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Stability in Distribution of Randomly Perturbed Quadratic Maps as Markov Processes

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STABILITY IN DISTRIBUTION OF RANDOMLY PERTURBED QUADRATIC MAPS AS MARKOV PROCESSES

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ABSTRACT

Iteration of randomly chosen quadtratic maps defines a Markov process: $X_{n+1} = \varepsilon_{n+1} X_n(1-X_n)$, where ε_n are i.i.d. with values in the parameter space $[0, 4]$ of quadratic maps $F_\theta(x) = \theta x(1-x)$. Its study is of significance not only as an important Markov model, but also for dynamical systems defined by the individual quadratic maps themselves. In this article a broad criterion is established for positive Harris recurrence of X_n, whose invariant probability may be viewed as an approximation to the so-called Kolmogorov measure of a dynamical system.

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1. **Introduction**

In this section we provide background and motivation for the problem at hand, including a survey of known results and a statement of our main result. Section 2 reviews some important facts about quadratic maps as dynamical systems. Section 3 contains the main result and its proof. The *quadratic maps* $F_\theta x \equiv F_\theta(x) = \theta x(1-x)$ on $[0,1](\theta \in [0,4])$ have been perhaps the most widely investigated family of dynamical systems in recent years (see, e.g., [10], [12], [14],–[17], [22],[23],[26],[27]). Their study has led to a great deal of understanding of chaotic phenomena which commonly occur in nature, including certain types of turbulence ([14], ([26]). The present article explores the problem of stability in distribution of randomly perturbed dynamical systems of random compositions the form

$$X_n = F_{\varepsilon_n} F_{\varepsilon_{n-1}} \cdots F_{\varepsilon_1} X_0 \ (n \geq 1), \tag{1.1}$$

where $\varepsilon_n (n \geq 1)$ is an i.i.d. sequence with values in the parameter space $[0,4]$ of the quadratic maps, and X_0 is independent of $\{\varepsilon_n : n \geq \ 1\}$. To avoid trivialities we will restrict the state space of the Markov process to $S = (0,1)$. By *stability in distribution* we mean the convergence in distribution of $\frac{1}{n}\sum_{m=0}^{n-1} p^m(x,dy)$ to the same limit $\pi(dy)$ for every initial state $x \in S$, with $p^{(n)}$ denoting the n-step transition probability of $\{X_n : n \geq 0\}$. Then π is the unique invariant probability of this Markov process. The significance of such a study may be viewed from three distinct perspectives which we describe below.

(i) *Kolmogorov - SRB measures for chaotic dynamical systems*
A chaotic dynamical system f with a compact state space $K \subset \mathbb{R}^d$, by definition, has sensitive dependence on initial conditions (Devaney(1989), p.50). Since an "exact" measurement of a state at some point of evolution (call it the 'initial state') is virtually impossible, states in the distant future are unpredictable. For most applications, however it is enough to know the large time statistical behavior of the trajectory $\{f^n x : n \geq 0\}$. That is, one needs to know if the empirical process $\frac{1}{n}\sum_{n=0}^{n-1} \delta_f n_x$ converges as $n \to \infty$ to some limit, say β, independent of x for almost all x(w.r.t Lebesgue measure), and if so what is this limit. If it exists, this limit is necessarily an invariant probability for the dynamical system: $\beta f^{-1} = \beta$, and it is ergodic. But there are infinitely many ergodic invariant probabilities for a chaotic dynamical system. In particular, the uniform distribution on a (necessarily repelling, or unstable) periodic orbit is an ergodic, i.e., extremal, invariant probability, and there are infinitely many such invariant probabilities on distinct periodic orbits (Devaney(1989), pp.49,50), none of which can be β. Long time ago Kolmogorov suggested that one should randomly perturb the dynamical system by adding an absolutely continuous noise component so that the resulting Markov process has a unique invariant probability, say π. The limit of π, as the noise goes to zero, should be β. Kolmogorov's conjecture has been proved for Axiom A diffeomerphisms independently by Sinai, Ruelle and Bowen (see Eckmann and Ruelle (1985), or Kifer (1988) for a precise statement), and the limit β is called the *SRB measure* in this case. We will refer to it more generally as the *Kolmogorov measure.* In our present context of quadratic maps F_θ, the existence of such a measure has been proved by Katok and Kifer (1986) for those values of θ which satisfy the Misiurewicz condition: F_θ has no stable periodic orbit and $\frac{1}{2}$

