely focused on the risk neutral environment, in which agents need not trade for any insurance purposes. With risk neutrality, when agents have homogeneous beliefs, even with short selling constraints, there cannot be any essential trade (agents are indifferent in terms of trading or not). This is the case even with heterogeneous signaling, as is implied by the "no-trade theorem," cf. Milgrom and Stokey (1982). Once we introduce heterogeneous beliefs the situation is not so clear. One could expect the heterogeneous priors would just make the most optimistic agent hold the asset forever. Obviously what might happen is that there is not a permanent optimist, but even if there is one, there is still some disagreement about the exact probabilities of various future events, not just fundamentals. Agents might be trying to exploit that disagreement to bet against each other, with the only tool to do that being to trade the asset. It seems very natural and following the popular understanding of the word *speculation* to call such a trade speculative. And my definition indeed exactly captures this type of trade in the risk neutral environment. I will discuss the issues associated with heterogeneous priors in more detail in the next subsection.

Note that in environments with risk aversion, one could easily construct examples in which the price is higher than the fundamental even with homogeneous beliefs. This is because the short-selling constraint makes the full insurance or smoothing impossible. Even though the price might be considered by agents higher than their marginal valuation, they cannot permanently reduce their asset holdings because of the binding short-selling constraints in the future. They also do not want to reduce the asset holding only in the current period because that would reduce the consumption smoothing. According to my definition however, this would also be a speculative trade, although clearly the mechanisms pushing agents to trade in this way would be different than in the case of heterogeneous belief-driven trade.

I will now go over the contents of this paper. The model developed here is based on the paper by Harrison and Kreps (1978). In their model the equilibrium price is above any agent's buy-and-hold forever valuation of the asset, hence there is a bubble according to the definition below (Harrison and Kreps call it a speculative premium rather than a bubble). This bubble is generated by a speculative trade driven by differences in subjective beliefs accompanied by a short sale constraint.

They consider an example with two groups of risk neutral agents. The agents trade one risky asset of supply 1. There is a short sale constraint. Dividends follow a 2-state Markov chain and can take value 0 or 1 in each period. The agents differ in their beliefs about the transition matrix. Agents of type one assign relatively high probabilities of switching between the states (dividends) while agents of type two perceive the states as more persistent. Both types are certain that their transition matrix is correct. The numerical values are chosen so that in each state the present expected value of the stream of all future dividends (the fundamental value) is higher according to the agents of group two.

This would suggest that it should be the agents of type two who permanently hold the asset in equilibrium and also, given risk neutrality, the price should be exactly their fundamental value. Surprisingly, only if the dividend is 1 agents of group two hold the asset. If the dividend is 0 agents of type one buy it. Why is it so? The mechanism is quite simple. Type one agents buy the asset in state 0 (when the dividend is 0), because they think there is a good chance of switching to the other state next period. Further, they (rationally) expect that the price will be high if the dividend is 1, since type two agents assign a high fundamental value to the asset in that state. Moreover we get that the equilibrium price is higher than any agent's fundamentals in each state. Throughout Sections 3 and 4 I will go over the Harrison and Kreps model in more detail, using it as an illustrative example for my model.

The result of Harrison and Kreps is simple and quite beautiful but unrealistic in some respects. The asymmetry of agents' subjective beliefs persists in spite of commonly observed histories. It is this feature which makes the speculation so highly persistent (indeed it lasts forever).

The problem in this example is that it is unclear how robust it would be to adding learning. One could expect that if agents were not dogmatic any initial disagreement would eventually disappear. This might be expect to remove speculation in two ways. One would be just asymptotic removal, coming from the convergence of posteriors. The other would be if the knowledge that other agents will be learning would not significantly reduce the size of speculation even in the beginning as agents would not be able to count for a profitable resale due to the anticipated learning process of the others. The difficulty with verifying these statements is that adding learning to the environment with heterogeneous beliefs makes solving for equilibrium much more difficult. Even though some work has been done in asset pricing with learning in the context of verifying the market selection hypothesis (cf. Blume and Easley (2006) Sandroni (2000), or more recently, Beker and Espino (2009)) that methodology only applies to frictionless environments.

This paper attempts to build a methodology to analyze the asset pricing with heterogeneous beliefs and the dynamic effects of learning in environment with market friction, namely short selling constraint. In this case the focus is on speculative bubbles and their dynamical features, but this methodology can be also used more broadly. It uses recursive techniques, so for that purpose the environment needs to be time homogeneous, which might be considered a significant constraint. To offset for that I propose an environment which is a little bit more general than standard Markov, which is sometimes called hidden Markov. It means agents believe there is some underlying regime process, which is Markov and cannot be directly observed. The current state of that process determines the current law of the dividend process. This way it is possible to capture various aspects of the way people tend to interpret the same signals. One can for instance consider an example where two regimes are most of the time generating very similar laws for dividends, and only differ on some rare, tail events. Conditional on observing these rare events, agents, who concentrate their priors on these different regimes, might react in terms of short run predictions in a very different ways even though for a long time they had been perceived to make similar forecasts. This might ultimately give rise to some pattern of speculative bubbles which arise as a result of rare events.

Using hidden Markov structure allows at the same time a relatively simple way of keeping track of beliefs updating. All the information about the agents' predictions for the future is completely described by their beliefs about the current position of the hidden state. This allows for the application of recursive equilibrium approach, in which agents' trading strategies (and prices) depend only on the current belief profile rather than the whole history. I formally introduce the notion of belief-based recursive equilibrium. To my knowledge this idea is pretty original.

This allows me to map the consumers' problems of such economies into the stochastic dynamic programming techniques a'la Stokey et al. (2004). The main result showing that mapping is Theorem 1. This provides a powerful tool for numerical computation of such equilibria and offers a useful analytical tool for checking up if a given environment features speculation.

An important supporting result is the one provided by Proposition 4, which states that recursive equilibrium is not just any sequential equilibrium, but it actually is a one with the lowest possible prices. This result is important in the context of analyzing speculation, namely if we have speculation in recursive equilibrium, then it must be the case in any other equilibrium.

In the end I propose two illustrative examples. In one example I prove that if there are only two hidden states then there is no speculation in any recursive equilibrium, no matter what initial beliefs are. In another setup with three hidden states I show that, for some particular initial belief, the equilibrium price has to feature a bubble.

An interesting feature of this example is that it is significantly different from the one of Harrison and Kreps. Here agents have exactly the same transition matrix. They only differ in their beliefs about the current position of the regime. The fact that this leads to speculation is pretty significant, because learning about the current regime is much more difficult than learning about the transition matrix. The thing is that the regime changes over time, while the transition matrix stays the same. This will lead to the proposed persistence of the speculative behavior.

Now, I will briefly discuss various issues associated with using heterogeneous prior beliefs. First, it is worth pointing out why it is crucial that the difference in beliefs must be coming from heterogeneous priors and not from heterogeneous signals.

In this class of models, as in Rational Expectation Equilibrium (REE) models, it follows from the famous *agreeing to disagree* (cf. Auman (1976)) and *no-trade* (cf. Milgrom Stokey (1982)) theorems that the price has to reflect any important private information, in particular ruling out agents' betting against their posterior beliefs, which in particular implies there cannot be any speculation. In other words and agent cannot take advantage of his private information because the rest of the market would learn that information instantly from the price.

The existence of speculative trade in the context of REE models is related to the value of information. There is a line of research that focuses on the value of information. The most influential attempt to give information a value was done by making a market game a positive sum game. Clearly if the game is of a positive sum, then the no trade theorem is not a problem anymore because the market participants do not need to use the private information against each other. They can coordinate in some fashion to exploit that information and share the surplus. Important papers in this spirit include Grossman and Stiglitz (1980), Hellwig (1980) and Kyle(1985). These papers consider a market game, in which there are noise traders, who trade for the reason external to the model. In other words, their utility function is not explicitly analyzed and they provide a stochastic supply, which is independent of the price. This gives that game a positive sum. If some of the agents have some information about the behavior of the noise traders they can use that information against them without making other strategic agents worse off.

Unsurprisingly, Grossman and Stiglitz (1982) and Hellwig (1982) obtain the result that as the noise goes to zero, the value of information goes to zero and the price becomes fully revealing. This just confirms that in the context of the models where heterogeneity is coming from differentiated signals, in order to get speculation one needs to significantly depart from the rationality of some agents.