does not belong to the closure of the trajectory $\{F_\theta^n \frac{1}{2} : n \geq 1\}$. It was shown by Misiurewicz (1981) that under this condition F_θ has a unique absolutely continuous invariant probability β, and that this condition is satisfied by uncountably many parameter values θ. It may be noted that, except in special cases, β is virtually impossible to compute analytically. On the other hand viewed as an approximation of β, π is more tractable and, at the least, has approximations $\frac{1}{N}\sum_{n=1}^{N} p^{(n)}(x, dy)$ where $p^{(n)}$ may be expressed analytically be recursion.

(ii) *Randomly perturbed dynamical systems as models of physical phenomena.* As has been pointed out by Eckmann and Ruelle (1985), physical systems are often "stochastically excited". For such systems and many social phenomena, the randomly perturbed dynamical system as a stochastic process is a more relevant object of study than the deterministic system. Although such phenomena are widespread in nature, we mention one particular application from economics that has provided at least a part of the motivation for this work. Consider a dynamic optimization problem in which one is given a *"production function"* $f : \mathbb{R}_+ \to \mathbb{R}_+$ and a *welfare function* $w : \mathbb{R}_+^2 \times A \to \mathbb{R}_+$ where A is a parameter set, say $A = [1, 4]$, parametrizing a family of economies. For an initial $x \geq 0$, a *program* $x_n : n \geq 0$, is a sequence such that $0 \leq x_0 = x \leq x_n \leq f(x_{n-1})$. The *consumption sequence* $\{c_n : n \geq 1\}$ is defined as $c_n = f(x_{n-1}) - x_n$. Given a *discount factor* $\delta > 0$, and a parameter value θ, one wishes to find an *optimal program* $\{\hat{x}_n : n \geq 0\}(\hat{x}_0 = x)$, which maximizes $\sum_{n=0}^{\infty} \delta^n w(x_n, c_{n+1}, \theta)$ over all programs $\{x_n : n \geq 0\}$ starting at $x_0 = x$. Under economically feasible assumptions one may find f and w such that an optimal program is given recursively by $\hat{x}_{n+1} = \theta\hat{x}_n(1 - \hat{x}_n), x \in [0, 1]$ (Majumdar and Mitra (2000)). That is, the optimal program is given by the trajectory of the dynamical system F_θ with initial state x. Since 'uncertainty' is inherent in economic systems, one may thus obtain a randomly perturbed quadratic system. Alternatively, one may at the out set consider a *stochastic dynamic programming problem* and directly arrive at a stationary optimal policy leading to an evolution of states of the form (1.1)(Mitra (1998)).

(iii) *Randomly perturbed quadratic maps as mathematically interesting Markov processes.* Last, but not least, randomly perturbed dynamical systems such as (1.1) are Markov processes and, conversely, every discrete parameter Markov process on a standard state space may be viewed as a randomly perturbed dynamical system (see, e.g., [8], p. 228). The particular class of such processes (1.1) and their large time properties have been the subject of study in [1], [2], [5]-[7], [9], reviewed below.

In the remainder of this section we review the literature on the process (1.1), leading up to the mathematical problem dealt with in this article.

As indicated following (1.1), we will consider the Markov process X_n on the state space $S = (0, 1)$ to avoid constantly having to exclude the uninteresting invariant probability δ_0 (the one-point mass distribution at 0) if the state space is taken as $[0, 1]$. Let μ denote *the smallest point and* λ *the largest point of the support* of the common distribution Q of ε_n. It is proved in [5]-[7] that there exists a unique invariant probability π of the Markov process X_n, which is then stable in distribution, if either

$$1 < \mu < \lambda \leq 2, \tag{1.2}$$

holds, or

$$2 < \mu < \lambda \leq 1 + \sqrt{5} \ \text{ and } \ \frac{8}{\lambda(4-\lambda)} \leq \mu < \lambda \tag{1.3}$$