Using unexplained heterogeneous priors allows avoiding the problems related to the no-trade theorem (agents trading with each other because they know that their beliefs differ, not their signals). Nevertheless, there are some objections in economic literature concerning using heterogeneous priors. Besides the obvious one, that if one is willing to assume heterogeneous priors one can explain pretty much everything, there is a more troubling point, called the market selection hypothesis. It is associated with the lack of learning in Harrison and Kreps mentioned above. The market selection hypothesis is a long standing tenet in economic theory, which states that even if agents start with heterogeneous priors, after a sufficiently long period of trading, the agents who started with beliefs that are the farthest from the truth should either learn or be removed from the market. In other words, in well–established markets the traders should have essentially the same beliefs. As compelling as it may sound, the market selection hypothesis was never convincingly proved or disproved in the economics literature, with many papers showing examples going either way (see Sandroni (2000) or Blume and Easley (2006) for a detailed market selection hypothesis literature overview).

Another aspect of using heterogeneous priors is necessity of dealing with the true data generating process on top of what agents believe it to be. This issue is related to the market selection hypothesis but going a little bit beyond, more into the nature of what economic modeling really is. If one takes a stand that economic modeling should be about how agents preferences affect their behavior in various circumstances, the idea of the true distribution seems to be somewhat irrelevant. We do observe only a single trajectory anyway. On the other hand if one believes in the stochastic nature of the physical world, than he should also be willing to accept that the stochastic nature of the reality should be added to the way agents adjust their beliefs. This objection is not exactly relevant to the model presented in this paper, which focuses on mechanics how current beliefs translate into prices, but any application of this model will require to specify what the true data generating process is in order to make any statistical predictions.

I will close this introduction with a brief overview of the most relevant literature on speculation, driven by heterogeneous beliefs.

Following Harrison and Kreps (1978) mentioned before, Morris (1996) considers the effects of allowing for learning in the original model. He considers a special case of Harrison and Kreps, with a iid dividend process. Using some parametric classes of invariant distributions for the priors (like β-distributions) he gets a nice explicit formula for the learning dynamics. Also he gets a nice criterion for having speculation in equilibrium. He indeed gets some speculative behavior but the numerical experiments show that the speed of the convergence of the equilibrium price to the fundamental value is very fast. Hence this paper addresses only the second of the two issues associated with adding learning mentioned before, namely the one about the current level of bubble being robust to adding learning.

An interesting attempt to control the convergence of equilibrium prices to the fundamentals is Bossaerts (1995). Here there is no dividend bearing, infinitely lived asset. There are only 1-period future contracts with risky return. The payoffs of these contracts are iid over time. There are countably many generations of agents. The beliefs are shared and updated within generations. Each period a new generation of agents joins the market and stays there forever. The new generation comes with its own initial beliefs, which are immediately updated by the up to date stream of returns. He assumes that the returns are normal with mean zero and the unknown variance, and that the beliefs about the variance are inverted gamma-2 distributions. This specification allows for an easy analytical treatment and a flexible control over the equilibrium price dynamics. It is easy to get that the initial price exceeds the rational expectation one. The conditions for the convergence of the prices to the rational expectation values are given. For some choice of beliefs we can get no convergence, which gives a powerful tool to control the rate of convergence. All of this is done at the expense of having new coming generations with more and more biased initial beliefs. Also an important role is played by the fact that the agents' problem is not dynamic (the future contract is only for 1 period).

The most recent paper in that spirit is the one of Scheinkman and Xiong (2003). In their model the (cumulative) dividend follows a diffusion process, with drift f_t, which is called a fundamental variable and is not observed by the agents, they only know it follows another diffusion process. Even though they use continuous time diffusion process techniques, their model can be treated almost as a special case of mine (after appropriate discretization of their setup or redesigning mine to cover the continuous time case). There is one crucial difference though. Given the normal environment of them, which is easy to deal with analytically, it is pretty hard to obtain a speculation in the case when the agents only observe the cumulative dividend as a common signal. To fix that problem Scheinkman and Xiong consider additional signal processes s_t^A and s_t^B, which are both diffusion processes with f_t as their drift part. As for the innovations part, A believes that the one of s_t^A is correlated with the one for the process f_t while agent B thinks that it is the one of s_t^B. So agent A even though he can observe both signals, he thinks his signal has a better quality than the signal of agent B and vice versa. The model stated in that way can be explicitly solved analytically and features speculation. The problem is that it requires an additional signalling structure (besides dividends), and also even though agents are updating their beliefs about the underlying fundamental process, f_t, they are not learning about the informativeness of the signals and always use their own one for updating.

2 The Model

2.1 Economy

There are 2 agents, who are endowed with zero units of consumption good at each time period, $t = 0, 1, \ldots$ they are both risk neutral and have a discount factor β.

There is one unit of risky asset in this economy, which agent can trade each period in general equilibrium fashion with no short sales allowed. The asset gives to its owner a dividend d_t each period. Each agent starts with some initial holding of the asset, $\bar{\gamma}_0^i$, such that $\bar{\gamma}_0^1 + \bar{\gamma}_0^2 = 1$.

There is an underlying regime process, a_t, taking value in some state space $\mathcal{A}$. We assume a_t is Markov and it cannot be directly observed by the agents.

The dividend, d_t, is generated independently each period from a distribution which depends on the current regime, $a_t \in \mathcal{A}$. The distribution associated with regime a_t we denote Φ_{a_t}.

Now let us turn to describing these processes formally.

First, to fix ideas, I will denote $(\Omega, \mathcal{F})$ an abstract measurable space over which all the random variables in this paper will be defined.

Let the set of possible regimes, $\mathcal{A}$, has a structure of the Polish space and the regime process, $(a_t)_{t=0,1,\ldots}$ be a stationary Markov process with the transition function $q : \mathcal{A} \to \Delta(\mathcal{A})$ assumed Borel-measurable, with $\Delta(A)$ denoting the linear space of all Borel probabilistic measures over $\mathcal{A}$ endowed with weak*-topology.

Let $\mathcal{D} \subseteq \mathbb{R}$ denote the set of possible dividends (assume it is Borel-measurable), and let $\{\Phi_a\}_{a \in \mathcal{A}}$ be a family of probability distributions over $\mathcal{D}$ (i.e. $\Phi_a \in \Delta(\mathcal{D})$ for each $a \in \mathcal{A}$)), such that $\Phi_\cdot : \mathcal{A} \to \Delta(\mathcal{D})$ is Borel-measurable (with respect to weak*-topology on $\Delta(\mathcal{D})$). We will also need to assume (in order to be able to use Bayes' rule) that Φ_a has a density function with respect to some regular measure on $\mathbb{R}$, μ (usually either discreet or Lebesgue). Denote this density by ϕ_a.

Having specified $\bar{a}_0 \in \mathcal{A}$, the family $\Phi \in \mathcal{B}(\mathcal{A}, \{\zeta \in \Delta(\mathcal{D}) | \zeta << \mu\})$, and the transition function $q \in \mathcal{B}(\mathcal{A}, \Delta(\mathcal{A}))$ (I follow the convention of denoting the linear space of Borel-measurable functions by $\mathcal{B}(\cdot, \cdot)$) we denote by $\Pr^{\bar{a}_0, \Phi, q}$ a probability measure over $(\Omega, \mathcal{F})$ which is consistent with the Markov structure of the process a_t and with the described structure of the process d_t (i.e. such that $a_0 = \bar{a}_0$ with probability 1, a_t is Markov with the transition function q and d_t is drawn independently each period from the distribution Φ_{a_t}). Formally, for each $A_0 \in \mathcal{B}(\mathcal{A}), \ldots, A_t \in \mathcal{B}(\mathcal{A})$, $D_1 \in \mathcal{B}(\mathcal{D}), \ldots, D_t \in \mathcal{B}(\mathcal{D})$ we have:

$$
\begin{aligned}
&\Pr^{\bar{a}_0, \Phi, q}(a_0 \in A_0, \ldots, a_t \in \mathcal{A}_t, d_1 \in D_1, \ldots, d_t \in \mathcal{D}_t) = \\
&\qquad = \int_{A_0, \ldots, A_t} \Phi_{a_1}(D_1) \ldots \Phi_{a_t}(D_t) \delta_{\{\bar{a}_0\}}(da_0) q(a_0, da_1) \ldots q(a_{t-1}, da_t) \\
&\qquad = \int_{\substack{A_0, \ldots, A_t \\ D_1, \ldots, D_t}} \phi_{a_1}(d_1) \ldots \phi_{a_t}(d_t) \delta_{\{\bar{a}_0\}}(da_0) q(a_0, da_1) \ldots q(a_{t-1}, da_t) \mu(dd_1) \ldots \mu(dd_t)
\end{aligned}
$$

Now let's turn to agent's information structure. They both can observe dividend d_t each period and none of them can observe a_t. Denote by $(\mathcal{F}_t^d)_t$ the filtration generated by the process d_t.

In particular the agents don't know the initial regime, a_0 and also they don't know the family of distributions $\Phi.$ or the transition function q. We will assume that $\Phi.$ and q can take values in some Borel sets of admissible values, $\boldsymbol{\Phi} \subseteq \mathcal{B}(\mathcal{A}, \{\zeta \in \Delta(\mathcal{D}) | \zeta << \mu\})$, and $\mathcal{Q} \subseteq \mathcal{B}(\mathcal{A}, \Delta(\mathcal{A}))$, respectively. Hence the agents formulate beliefs about the value of $(a_0, \Phi., q)$.

Let $\Pr^\pi$ be a measure that a player with beliefs $\pi \in \Delta(\mathcal{A} \times \boldsymbol{\Phi} \times \mathcal{Q})$ assigns to $(\Omega, \mathcal{F})$. This is clearly given by:

$$
\Pr^\pi(F) = \int_{\mathcal{A} \times \boldsymbol{\Phi} \times \mathcal{Q}} \Pr^{\bar{a}_0, \Phi, q}(F) \pi(d\bar{a}_0, d\Phi, dq) \tag{1}
$$

for any $F \in \mathcal{F}$. Let E^{π} be the expected value operator associated with that measure. Denote the initial beliefs of player i by $\pi_0^i \in \Delta(\mathcal{A} \times \mathbf{\Phi} \times \mathcal{Q})$.

The timing is as follows. In the beginning of period t the new position in the Markov process a_t is established, then dividend d_t is generated and paid to the current owner of the asset. After the dividend is paid the agents can trade on the centralized market for price p_t, subject to the short sale constraint.

Now we are ready to define (competitive) equilibrium.

Definition. An *equilibrium* (given initial beliefs, π_0^1, π_0^2) consists of the processes: an allocation, $(\hat{c}_t^i)_t$, asset holdings, $(\hat{\gamma}_t^i)_t$, and prices $(p_t)_t$ such that prices are $\mathcal{F}_t^d$-adapted, and:

- For $i = 1, 2$, taking $(p_t)_t$ as given, $(\hat{c}_t^i)_t$ and $(\hat{\gamma}_t^i)_t$ solve:

$$
\begin{aligned}
&\max E^{\pi_0^i} \sum \beta^t c_t^i \\
&\text{s.t. } c_t^i + p_t \gamma_{t+1}^i \leq p_t \gamma_t^i + \gamma_t^i d_t \\
&\text{s.t. } c_t^i, \gamma_{t+1}^i \text{ — } \mathcal{F}_t^d\text{-measurable} \\
&\text{s.t. } \gamma_0^i = \bar{\gamma}_0^i, \gamma_t^i \geq 0
\end{aligned}
\tag{2}
$$

- Asset market clears: $\gamma_t^1 + \gamma_t^2 = 1$

It is worth noting at this point, that it is an implicit feature of this general equilibrium environment, that agents are facing prices as functions of all potential histories not beliefs. Even though the equilibrium prices, in order to clear the market, have to convey the information about all the agents beliefs, agents do not need to know the beliefs of the others. That information, however, can be often inferred from the prices. In either case in this Walrasian type of equilibrium, where agents take prices as given the structure of higher order beliefs seems to be irrelevant. In order to consider this equilibrium as a rational expectations equilibrium, the whole hierarchy of beliefs needs to be specified. Specifically, we can assume the common knowledge of beliefs.

Another technical issue associated with the definition above is that there might be a set of future contingencies, which is believed by an agent to be of zero probability, on which his behavior is inconsistent with Bayesian learning. In this paper I will be always assuming to use the version of conditional probability which is consistent with Bayes' rule, so any resulting equilibria will not have this problem.

2.2 Speculative trade

In this setup, it is the most natural way to define speculative trade as the situation in which the equilibrium price exceeds the fundamental valuation of the asset for the agent is an actual holder. By fundamental valuation of agent i we understand the discounted stream of all the future dividends expected by agent i. This is the highest price he would be willing to pay for the asset if he was forced to keep it forever after the purchase.

It is a natual definition because agents are risk neutral and are sharing the discount factor, hence agents do not need the asset to smooth consumption or for insurance. If they decide to purchase the asset for the price which is higher than their subjective fundamental value it is because they use it as a betting device against the market which they perceive as not pricing the asset properly in the future. They understand the market does not price the asset properly because there are some other traders with fundamentally wrong beliefs.

This definition however is not the most practical one because to check if there is a speculative trade one would need to analyze the behavior of all the traders. A simpler statistic which can measure the strength of speculative trade is a speculative bubble.

Definition. The fundamental value of the asset for agent i at time t, given history $d^t \equiv (d_1, \dots, d_t)$ is:

$$V_t^i(d^t) \equiv E^{\pi^i}(\sum_{\tau=t+1}^{\infty} \beta^t d_t | d^t)$$

Definition. Speculative bubble is a current excess of the equilibrium prices over the market fundamental value,

$$s(d^t) \equiv p(d^t) - \max_i V_t^i(d^t)$$

Note that whenever $s > 0$ we do have a speculative trade, but not necessarily the other way round. Also note that the magnitude of the bubble is measuring how much the agents value the opportunity to bet against each other provided by the asset, and not necessary the volume of the speculative trade. For the remaining of this work I will be focusing on the speculative trade which is associated with a positive bubble, and i will often abuse the terminology by using speculative trade and a bubble interchangeably.

Example. Now let us see how the example of Harrison and Kreps fits into this notation.

In their model the dividend itself follows a 2-state Markov chain (can be either 0 or 1). Agents differ in what they think the transition matrix is. Agent 1 thinks the transition matrix is:

$$Q^1 = \begin{bmatrix} 1/2 & 1/2 \\ 2/3 & 1/3 \end{bmatrix}$$

agent 2 thinks it is:

$$Q^2 = \begin{bmatrix} 2/3 & 1/3 \\ 1/4 & 3/4 \end{bmatrix}$$

To map it into my notation it is enough to take $\mathcal{A} = \mathcal{D} = \{0, 1\}$, $\mathbf{\Phi} = \{\Phi\}$ with $\Phi_0 = \delta_{\{0\}}$ $\Phi_1 = \delta_{\{1\}}$ (i.e. the current regime coincides with the current dividend and both agents agree about it). We also have $\mathcal{Q} = \{Q^1, Q^2\}$. As for a_0 we may assume the agents know it as it coincides with the dividend hence the agent's beliefs about this one coincide and put the whole measure on its true value. Hence the initial beliefs are just measures over $\mathcal{A} \times Q$, specifically π_0^1 puts all the measure on $(\bar{a}_0, Q^1)$ and π_0^2 puts all the measure on $(\bar{a}_0, Q^2)$ (where a_0 is the true value of a_0), i.e. $\pi_0^i = \delta_{\bar{a}_0} \otimes \delta_{Q^i}$

We can see that since agents have disjoint supports in their beliefs they will not be learning the transition matrix over time. We will see in the next section that this lack of dynamics in the beliefs is crucial for having an explicit solution for equilibrium prices in the Harrison and Kreps example. In this case it is also straightforward to compute the fundamental values. It is clearly only the function of last period dividend (for each agent) because it is the sole factor to determine the future distribution of dividends: $V^i(d^t) = V^i(d_t)$. Denoting $V^i \equiv [V^i(0), V^i(1)]'$ and using recursiveness and Markov property we get that it has to satisfy:

$$V^i = \beta Q^i V + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so

$$V^i = \beta(I - \beta Q^i)^{-1} Q^i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In case of $\beta = 3/4$ we get:

$$V^1 = \begin{bmatrix} 4/3 \\ 11/9 \end{bmatrix} = \begin{bmatrix} 1.33 \\ 1.22 \end{bmatrix} \qquad V^2 = \begin{bmatrix} 16/11 \\ 21/11 \end{bmatrix} = \begin{bmatrix} 1.45 \\ 1.91 \end{bmatrix} \tag{3}$$

3 Recursive Equilibrium

Since we assume rationality of agents (given their own initial beliefs) they must do learn from the observed signal in the Bayesian way. This creates certain dynamics of beliefs. I want to make new beliefs be only dependent on the last period beliefs and the current period dividends (rather than the whole history). To achieve that we introduce some new notation for the updated beliefs, and understand the current beliefs to be a distribution over the current position of the Markov process, a_t rather than the initial one. This will lead me to the notion of recursive equilibrium, which will appear to be a powerful tool in analysis of the equilibria in this model. That doing so makes the model consistent with the Bayesian learning requires some formal argument. It will be done in this section by proving that a recursive equilibrium can be translated to an equilibrium.