hold. In the case (1.2), the maps $F_\theta, \mu \leq \theta \leq \lambda$, are monotone increasing on the invariant interval $[p_\mu, p_\lambda]$, where $p_\theta = 1 - 1/\theta$ is a fixed point of $F_\theta (1 < \theta \leq 4)$. In the case (1.3), the maps F_θ, $\mu \leq \theta \leq \lambda$, are monotone decreasing on the invariant interval $[\frac{1}{2}, \lambda/4]$. One may then apply a theorem of Dubins and Freedman (1966) to prove the existence and uniqueness of an invariant probability of the process (1.1) (Bhattacharya and Rao (1993), Bhattacharya and Majumdar (1999) and Bhattacharya Waymire (2002)). This technique may also be extended to the case where F_μ and F_λ have attractive 2^n-period orbits, for some $n \geq 1$, such that the second condition in (1.3) holds (which guarantees that $[\frac{1}{2}, \lambda/4]$ is invariant under F_θ for $\mu \leq \theta \leq \lambda$) and μ and λ are close enough so that the line segment joining the largest points of their attractive 2^n-period orbits does not include any other periodic fixed point of either F_μ or F_λ (See [7]). An example of this latter kind is provided (for the case $n = 2$) by $\mu = 3.15$ and $\lambda = 3.20$ (see [5]).

It has been recently shown by Carlsson (2002), using a "weak contractivity" criterion, that the interval (1.2) may be extended to $1 < \mu < \lambda \leq 3$ for the existence and uniqueness of in invariant probability π on $(0, 1)$. Assuming Q is absolutely continuous on $[\mu, \lambda]$, $1 < \mu < \lambda < 4$, with a density bounded away from zero on an open interval contained in (1.3), and $F_\mu(\lambda/4) \geq \frac{1}{2}$, Dai (2000) showed that X_n is Harris recurrent and has a unique invariant probability.

In a somewhat different vein, Athreya and Dai (2000) have shown that a *necessary condition* for the *existence* of an invariant probability on $(0, 1)$ is

$$E \log \varepsilon_1 > 0. \tag{1.4}$$

To outline a proof of this, express (1.1) by recursion: $X_{n+1} = \varepsilon_{n+1} X_n (1 - X_n)$, so that $\log X_{n+1} = \log \varepsilon_{n+1} + \log X_n + \log(1 - X_n)$. Note that if an invariant π exists then for the stationary process $\{X_n : n \geq 0\}$ with initial distribution π one may take expectations on both sides of the last equation to get $E \log X_{n+1} = E \log \varepsilon_{n+1} + E \log X_n + E \log(1 - X_n)$. Since $E \log X_{n+1} = E \log X_n$, one arrives at $E \log \varepsilon_{n+1} = -E \log(1 - X_n) > 0$, yielding the 'necessary' criterion (1.4). Of course, since on the *compact* state space [0,1], δ_0 is invariant, it follows that if $E \log \varepsilon_1 \leq 0$ (contrary to (1.4)), then X_n must converge in probability to zero no matter what X_0 may be. Athreya and Dai (loc.cit.) also showed that a *sufficient condition* for the *existence* of an invariant probability π on $(0, 1)$ is that, in addition to (1.4), one has

$$E|\log(4 - \varepsilon_1)| < \infty. \tag{1.5}$$

Conditions (1.4), (1.5) together are *not necessary and sufficient* for the existence of an invariant probability, as is shown, with $Q = \delta_4$ by the famous example of von Neumann and Ulam (1947): F_4 has ergodic, or extremal, absolutely continuous invariant probability whose density is $(1/\pi)[x(1-x)]^{-1/2}$.