First let us define the belief update operator, λ, which is crucial for applying the stochastic dynamic programming techniques. It maps the previous period beliefs into the current one, taking into account the current realization of the dividend d.

Definition 1 *Given $\pi \in \Delta(\mathcal{A}\times\mathbf{\Phi}\times\mathcal{Q})$, and $d \in \mathcal{D}$ define a measure $\lambda^d(\pi) \in \Delta(\mathcal{A}\times\mathbf{\Phi}\times\mathcal{Q})$, by*

$$\lambda^d(\pi)(A \times \Phi \times Q) \equiv \Pr^{\pi}(a_1 \in A \wedge \phi \in \Phi \wedge q \in Q | d_1 = d)$$

for each measurable $A \subseteq \mathcal{A}, \Phi \subseteq \mathbf{\Phi}, Q \subseteq \mathcal{Q}$, and some particular version of the conditional probability. Without loss of generality, throughout this paper I will be always using the version of the conditional probability which is given by Bayes' rule.

Definition 2 *A (symmetric)* recursive equilibrium *consists of a value function $V : [0,1] \times \Delta(\mathcal{A}\times\mathbf{\Phi}\times\mathcal{Q})^2 \to \mathbb{R}$, policy function, $\Gamma : [0,1] \times \Delta(\mathcal{A}\times\mathbf{\Phi}\times\mathcal{Q})^2 \to \mathbb{R}_+$, and pricing function, $p : \Delta(\mathcal{A}\times\mathbf{\Phi}\times\mathcal{Q})^2 \to \mathbb{R}$, such that:*

$$V(\gamma, \pi^1, \pi^2) = \max_{\gamma' \geq 0} \left\{ (\gamma - \gamma')p(\pi^1, \pi^2) + \beta E^{\pi^1} \left(V(\gamma', \lambda^{d_1}(\pi^1), \lambda^{d_1}(\pi^2)) + \gamma' d_1 \right) \right\}$$

$$\Gamma(\gamma, \pi^1, \pi^2) \in \underset{\gamma' \geq 0}{argmax} \left\{ (\gamma - \gamma')p(\pi^1, \pi^2) + \beta E^{\pi^1} \left(V(\gamma', \lambda^{d_1}(\pi^1), \lambda^{d_1}(\pi^2)) + \gamma' d_1 \right) \right\}$$

and for each γ, π^1, π^2, d we have:

$$\Gamma(\gamma, \pi^1, \pi^2) + \Gamma(1 - \gamma, \pi^2, \pi^1) = 1$$

It should be noted that in this case the symmetry reflects the fact that only one value function is used for both agents. When it is used for agent 1 it reads $V(\gamma, \pi^1, \pi^2)$ and when for agent 2 it becomes $V(\gamma, \pi^2, \pi^1)$. Also note that whenever d_1 appears under the expectation associated with measure π, (e.i. $E^{\pi} f(d_1)$ with f being any measurable real function) it refers to the first period dividend distributed according to the probability distribution Pr^{π} defined in (1).

For notational simplicity it is useful to denote the beliefs process by π_t^i, which is defined recursively via:

$$\pi_1^i \equiv \lambda^{d_1}(\pi_0^i)$$
$$\pi_{t+1}^i \equiv \lambda^{d_{t+1}}(\pi_t^i)$$

Theorem 3 *Given some initial beliefs, $\pi_0^1, \pi_0^2 \in \Delta\big(\mathcal{A} \times \mathbf{\Phi} \times \mathcal{Q}\big)^2$, if $V : [0,1] \times \Delta\big(\mathcal{A} \times \mathbf{\Phi} \times \mathcal{Q}\big)^2 \to \mathbb{R}_+$, Γ, $p : \Delta\big(\mathcal{A} \times \mathbf{\Phi} \times \mathcal{Q}\big)^2 \to \mathbb{R}$ constitute a recursive equilibrium then the processes: $p_t^* \equiv p(\pi_t^1, \pi_t^2)$, $\gamma_{t+1}^{*i} \equiv \Gamma(\pi_t^i, \pi_t^{-i})$, $c_t^{*i} \equiv p_t(\gamma_t^{i*} - \gamma_{t+1}^{i*}) + \gamma_t^{i*} d_t$ constitute a sequential market equilibrium, given beliefs π_0^1, π_0^2 if the following transversality condition holds:*

$$\lim_{t\to\infty} \beta^t E^{\pi_0^i} \left(V(\gamma_t^{i*}, \pi_t^i, \pi_t^{-i}) + \gamma_t^{i*} d_t \right) = 0$$

for $i = 1, 2$.

The proof will follow from the following lemma:

Lemma 4 *For each $s \in \mathbb{N}$, $d \in \mathcal{D}$, $A_0, \ldots, A_s \subseteq \mathcal{A}$, $\Phi \subseteq \mathbf{\Phi}$, $Q \subseteq \mathcal{Q}$, such that $A_0, \ldots, A_s, \Phi, Q$ are measurable subsets, we have almost surely:*

$$\begin{aligned} &\Pr^{\pi}(a_1 \in A_0 \wedge \ldots \wedge a_{s+1} \in A_s \wedge \phi \in \Phi \wedge q \in Q | d_1) \\ &\quad = \Pr^{\lambda^{d_1}(\pi)}(a_0 \in A_0 \wedge \ldots \wedge a_s \in A_s \wedge \phi \in \Phi \wedge q \in Q) \end{aligned}$$

This lemma states that all the future distribution of the relevant processes at the next period is, from the perspective of player i, completely described by the updated measure $\pi_1^i = \lambda^{d_1}(\pi_0^i)$. This argument extends by induction to any future period: all the information about agent i's subjective future distributions of all the processes is completely encoded in π_t^i. This legitimates the introduction of the recursive equilibrium in this environment.

The proof of Theorem 3 as well as that of Lemma 4 are relegated to the appendix as they are purely technical.

3.1 Characterization of recursive pricing rule

Using linearity of preferences and no short sales condition, we can argue that a pricing function $p : \Delta\big(\mathcal{A} \times \mathbf{\Phi} \times \mathcal{Q}\big)^2 \to \mathbb{R}$ is the pricing rule of some recursive equilibrium iff it satisfy the following first order condition to the Bellman equation:

$$p(\pi^1, \pi^2) = \max_{i=1,2} \beta E^{\pi_i}(p(\lambda^{d_1}(\pi^1), \lambda^{d_1}(\pi^2)) + d_1)$$

In order to organize the notation let's define the following operators, $T, T^{(1)}, T^{(2)}$: $\mathcal{B}\big(\Delta\big(\mathcal{A} \times \mathbf{\Phi} \times \mathcal{Q}\big)^2, \mathbb{R}\big) \to \mathcal{B}\big(\Delta\big(\mathcal{A} \times \mathbf{\Phi} \times \mathcal{Q}\big)^2, \mathbb{R}\big)$, with:

$$\begin{aligned} T^{(i)}p(\pi^1, \pi^2) &\equiv \beta E^{\pi_i}\left\{d_1 + p\big(\lambda^{d_1}(\pi^1), \lambda^{d_1}(\pi^2)\big)\right\} \qquad i = 1, 2 \\ Tp &\equiv \max_{i=1,2} T^{(i)}p \end{aligned}$$

With this notation, the equation for prices becomes:

$$p = Tp \tag{4}$$

so we are just looking for a fixed point of T.

Also note, that the the fixed point of operator $T^{(i)}$ is the fundamental value for agent i, V^i.

Example of Harrison and Kreps (1978) – cont. Now we can see how much the lack of learning facilitates the solution of the functional equation. The formula for operator $T^{(i)}$

becomes (note that without loss of generality we can treat prices as a function of a current dividend since so are the beliefs):

$$T^{(i)}p(d) = \beta E^{\pi_i}\left\{d_1 + p\big(d_1\big)\right\}$$
$$= \begin{cases} \beta\left\{(1/2)p(0) + (1/2)(1+p(1))\right\} & \text{for } d=0, i=1 \\ \beta\left\{(2/3)p(0) + (1/3)(1+p(1))\right\} & \text{for } d=0, i=2 \\ \beta\left\{(2/3)p(0) + (1/3)(1+p(1))\right\} & \text{for } d=1, i=1 \\ \beta\left\{(1/4)p(0) + (3/4)(1+p(1))\right\} & \text{for } d=1, i=2 \end{cases}$$

This allows us for almost immediate guess the solution to (4), which in the case of $\beta = 3/4$ is $p^*(0) = 24/13 = 1.85$ and $p^*(1) = 27/13 = 2.04$, which is clearly higher than the corresponding maximal fundamental values derived in (3).

Let us investigate some properties of T.

Lemma 5 *Operators T,$T^{(i)}$ are all β-contractions w.r.t. the* sup*-norm.*

Proof. A direct application of Blackwell's sufficient conditions. ■

Lemma 6 *If p is continuous, then Tp is continuous.*

As a corollary we get:

Theorem 7 *If $\beta < 1$ then there is the unique bounded solution, p^* to the functional equation (4). Moreover, p^* is continuous and $p^* = \lim_{t\to\infty} T^t p$ (in the* sup *norm).*

It is worth noting, that there is no hope for more general regularity conditions for the price system beyond continuity. The following examples will show the lack of differentiability, while any kind of convexity seems meaningless in this setup (at least in general). Nevertheless, the following monotone property of operator T appears to be useful.

Lemma 8 *If $Tp \geq p$ for some price system p, then $T^2 p \geq Tp$. Hence also $p^* \geq p$*

Proof. For any beliefs π^1, π^2 we have:

$$\begin{aligned} T^2 p(\pi^1, \pi^2) &= \beta \max_{i=1,2} \int_{\mathcal{D}} p^{\pi^i}(d')(d' + Tp(\lambda^{d'}(\pi^1), \lambda^{d'}(\pi^2)))dd' \\ &\geq \max_{i=1,2} \int_{\mathcal{D}} p^{\pi^i}(d')(d' + p(\lambda^{d'}(\pi^1), \lambda^{d'}(\pi^2)))dd' \\ &= Tp(\pi^1, \pi^2) \end{aligned}$$

with $p^{\pi^i}(d')$ denoting probability density of next period dividend according to an agent with beliefs π^i i.e.

$$p^{\pi^i}(d') = \int_{\mathcal{A} \times \mathbf{\Phi} \times \mathcal{Q}} \int_{\mathcal{A}} \phi_{a'}(d') q(a, da') \pi^i(da, d\phi, dq)$$

■

As a corollary we get a useful fact.

Theorem 9 *If we define the fundamental pricing rule by $p^F = \max_{i=1,2} V^i$ then if for some beliefs (π^1, π^2), $Tp^F(\pi^1, \pi^2) > p^F(\pi^1, \pi^2)$, then $p^*(\pi^1, \pi^2) > p^F(\pi^1, \pi^2)$, i.e. we have speculation in equilibrium for those initial beliefs.*

Proof. It is clear, that for fundamental pricing we have $Tp^F \geq p^F$. From the previous lemma we get, that $T^2 p^F \geq Tp^F$ hence using this lemma again we get that $p^* \geq Tp^f$. Hence by our assumption $p^*(\pi^1, \pi^2) \geq Tp^F(\pi^1, \pi^2) > p^F(\pi^1, \pi^2)$ ■

This theorem gives us an easy tool to check if we have a speculative bubble in a given economy. Just take an initial guess for pricing system to be $p^f = \max_{i=1,2} V^i$ (the highest fundamental value). Then iterate it once. Obviously we must have $Tp^f \geq p^f$ (if the agents are promised to be able to resell the asset at the highest fundamental price next period then in the current period they must be willing to pay at least their fundamental values). If we get $Tp^f = p^f$ then p^f is the solution to the functional equation (4) i.e. $p^* = p^f$ and the bubble is always zero. Otherwise there are some beliefs, π^1, π^2 for which $Tp^f(\pi^1, \pi^2) > p^f(\pi^1, \pi^2)$, which by proposition 2 implies that $p^*(\pi^1, \pi^2) \geq Tp^f(\pi^1, \pi^2) > p^f 0(\pi^1, \pi^2)$, which means that we have a positive bubble, and hence a speculative trade, in equilibrium.

Now, I will show that the price in the bounded recursive equilibrium provides a lower bound for the set of all sequential market equilibria. This justifies the use of applying recursive equilibrium in the contest of speculative trade. If there is a speculative bubble in the recursive equilibrium it is also positive in any sequential market equilibrium.

Proposition 1 *For any initial beliefs π_0, if p is a sequential equilibrium price system, and p^* is the bounded recursive equilibrium price, then for almost every history, d^t, we have $p(d^t) \geq p^*(\pi_t(d^t))$.*

Proof. If $(p_t(d^t))$ is an equilibrium price system, then it has to satisfy the first order conditions of (2), which, taking into account the fact that the market clearing condition must hold (i.e., $\gamma_t^i > 0$ for at least one agent), leads to:

$$p_t(d^t) = \max_{i=1,2} \beta E^{\pi_0^i}\left[p_{t+1}(d^{t+1}) + d_{t+1}|d^t\right]. \tag{5}$$

Define inductively a sequence of functions, $p_n : \Delta\big(\mathcal{A} \times \mathbf{\Phi} \times \mathcal{Q}\big)^2 \to \mathbb{R}$, by

$$p_0^*(\pi) \equiv 0$$
$$p_{n+1}^*(\pi) \equiv \max_{i=1,2} \beta E^{\pi}\left[d_1 + p_n^*\big(\lambda^{d_1}(\pi)\big)\right].$$

Then by Proposition 7, we have $p^* = \lim_{n\to\infty} p_n^*$. I will show by induction that $p_t(d_t) \geq p_n^*(d_t, \pi_t^1(d_t), \pi_t^2(d^t))$, for each n, d^t.

For $n = 0$ this is obvious. Suppose that $p_t(d^t) \geq p_n^*(d^t, \pi_t(d^t))$ for some n and all t and d^t. Then, using (5) and Lemma 4 (recursively, t times) we get

$$
\begin{aligned}
p_t(d^t) &= \max_{i=1,2} \beta E^{\pi_0^i}\left[d_{t+1} + p_{t+1}\left(d^{t+1}\right)|d^t\right] \\
&\geq \max_{i=1,2} \beta E^{\pi_0^i}\left[d_{t+1} + p_n^*\left(\pi_{t+1}(d^{t+1})\right)|d^t\right] \\
&= \max_{i=1,2} \beta E^{\pi_t^i(d^t)}\left[d_1 + p_n^*\left(\lambda^{d_1}(\pi_t(d^t))\right)\right] \\
&= p_{n+1}^*(\pi_t(d^t)).
\end{aligned}
$$

■

4 Examples

In this section I will consider the environment in which the agents' only potential disagreement is about the current regime.

I present two examples. The first one has 2-state regime process where the states are stable in the sense that in each of them probability of staying in it is bigger then moving out of it. In that case I prove that no speculative trade (and hance no bubble) can exist.

In the second example I consider 3-state regime process. There are two "good" states and one "bad" state. In good states agents get relatively high probability of dividend (around 2/3) but if the state is bad the probability of dividend is 0. The "high" regimes are different in terms of probability of switching to the "bad" regime. These probabilities are both low but one is smaller than the other. Agents initially think they are in a "good" regime but of a different type. Here I will be able to show how Theorem 9 can be easily applied to show that there is a generic speculative pattern (for some open set of parameter values). In that example we will also see that the speculation may persist arbitrarily long even though the agents are learning.

4.1 A Simplified Environment

Here the agents agree upon the value of the transition function as well as upon the distribution of dividends. They just disagree about the current regime. Formally this means that we consider the class of models in which: $\mathcal{A} = \{a_1, \ldots, a_n\}$, $\mathcal{D} = \{0, 1\}$, $\mathbf{\Phi} = \{\phi\}$ with $\phi_{a_j}(1) = 1 - \phi_{a_j}(0) \equiv \phi_j$, $\mathcal{Q} = \{q\}$, with $q = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{bmatrix}$ (Markov chain transition matrix).

In words this setup means that there is an underlying $\mathcal{A}$-valued Markov chain in the economy, $a_0, a_1, \ldots, a_t, \ldots$ with the known transition matrix, q. Each period dividend d_t is paid with probability ϕ_j which is determined solely by the current position in the Markov chain, $a_t = a_j$. The agents formulate beliefs about the initial position of the Markov chain, a_0. They both know the value of ϕ and q, and over time they beliefs evolve in the Bayesian fashion. The current beliefs of agent i are $\pi^i = [\pi^i_1, \ldots, \pi^i_n] \in \Delta(\mathcal{A}) \simeq \Delta^{n-1}$.