The conditions (1.4) and (1.5) of Athreya and Dai (loc. cit.) only imply existence, and not *uniqueness*, of an invariant probability π on $(0, 1)$. Indeed, Athreya and Dai (2002) provide an example with at least two extremal invariant probabilities on $(0,1)$ with a Q that has a two-point support $\{(2/3+\delta)^{-1}, (1/3-\delta)^{-1} = \{\mu, \lambda\}$ for a sufficiently small $\delta > 0$, and with $\eta = Q(\{\mu\})$ sufficiently small. Since the set $\{\frac{1}{3}-\delta, \frac{2}{3}+\delta\} = \{a, b\}$ is invariant under both F_μ and F_λ one easily shows that $\pi_1(\{a\}) = 1-\eta, \pi_1(\{1\}) = \eta$ defines an invariant probability. Since b is a repelling fixed point of F_λ, which has an attractive period-two orbit $\{c, d\}$ say, if $\lambda \equiv (\frac{1}{3}-\delta)^{-1} < 1+\sqrt{6}$, (see, e.g., Sandefur (1990), pp. 172-181), and since η is small, one can prove the existence of an invariant probability π_2 whose support contains $\{c, d\}$. Note that in this case $1 < \mu < 2$, while λ is only slightly larger than 3. That is, $\{\mu, \lambda\}$ does not satisfy (1.2) or (1.3), and barely misses the sufficiency criterion $1 < \mu < \lambda \leq 3$ of Carlsson (loc. cit.) mentioned above. We believe that such *nonuniqueness* and the consequent lack of stability in distribution, are quite common for discrete Q. Our main objective in the present article is to show that if Q has a density component w.r.t Lebesgue measure on $[0,4]$ (which is bounded away from zero on some interval), and if $\{(1/N)\sum_{n=1}^{N} p^{(n)}(x, dy) : N \geq 1\}$ is tight for some $x \in (0,1)$, then X_n is Harris recurrent and, therefore, has a unique invariant probability π to which $(1/N)\sum_{n=1}^{N} p^{(n)}(x)$, converges, in total variation distance (Theorem 3.1). The tightness condition is guaranteed, e.g., by (1.4) and (1.5).

Finally, Figures 1(a)-(h) show computer simulations of the empirical measure $(1/N)\sum_{n=0}^{N-1} \delta_{F_\theta^n x}$ of the dynamical system F_θ for several different values of θ, starting with $x_0 = .5$. In $1(a)-(c)$ with $N = 10,000$, one clearly sees a transition from a chaotic $F_{3.82}$ to $F_{3.83}$ and $F_{3.84}$ with a stable period-three orbit (and a uniform distribution on this orbit). Then at $\theta = 3.85$ the histogram shows quasi-periodic behavior (not chaotic), passing to a chaotic behavior at $\theta = 3.86,\ 3.87$. Chaotic behavior is also seen for $\theta = 3.92,\ 3.93$. To enhance the asymptotics, in Figure, 1(d)-(h), of $70,000$ iterations the first $m = 20,000$ are omitted in constructing the histogram, but no significant difference was noticeable from the corresponding histogram based only on the first $20,000$ iterations (The latter are not shown). It is known, e.g., that there is at least one θ in $[3.92,\ 3.93]$ for which F_θ has an absolutely continuous invariant probability (Bala and Majundar (1992)).

Figures 2(a)-(f) show histograms of empirical measures for the Markov process $\{X_n : 0 \leq n \leq 10,000\}$ with Q as the uniform distribution on the indicated interval.

2. **The quadratic family of maps**

We recall in this section some basic facts about the family of quadratic maps $\{F_\theta : \theta \in [0,4]\}$. See Collet and Eckman (1980), Devaney (1989), Eckmann and Ruelle (1985), and Ruelle (1989) for details concerning general properties of these maps. F_θ is said to have a *period-m orbit* if there are m distinct points $x^1, x^2, \cdots, x^m$ such that $F_\theta x^i = x^{i+1},\ 1 \leq i \leq m, x^{m+1} := x^1$. The case $m = 1$ corresponds to a *fixed point* x^1 of $F_\theta \cdot A$ period-m orbit of F_θ is *attractive* if it has an open neighborhood U such that, for every $x \in U$, $F_\theta^n x$ converges to this orbit as $n \to \infty$. A period-m orbit $\{x^1, \cdots, x^m\}$ of F_θ is *repelling*, or *unstable*, if it has an open neighborhood U such that if $x \in U \backslash \{x^1, \ldots, x^m\}$ then $F_\theta^n x \in U^c$ for some $n \geq 1$. A period-m orbit $\{x^1, \ldots, x^m\}$ of F_θ is said to be *hyperbolic* if $|(F_\theta^m)'(x^1)| \neq 1$.