In this setup the updating dynamics as well as finding the fundamental values become relatively easy. It is straightforward to see that the fundamental value of agent i, $V^i = \pi^i \cdot V$, where $V = [V_1, \ldots, V_n]'$ is the vector of fundamental values for each initial position in the Markov chain. V is the solution to

$$(I - \beta q)V = \beta q \phi$$

The fundamental price becomes:

$$p^f = \max_{i=1,2} \pi^i V$$

Given the initial beliefs π^i and the current period dividend, d' the new beliefs are (using

Bayes' rule):

$$\lambda(d'|\pi^i) = \begin{cases} \left[\frac{\phi_1 \sum_{j=1}^n q_{j1}\pi_j^i}{\sum_{j'=1}^n \phi_{j'} \sum_{j=1}^n q_{jj'}\pi_j^i}, \ldots, \frac{\phi_n \sum_{j=1}^n q_{jn}\pi_j^i}{\sum_{j'=1}^n \phi_{j'} \sum_{j=1}^n q_{jj'}\pi_j^i}\right] & \text{if } d' = 1 \\ \left[\frac{(1-\phi_1) \sum_{j=1}^n q_{j1}\pi_j^i}{\sum_{j'=1}^n (1-\phi_{j'}) \sum_{j=1}^n q_{jj'}\pi_j^i}, \ldots, \frac{(1-\phi_n) \sum_{j=1}^n q_{jn}\pi_j^i}{\sum_{j'=1}^n (1-\phi_{j'}) \sum_{j=1}^n q_{jj'}\pi_j^i}\right] & \text{if } d' = 0 \end{cases} \tag{6}$$

Also it will be useful to have the explicit expression for the probability of dividend next period, $d' = 1$, given the current beliefs are π^i:

$$\Pr^{\pi^i}\{d' = 1\} = \pi^i q\phi \tag{7}$$

The formula for operator T becomes:

$$Tp(\pi^1, \pi^2) = \beta \max_{i=1,2} \Big[\Pr^{\pi^i}\{d' = 1\}(1 + p\left(\lambda\left(1|\pi^1\right), \lambda\left(1|\pi^2\right)\right) \\ + \left(1 - \Pr^{\pi^i}\{d' = 1\}\right) p\left(\lambda(0|\pi^1), \lambda\left(0|\pi^2\right)\right) \Big]$$

4.2 Example with no speculation

Here we will see a situation in which we will actually solve the functional equation. The solution will be (as one may expect) the fundamental valuation by the agent for whom it's the highest (given beliefs).

Proposition 3. If $\mathcal{A} = \{h, l\}$ and $q = \begin{bmatrix} q_{hh} & q_{hl} \\ q_{lh} & q_{ll} \end{bmatrix} = \begin{bmatrix} 1-\epsilon^1 & \epsilon^1 \\ \epsilon^2 & 1-\epsilon^2 \end{bmatrix}$ satisfies $\epsilon^1 + \epsilon^2 \leq 1 < 1$, then the equilibrium price is equal to the fundamental price: $p^* = p^F$ for all beliefs.

This proposition states that if we have only 2 regimes there will be no speculation. It seems that only two regimes cannot give enough room for disagreement if we have only 2 signals (the dividend either paid or not).

Proof.

Since we have only 2-state Markov chain, the beliefs can be just represented by one number: probability of being in a given state. To fix ideas let it be state h. Hence the beliefs are: $\pi^i \in [0,1]$, $i = 1,2$ and satisfy: $\Pr^{\pi^i}\{a_0 = h\} = \pi^i$.

As usual, denote: $\phi = [\phi_h, \phi_l]'$ to be the vector of the probabilities of the dividend in regime.

I will show that the equilibrium price is just the fundamental price, i.e. $Tp^f = p^f$ (for all beliefs).

Without loss of generality assume that the fundamental vector $V = [V_h, V_l]'$ satisfies $V_h > V_l$ (otherwise just relabel the states). We have then

$$
\begin{aligned}
p^f(\pi^1, \pi^2) &= \max_{i=1,2}\{\pi^i V_h + (1-\pi^i)V_l\} \\
&= (\max_{i=1,2}\pi^i)V_h + (1 - \max_{i=1,2}\pi^i)V_l
\end{aligned}
$$

by symmetry of the agents, wlog I can always assume $\pi^1 \geq \pi^2$ (otherwise just renumber them), which leads us to

$$
p^f(\pi^1, \pi^2) = \pi^1 V_h + (1-\pi^1)V_l \tag{8}
$$

Now I will show that $\lambda^d(\pi)$ is increasing in π for $d = 0,1$ (in words: agent who was more optimistic in the first period will always remain more optimistic in the next period, no matter which dividend occured). By (6) we have:

$$
\lambda^d(\pi) = \begin{cases} \frac{\phi_h\left[(1-\epsilon^1)\pi+\epsilon^2(1-\pi)\right]}{\phi_h[(1-\epsilon^1)\pi+\epsilon^2(1-\pi)]+\phi_l[\epsilon^1\pi+(1-\epsilon^2)(1-\pi)]} & \text{for } d = 1 \\ \frac{\phi_l\left[(1-\epsilon^1)\pi+\epsilon^2(1-\pi)\right]}{\phi_l[(1-\epsilon^1)\pi+\epsilon^2(1-\pi)]+\phi_h[\epsilon^1\pi+(1-\epsilon^2)(1-\pi)]} & \text{for } d = 0 \end{cases} \tag{9}
$$

A bit of algebra gives:

$$
\frac{\partial}{\partial \pi}\lambda^d(\pi) = \frac{1-\epsilon^1-\epsilon^2}{(\text{appropriate denominator expression} \neq 0)^2} > 0 \qquad d = 0,1
$$

so indeed for $\epsilon^1, \epsilon^2 < 1/2$ $\lambda^d(\pi)$ is increasing in π. Hence $\pi^1 \geq \pi^2$ implies that also $\lambda^d(\pi^1) \geq \lambda^d(\pi^2)$.

Another thing I am going to need is $\Pr^{\pi^1}(d' = 1) \geq \Pr^{\pi^2}(d' = 1)$ (always assuming $\pi^1 \geq \pi^2$), which is easy to verify using (7). Also I will need $\lambda^1(\pi) > \lambda^0(\pi)$, which is intuitively obvious and straightforward to check from (9).

This gives us:

$$
\begin{aligned}
Tp^f &= \beta \max_{i=1,2} \Big\{ \Pr^{\pi^i}\{d=1\} \left[1 + p^f \left(\lambda^1(\pi^1), \lambda^1(\pi^2)\right)\right] \\
&\qquad + \Big(1 - \Pr^{\pi^i}\{d=0\}\Big) p^f(\lambda^0(\pi^1), \lambda^0(\pi^2)) \Big\} \\
&= \beta \max_{i=1,2} \Big\{ \Pr^{\pi^i}\{d=1\} \left(1 + \lambda^1(\pi^1)V_h + (1-\lambda^1(\pi^1))V_l\right) \\
&\qquad + \Big(1 - \Pr^{\pi^i}\{d=0\}\Big) \left(\lambda^0(\pi^1)V_h + (1-\lambda^0(\pi^1))V_l\right) \Big\} \\
&= \beta \Pr^{\pi^1}\{d=1\} \left(1 + \lambda^1(\pi^1)V_h + (1-\lambda^1(\pi^1))V_l\right) \\
&\quad + (1 - \Pr^{\pi^1}\{d=0\}) \left(\lambda^0(\pi^1)V_h + (1-\lambda^0(\pi^1))V_l\right) \\
&= T^1 V^1 \\
&= V^1 \\
&= \max_{i=1,2}\{V^1, V^2\} \\
&= p^f
\end{aligned}
$$

where the second line comes from (8), the third uses the fact that $\Pr^{\pi^1}(d' = 1) \geq \Pr^{\pi^2}(d' = 1)$ and $\lambda^1(\pi) > \lambda^0(\pi)$. Then we use the fact that V^i is the fixed point of operator T^i. this allows us to conclude that p^F is the equilibrium price for all beliefs. ■

4.3 Example with speculation

Let $\mathcal{A} = \{h_1, h_2, l\}$, $\phi = [\Phi_1, \Phi_2, 0]'$ and

$$q = \begin{bmatrix} q_{h_1h_1} & q_{h_1h_2} & q_{h_1l} \\ q_{h_2h_1} & q_{h_2h_2} & q_{h_2l} \\ q_{lh_1} & q_{lh_2} & q_{ll} \end{bmatrix} = \begin{bmatrix} 1-\epsilon^1 & 0 & \epsilon^1 \\ 0 & 1-\epsilon^2 & \epsilon^2 \\ 0 & 0 & 1 \end{bmatrix}$$

This setup means that we have 2 'good' regimes, h_1, h_2, and one 'bad' regime, l. In regime h_1 the probability of dividend is $\Phi_1 > 0$ and the probability of switching to the bad regime is ϵ^1. In regime h_2 the probability of dividend is Φ_2 and the probability of switching to the bad regime is ϵ^2. In bad regime l there are no dividends ($\Phi_3 = 0$) and this state is absorbing.