Since $F_\theta(0) = 0$ for every $\theta, 0$ is a fixed point of F_θ for $\theta \in [0,4]$. For $0 \leq \theta \leq 1, 0$ is an attractive fixed point of F_θ, and for $\theta > 1$ it is repelling. For $\theta > 1$ a new fixed point of F_θ occurs at $p_\theta = 1 - 1/\theta$. This fixed point is attractive for $1 < \theta \leq 3$ and repelling for $\theta > 3$. For $\theta > 3, F_\theta$ has a period-2 orbit, which is attractive for

$3 < \theta \le 1+\sqrt{6}$ and repelling for $\theta > 1+\sqrt{6}$. A period-4 orbit of F_θ appears for $\theta > 1+\sqrt{6}$ which is attractive on some interval $(1+\sqrt{6}, \theta_4]$ and repelling for $\theta > \theta_4$. Next a period-8 orbit shows up for $\theta > \theta_4$, which remains attractive on an interval $(\theta_4, \theta_8]$ and becomes repelling for $\theta > \theta_8$. This *period doubling bifurcation* continues indefinitely, with a period-2^n orbit of F_θ arising for $\theta > \theta_{2^{n-1}}$, say, which is attractive on an interval $(\theta_{2^{n-1}}, \theta_{2^n}], (n \ge 1), \theta_\infty := \lim_{n\to\infty} \theta_{2^n} = 3.57\cdots$. What happens at θ_∞ and immediately after is not entirely clear. But, according to a celebrated theorem of Sarkovskii (Devaney (1989),pp. 60-62), periods in reverse order $\triangleright$ of their first appearance may be arranged as

$$\begin{aligned}&3 \triangleright 5 \triangleright 7 \triangleright 9 \triangleright \cdots \triangleright 2\cdot 3 \triangleright 2\cdot 5 \triangleright 2\cdot 7 \triangleright 2\cdot 9 \triangleright \cdots \triangleright 2^n\cdot 3 \triangleright 2^n\cdot 5 \triangleright 2^n \cdot 7 \triangleright \\ &\cdots\cdots \triangleright 2^n \triangleright 2^{n-1} \triangleright \cdots \triangleright 2^3 \triangleright 2^2 \triangleright 2 \triangleright 1.\end{aligned} \tag{2.1}$$

Period 3 appears last in this list at $\theta^* = 3.8284\cdots$. For $\theta \ge \theta^*$ the map F_θ has periodic orbits of all orders. Notice that for $\theta \ge \theta^*$ period-doubling bifurcations occur giving rise to successive periods $3.2, 3{\cdot}2^2, \cdots, 3{\cdot}2^n, \cdots$, with $\lim_{n\to\infty} \theta_{3.2^n} = 3.8495\cdots$. In particular, attractive periodic orbits appear again and again as θ increases. A deep recent result of Graczyk and Swiatek (1997) says that the open set of θ's in $[0,4]$ for which F_θ has a hyperbolic attractive periodic orbit is dense in $[0,4]$. This result will prove crucial for our main result Theorem 3.1 in the next section. Consider an F_θ which has no attractive periodic orbit. Then F_θ may be *quasi-periodic*, i.e., for all x outside a set of Lebesgue measure zero, the empirical measure of the trajectory, namely, $\frac{1}{N}\sum_{n=0}^{N-1} \delta_{F_\theta{}^n x}$ converges to an invariant probability measure π (independent of x) whose support is a Cantor set of Lebesgue measure zero. Also, in this case F_θ has no sensitive dependence on initial conditions, as defined below, i.e., F_θ is not chaotic.

Another possibility is that F_θ has no attractive periodic orbit and it has *sensitive dependence on initial conditions*. This means there exists $\delta > 0$ such that, however close two initial points $x \ne y$ may be, $|F_\theta^n x - F_\theta^n y| > \delta$ for some n. F_θ is then said to be *chaotic*. An important result of Jakobson (1981) says that the set of θ's for which F_θ has an (extremal) absolutely continuous invariant probability (and, in particular, F_θ is chaotic) has positive Lebesgue measure. In deed, Jakobson (loc.cit.) proves that the set Λ of such θ's have Lebesgue density 1 at $\theta = 4$: $\lim_{\varepsilon\downarrow 0} \dfrac{\text{Leb}(\Lambda \cap [4-\varepsilon, 4])}{\varepsilon} = 1$, where Leb (A) is Lebesgue measure of A.

Figures 1(a),(e)-(h), correspond to chaotic maps F_θ. Figure 1(b), 1(c) correspond to F_θ having a stable period-3 orbit, while Figure 1(d) is indicative of a quasi-periodic F_θ.

3. **Harris recurrence and ergodicity**

On the state space $S = (0,1)$ consider the Markov process defined recursively by

$$X_{n+1} = F_{\varepsilon_{n+1}} X_n \quad (n = 0,1,2,\cdots) \tag{3.1}$$

where $\{\varepsilon_n : n \ge 1\}$ is a sequence of i.i.d. random variables with values in $(0,4)$ and, for each value $\theta \in (0,4), F_\theta$ is the quadratic function (on S):

$$F_\theta x \equiv F_\theta(x) = \theta x(1-x) \qquad 0 < x < 1. \tag{3.2}$$

As always, the initial random variable X_0 is independent of $\{\varepsilon_n : n \geq 1\}$. Our main result provides a criterion for Harris recurrence and the existence of a unique invariant probability for the process $\{X_n : n \geq 0\}$.

Let $p(x, dy)$ denote the (one-step) transition probability of $\{X_n : n \geq 1\}$ and $p^{(m)}(x, dy)$ the corresponding $m-$step transition probability.

Theorem 3.1. *Assume that the distribution of ε_1 has a nonzero absolutely continuous component (w.r.t. Lebesgue measure on $(0,4)$) whose density is bounded away from zero on some nondegenerate interval in $(0,4)$. If, in addition, $\{\frac{1}{N}\sum_{n=1}^{N} p^{(n)}(x, dy) : N \geq 1\}$ is tight on $S = (0,1)$ for some x, then (i)$\{X_n : n \geq 0\}$ is Harris recurrent and has a unique invariant probability π and (ii) $\frac{1}{N}\sum_{n=1}^{N} p^{(n)}(x, dy)$ converges to π in total variation distance, for every x, as $n \to \infty$.*

Using the sufficiency conditions (1.4), (1.5) of Athreya and Dai (2000), the following corollary is an immediate consequence of the theorem.

Corollary 3.2. *If ε_1 has a nonzero density component which is bounded away from zero on some nondegenerate interval and if, in addition,*

$$E\log\varepsilon_1 > 0 \text{ and } E|\log(4-\varepsilon_1)| < \infty, \tag{3.3}$$

then $\{X_n : n \geq 0\}$ has a unique invariant probability π on $S = (0,1)$ and $(1/N)\sum_{n=1}^{N} p^{(n)}(x, dy) \to \pi$ in total variation distance, for every $x \in (0,1)$.

Note that if the support of the distribution Q of ε_1 is contained in $[\mu, \nu]$ where $1 < \mu < \nu < 4$, then $[a, b] \equiv [\min\{1 - \frac{1}{\mu}, F_\mu(\frac{\nu}{4})\}, \frac{\nu}{4}]$ is an invariant interval for the Markov process (3.1) (See Bhattacharya and Rao (1993), Bhattacharya and Waymire (2002)). Since the transition probability has the Feller property, whatever be Q, there exists an invariant probability with support contained $[a, b] \subset (0,1)$. The result of Athreya and Dai(2000) is an important generalization of this.

We will need some lemmas for the proof of this theorem.