In this setup we can readily get an interesting result.

Proposition 2 *If $\Phi_1, \epsilon^1, \Phi_2, \epsilon^2$ are such that $V_1 = V_2$, $\pi^1 = (1, 0, 0)$, $\pi^2 = (0, 1, 0)$, and $\Phi_1 \neq \Phi_2$ (the agents' valuations are exactly the same but the beliefs differ) then we have speculation, namely $p^*(\pi^1, \pi^2) > p^F(\pi^1, \pi^2) = V_1 = V_2$.*

This proposition says, that whenever the parameters are such that both agents' valuations of the asset are exactly the same but their beliefs about the probabilistic structure of dividends differers in any way, then there must be some speculation going on. The intuition behind this result is that if agents agree upon the discounted present value of the stream of the future dividends, then in order to have different probabilistic structure of them one of them must have a higher probability of dividend in 'his' good state (Φ_i), which must be compensated by a lower higher probability of switching into the low state (ϵ_i). Also, a simple algebra shows that the agent with the higher Φ_i must also have a higher probability of seeing a dividend the next period. This means that his willingness of buying the asset today and selling it for its fundamental value tomorrow must be higher than that of the other agent. We know that any agent's willingness to buy it today with the option of resell it tomorrow must be at least his own fundamental value, V_i. Finally, using that $V_1 = V_2$ we conclude that the agent with the higher Φ_i must be willing to pay for the asset more than $V_1 = V_2$, hence we must have speculation.

Proof Without loss of generality assume $\Phi_1 > \Phi_2$. We have:

$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} \frac{\beta(1-\epsilon^1)\Phi_1}{1-\beta(1-\epsilon^1)} \\ \frac{\beta(1-\epsilon^2)\Phi_1}{1-\beta(1-\epsilon^2)} \\ 0 \end{bmatrix}$$

Hence the condition $V_1 = V_2$ implies (after some rearrangements):

$$\frac{\Phi_2}{\Phi_1} = \frac{\frac{1}{1-\epsilon^2} - \beta}{\frac{1}{1-\epsilon^1} - \beta}$$

Then by $\Phi_1 > \Phi_2$ we get that:

$$\frac{1}{1-\epsilon^2} - \beta < \frac{1}{1-\epsilon^1} - \beta$$

which implies $\epsilon^1 > \epsilon^2$. Also note, that we can rearrange the condition $V_1 = V_2$ in another way to get:

$$\frac{(1-\epsilon^1)\Phi_1}{(1-\epsilon^2)\Phi_2} = \frac{1-\beta(1-\epsilon^1)}{1-\beta(1-\epsilon^2)}$$

But because $\epsilon^1 > \epsilon^2$, the RHS of the above is bigger than 1 hence we have:

$$(1-\epsilon^1)\Phi_1 > (1-\epsilon^2)\Phi_2$$

Now we will see that $T^1 p^F(\pi^1,\pi^2) > T^2 p^F(\pi^1,\pi^2)$. Using the definition of T^i we get:

$$
\begin{aligned}
T^i p^F(\pi^1,\pi^2) &= \beta\left[\Pr^{\pi^i}(d'=1)(1+\max\{\lambda^1(\pi^1)V,\lambda^1(\pi^2)V\}) + \Pr^{\pi^i}(d'=0)\max\{\lambda^0(\pi^1)V,\lambda^0(\pi^2)V\}\right] \\
&= \beta\left[(1-\epsilon^i)\Phi_i(1+V_1) + (1-(1-\epsilon^i)\Phi_i)\max\left\{\frac{(1-\epsilon^1)(1-\Phi_1)}{1-(1-\epsilon^1)\Phi_1}V_1, \frac{(1-\epsilon^2)(1-\Phi_2)}{1-(1-\epsilon^2)\Phi_2}V_2\right\}\right] \\
&= \beta\left[(1-\epsilon^i)\Phi_i(1+V_1) + (1-(1-\epsilon^i)\Phi_i)\max\left\{\frac{(1-\epsilon^1)(1-\Phi_1)}{1-(1-\epsilon^1)\Phi_1}, \frac{(1-\epsilon^2)(1-\Phi_2)}{1-(1-\epsilon^2)\Phi_2}\right\}V_1\right]
\end{aligned}
$$

Since $\frac{(1-\epsilon^1)(1-\Phi_1)}{1-(1-\epsilon^1)\Phi_1}, \frac{(1-\epsilon^2)(1-\Phi_2)}{1-(1-\epsilon^2)\Phi_2} < 1$, and $(1-\epsilon^1)\Phi_1 > (1-\epsilon^2)\Phi_2$ then indeed we must have $T^1 p^F(\pi^1,\pi^2) > T^2 p^F(\pi^1,\pi^2)$. We also must have that $T^2 p^F(\pi^1,\pi^2) \geq V_2$ (if agent 2 is promised to be paid at least his own valuation tomorrow, then his willingness to pay for the asset must be at least his own valuation today, which is V_2). Hence we have shown that $T^1 p^F(\pi^1,\pi^2) > T^2 p^F(\pi^1,\pi^2) \geq V_2 = V_1 = p^F(\pi^1,\pi^2)$. Now by Proposition 2 we get $p^*(\pi^1,\pi^2) > p^F(\pi^1,\pi^2)$ hence we have speculation in equilibrium. ∎

One can expect that this speculation may persist very long. Each time the agents observe $d=1$ they know they cannot be in the state l, so system is reset (we are back to the initial beliefs, which we know lead to speculation). Indeed this speculation will last till we are finally settled in state l.

5 Conclusion

In this paper I construct a model of speculation which looks like a promising tool in modeling long lasting speculative behavior when investors are learning from data. The idea is that investors, even though they learn from data, sometimes have to wait for some particular stream of signals to learn about certain aspects of the underlying regime. In the last example we saw that each time the agents observe dividend 1 the system is almost reset. Hence in order to achieve convergence of beliefs they need to observe a sufficiently long stream of zeros, so that both agents get convinced that a bad regime really occurred (once it happens their beliefs are pretty much the same because we have only one bad regime).

It is clear that when calibrating this model we can use more regimes in order to be able to capture some more sophisticated states which can be distinguished only after observing some very specific sequence of signals. For such signals we may need to wait very long. This creates the persistence of speculation, which is somehow hidden before that particular sequence occurs. This would be a good explanation for bursting of market bubbles. Doing so seems like a natural direction of future research.

Appendix

Remaining proofs

Proof of Lemma 4. First note, that since we defined:

$$\lambda^d(\pi)(A \times \Phi \times Q) = \Pr^\pi(a_1 \in A \wedge \phi \in \Phi \wedge q \in Q | d_1 = d)$$
$$= \frac{\int_{\phi\in\Phi}\int_{q\in Q}\int_{a_1\in A}\int_{a_0\in\mathcal{A}} \pi(da_0,d\phi,dq)q(a_0,da_1)\phi_{a_1}(d_1)}{\int_{\phi\in\mathbf{\Phi}}\int_{q\in\mathcal{Q}}\int_{a_0,a_1\in\mathcal{A}^2} \pi(da_t,d\phi,dq)q(a_0,da_1)\phi_{a_1}(d_1)}$$

for each measurable $A \subseteq \mathcal{A}, \Phi \subseteq \mathbf{\Phi}, Q \subseteq \mathcal{Q}$, then each time we integrate with respect to the measure $\lambda^d(\pi)$, we can do the following replacement under any integral (the quotes will not be needed under an actual integral):

$$\lambda^d(\pi)(da_1,d\phi,dq) \text{"="} \frac{\int_{a_0\in\mathcal{A}} \pi(da_1,d\phi,dq)q(a_0,da_1)\phi_{a_1}(d_1)}{\int_{\phi\in\mathbf{\Phi}}\int_{q\in\mathcal{Q}}\int_{a_0,a_1\in\mathcal{A}^2} \pi(da_1,d\phi,dq)q(a_0,da_1)\phi_{a_1}(d_1)}$$