Lemma 3.3. *Suppose the distribution Q of ε_1 on $(0,4)$ has a nonzero absolutely continuous component (w.r.t. Lebesgue measure λ) whose density $h(\theta)$ is bounded away from zero on an interval $[c, d], c < d$. Then there exists a nonempty open interval $J \subset (0,1)$, a number $\delta > 0$, and a positive integer m such that*

$$\inf_{x\in J} p^{(m)}(x, B) \geq \delta\lambda(B)\ \forall \text{ Borel } B \subset J. \tag{3.4}$$

Proof. First assume Q is absolutely continuous with a continuous density h. Let $\theta_0 \in (0,4)$ be such that $h(\theta_0) > 0$ and F_{θ_0} has an attractive periodic orbit of period m. Such a point θ_0 exists, since the set of points θ for which F_θ has an attractive fixed point or an attractive periodic orbit is dense in $(0,4)$, by a result of Graczyk and Swiatek (1997). The n-step transition probability density is continuous in (x, y) and is given recursively, by

$$\begin{aligned} (3.5)\quad p(x,y) \equiv p^{(1)}(x,y) &= \frac{1}{x(1-x)} h\left(\frac{y}{x(1-x)}\right), \\ p^{(n+1)}(x,y) &= \int_{(0,1)} \frac{1}{z(1-z)} h\left(\frac{y}{z(1-z)}\right) p^{(n)}(x,z)dz, \\ &\qquad x, y \in (0,1), (n \geq 1). \end{aligned}$$

Let $\{x_0, x_1, \cdots, x_{m-1}\}$ be the attractive periodic orbit of $F_{\theta_0}: F_{\theta_0}\, x_{i-1} = x_i$ $(i = 1, \ldots, m)$, $x_m \equiv x_0$. Then

$$p^{(1)}(x_{i-1}, x_i) = \frac{1}{x_{i-1}(1-x_{i-1})} h\left(\frac{x_i}{x_{i-1}(1-x_{i-1})}\right) \tag{3.6}$$

$$= \frac{h(\theta_0)}{x_{i-1}(1-x_{i-1})} > 0 \ (1 \le i \le m),$$

since $x_i = \theta_0 x_{i-1}(1-x_{i-1}) \equiv F_{\theta_0} x_{i-1}$. By (3.6) and the continuity of $(x,y) \to p^{(1)}(x,y)$, there exist $h_i > 0$ such that

$$g(y_1, \cdots, y_{m-1}) := p^{(1)}(x_0, y_1) p^{(1)}(y_1, y_2) \cdots p^{(1)}(y_{m-2}, y_{m-1}) \cdot p^{(1)}(y_{m-1}, x_0)$$

$$> 0 \ \forall \ y_i \in [x_i - h_i, x_i + h_i] (1 \le i \le m-1),$$

so that

$$p^{(m)}(x_0, x_0) \ge \int \cdots \int g(y_1, \cdots, y_{m-1}) dy_1 \ldots dy_{m-1} > 0, \tag{3.7}$$

where the integration is over the rectangle $[x_1 - h_1, x_1 + h_1] \times \cdots \times [x_{m-1} - h_{m-1}, x_{m-1} + h_{m-1}]$. By the continuity of $(x,y) \to p^{(m)}(x,y)$, it follows that there exists an open neighborhood J of x_0 such that

$$p^{(m)}(x,y) \ge \delta > 0 \qquad \forall \ x, y \in \bar{J}, \tag{3.8}$$

where $\bar{J}$ is the closure of J in $(0,1)$. This proves (3.4) assuming that Q is absolutely continuous with a continuous density. In the general case let $I \subset (0,4)$ be a nondegenerate closed interval such that $h(\theta) \ge \delta' > 0 \ \forall \ \theta \in I$. There exists a nonnegative continuous function $\underline{\mathrm{h}}$ on $(0,4)$ such that $\underline{\mathrm{h}}(\theta) \ge \delta'/2 \ \forall \ \theta \in I$, and $\underline{\mathrm{h}}(\theta) \le h(\theta) \ \forall \ \theta \in (0,4)$. Define $p^{(n)}(x,y)$ in place of $\underline{\mathrm{p}}^{(n)}(x,y)(n \ge 1)$ in (3.5) by replacing h by $\underline{\mathrm{h}}$. Let θ_0 be a point in the interior of I such that F_{θ_0} has an attractive periodic orbit of period m, say, $\{x_0, x_1, \cdots, x_{m-1}\}$. Then the same argument as given above shows that there exists an open neighborhood J of x_0 such that $\underline{\mathrm{p}}^{(m)}(x,y) \ge \delta \ \forall \ x, y \in \underline{\mathrm{J}}$, for some $\delta > 0$. Since $h \ge \underline{\mathrm{h}}, p^{(m)}(x,y) \ge \underline{\mathrm{p}}^{(m)}(x,y) \ge \delta \ \forall \ x, y \in \bar{J}$, and the proof of (3.4) is complete. □