Using this we have:

$$\Pr^{\lambda^d(\pi)}(a_0 \in A_0 \ldots, a_s \in A_s, \phi \in \Phi, q \in Q) =$$
$$= \int_{\phi\in\Phi}\int_{q\in Q}\int_{(a_0,\ldots,a_s)\in A_0\times A_s} \lambda^d(\pi)(da_0,d\phi,dq)q(a_0,da_1)$$
$$= \int_{\phi\in\Phi}\int_{q\in Q}\int_{(a_1,\ldots,a_{s+1})\in A_0\times\ldots\times A_s} \frac{\int_{a_0\in\mathcal{A}} \pi(da_0,d\phi,dq)q(a_0,da_1)\phi_{a_1}(d_1)}{\int_{\phi\in\mathbf{\Phi}}\int_{q\in\mathcal{Q}}\int_{a_0,a_1\in\mathcal{A}^2} \pi(da_1,d\phi,dq)q(a_0,da_1)\phi_{a_1}(d_1)}$$
$$= \frac{\int_{\phi\in\Phi}\int_{q\in Q}\int_{a_0\in\mathcal{A}}\int_{(a_1,\ldots,a_{s+1})\in A_0\times\ldots\times A_s} \pi(da_0,d\phi,dq)q(a_0,da_1)\phi_{a_1}(d_1)}{\int_{\phi\in\mathbf{\Phi}}\int_{q\in\mathcal{Q}}\int_{a_0,a_1\in\mathcal{A}^2} \pi(da_1,d\phi,dq)q(a_0,da_1)\phi_{a_1}(d_1)}$$
$$= \Pr^\pi(a_1 \in A_0, \ldots, a_{s+1} \in A_s, \phi \in \Phi, q \in Q | d_1 = d)$$

■ **Proof of Theorem 3.** It is straightforward to check that the proposed allocation satisfies feasibility, budget feasibility as well as measurability assumptions. The only thing which requires an argument is that the proposed agents' plans, $(\gamma_t^{*i})_t$ maximize their utilities, given prices. We shall do it only for agent 1 (the other follows by symmetry). The proof here follows along the lines of the proof of theorem 9.2 of Stokey et al. (2004) with an adjustment for slight change in their Markov environment (our environment is technically not Markov but thanks to Lemma 1 we may treat it as if it was).

Clearly in any solution to an agent's problem the budget constraint is satisfied with the equalities, therefore wlog we may assume agent 1 is choosing only $\gamma^1 \equiv$

$(\gamma_1^1, \gamma_2^1(d^1), \gamma_3^1(d^2)\ldots) \geq 0$ (following Stokey et al. (2004) we call it a plan) to maximize:

$$u(\gamma, \gamma_0^1, \pi^1, \pi^2) \equiv \lim_{T\to\infty} u_T(\gamma, \gamma_0^1, \pi^1, \pi^2)$$

where $u_T(\gamma, \gamma_0^1, \pi^1, \pi^2) \equiv \mathrm{E}^{\pi_0^1} \sum_{t=0}^{T} \beta^t [p_t(\gamma_t^1 - \gamma_{t+1}^1) + \gamma_t d_t]$ taken as given $\gamma_0^1 = 0$.

Denote Γ to be the set of feasible plans for asset holdings for agent 1 (i.e. satisfying $\gamma_t^1 \geq 0$, $\mathcal{F}_t^d$-measurability and such that u is well defined, potentially allowing for $\pm\infty$).

Following the notation of Stokey et al. (2004) we denote:

$$V^*(\gamma_0^1, \pi_0^1, \pi_0^2) = \sup_{\gamma\in\Gamma} u(\gamma, \pi_0^1, \pi_0^2) \tag{10}$$

for each $\gamma_0^1 \geq 0$.

We will show that $V(\gamma_0^1, \pi_0^1, \pi_0^2) = V^*(\gamma_0^1, \pi_0^1, \pi_0^2)$ and that proof will imply that γ^{*1} attains the sup in (10). First we prove that

$$V(\gamma_0^1, \pi_0^1, \pi_0^2) \geq u(\gamma, \gamma_0^1, \pi_0^1, \pi_0^2) \tag{11}$$

for all $\gamma \in \Gamma$, and then we will see that

$$V(\gamma_0^1, \pi_0^1, \pi_0^2) = u(\gamma^{*i}, \gamma_0^1, \pi_0^1, \pi_0^2) \tag{12}$$

We have for any $\gamma^1 \in \Gamma$,

$$\begin{aligned}
V(\gamma_0^i, \pi_0^1, \pi_0^2) &= \max_{\gamma'\geq 0} \Big\{ (\gamma_0^i - \gamma')p(\pi_0^1, \pi_0^2) + \beta \mathrm{E}^{\pi_0^1}\big(V(\gamma', \lambda(d_1|\pi_0^1), \lambda(d_1|\pi_0^2)) + \gamma' d_1\big)\Big\} \\
&\geq (\gamma_0^i - \gamma_1^1)p(\pi_0^1, \pi_0^2) + \beta \mathrm{E}^{\pi_0^1}\big(V(\gamma_1^1, \lambda(d_1|\pi_0^1), \lambda(d_1|\pi_0^2)) + \gamma_1^1 d_1\big) \\
&= (\gamma_0^i - \gamma_1^1)p(\pi_0^1, \pi_0^2) + \beta \mathrm{E}^{\pi_0^1}\left(\max_{\gamma'\geq 0}\Big\{ (\gamma_1^1 - \gamma')p(\pi_0^1, \pi_0^2) + \beta \mathrm{E}^{\lambda(d^1|\pi_0^1)}\Big(V(\gamma', \lambda(d_1|\pi_0^1), \lambda(d_1|\pi_0^2)) + \gamma' d_1\Big)\Big\} + \gamma_1^1 d_1 \right) \\
&= (\gamma_0^i - \gamma')p(\pi_0^1, \pi_0^2) + \beta \mathrm{E}^{\pi_0^1}\left(\max_{\gamma'\geq 0}\Big\{ (\gamma_1^1 - \gamma')p(\pi_0^1, \pi_0^2) + \beta\Big(V(\gamma', \lambda(d_2|\pi_0^1), \lambda(d_2|\pi_0^2)) + \gamma' d_2\Big)\Big\} + \gamma_1^1 d_1 \right) \\
&\geq (\gamma_0^i - \gamma')p(\pi_0^1, \pi_0^2) + \beta \mathrm{E}^{\pi_0^1}\Big((\gamma_1^1 - \gamma_2^1)p(\pi_0^1, \pi_0^2) + \beta\Big(V(\gamma_2^1, \lambda(d_2|\pi_0^1), \lambda(d_2|\pi_0^2)) + \gamma_2^1 d_2\Big) + \gamma_1^1 d_1\Big) \\
&= u_1(\gamma^1, \gamma_0^1, \pi^1, \pi^2) + \beta^2 \mathrm{E}^{\pi_0^1}\big(V(\gamma_2^1, \pi_1(d_2|\pi_0^1), \lambda(d_2|\pi_0^2)) + \gamma_1^2 d_2\big)
\end{aligned}$$

Here, line 4 follows from Lemma 1 and the law of iterated expectations. Note some notational complication in line 3 caused by the fact that d_1 under the second expectation is a dummy variable for that expectation and is a different d_1 then that out of that expectation. Indeed d_1 under the second expectation refers to the period 2 from the perspective of initial beliefs, but it is the first period from the perspective of updated second period beliefs — actually thanks to Lemma 1 we can replace that d_1 with d_2 in line 4.

Now we may continue this process to obtain by induction that

$$V(\gamma_0^i, \pi_0^1, \pi_0^2) \geq u_T(\gamma^1, \gamma_0^1, \pi^1, \pi^2) + \beta^T \mathrm{E}^{\pi_0^1}\big(V(\gamma_T^1, \lambda(d_2|\pi_0^1), \lambda(d_2|\pi_0^2)) + \gamma_T^1 d_T\big)$$

for all T. Now, using the assumption that $V \geq 0$ we conclude that (11) holds. Having

(11) we can go over the above derivation replacing in each line γ' with the respective γ_t^{*1} (now getting the equality in each line by the construction of γ_t^{*1} which comes from the policy function for V) to obtain (12). To do so we need to use the assumed transversality condition. But this means that the plan γ^{*1} attains the maximum for agent one's problem. ■

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