Our final lemma adds greater specificity to Lemma 3.3 and to the proof of Theorem 3.1

Lemma 3.4. *Assume the hypothesis of Lemma* 3.3. *There exist* $\gamma_1, \gamma_2 (c < \gamma_1 < \gamma_2 < d)$ *and* $m \ge 1$ *such that* (a) F_θ *has an attractive periodic orbit of period* m *for every* $\theta \in (\gamma_1, \gamma_2)$, *and* (b) *if* $q(\theta)$ *denotes the largest point of the attractive periodic orbit of* $F_\theta (\theta \in (\gamma_1, \gamma_2))$, *then there exists an open interval* $J \subset (0,1)$ *for which* $(i)(3.4)$ *holds,* (ii) $q(\theta) \in J \ \forall \ \theta \in (\gamma_1, \gamma_2), (iii)$ $\theta \to q(\theta)$ *is a diffeomorphism on* (γ_1, γ_2) *onto* J.

Proof. As in the proof of Lemma 3.3, let $\theta_0 \in (c,d)$ be such that F_{θ_0} has an attractive periodic orbit of some period, say, m. Apply the inverse function theorem to the function $(\theta, x) \to F_\theta^m x - x$ in a neighborhood of $(\theta_0, q(\theta_0))$. For this note that

$$\left(\frac{d}{dx}\{F_\theta^m x - x\}\right)_{\theta=\theta_0, x=q(\theta_0)} < 0, \tag{3.9}$$

in view of the property $|\frac{d}{dx}F^m_{\theta_0}x|_{x=q(\theta_0)} < 1$ (since $q(\theta_0)$ is an attractive fixed point of $F^m_{\theta_0}$). Hence there exists $\underline{\theta} < \theta_0 < \bar{\theta}$ such that $\theta \to q(\theta)$ is a diffeomorphism on $(\underline{\theta}, \bar{\theta})$ onto an open interval $I \subset (0,1)$. Now apply Lemma 3.3 to find an open interval $J = (u_1, u_2) \subset I, u_1 < q(\theta_0) < u_2$, such that (3.4) holds, and let $\gamma_i = q^{-1}(u_i)(i = 1, 2)$. □

Proof. of Theorem 3.1. Let π be an ergodic (i.e., extremal) invariant probability on $S = (0,1)$, which exists by the assumption of tightness. We will first show that $\pi(J) > 0$ for the set J in Lemma 3.4. Fix $x \in (0,1)$. There exists a point in the interval $(F_{\gamma_1}x, F_{\gamma_2}x)$ which is attracted to the (attractive) periodic orbit of F_{θ_0}, where θ_0 is as in the proof of Lemma 3.4. Note that, outside a set of Lebesgue measure zero, every point of (0,1) is so attracted (see, e.g., Collet and Eckmann (1980), p.13). Thus there exist n and $\theta_1^0, \theta_2^0, \ldots, \theta_n^0 \in (\gamma_1, \gamma_2)$ (with $\theta_i^0 = \theta_0, 2 \leq i \leq n$) such that $F_{\theta_1^0}F_{\theta_2^0}\cdots F_{\theta_n^0}x \in J$. Consider the open subset of $(0,1) \times (\gamma_1, \gamma_2)^n$ given by $\{(y, \theta_1, \theta_2, \ldots, \theta_n) : F_{\theta_1}F_{\theta_2}\cdots F_{\theta_n}y \in J\}$. Since $(x, \theta_1^0, \theta_2^0, \ldots, \theta_n^0)$ belongs to this open set, there exists a neighborhood of this point, say, $(y_1, y_2) \times (\theta_{11}, \theta_{12}) \times \cdots \times (\theta_{n1}, \theta_{n2}) \subset (0,1) \times (\gamma_1, \gamma_2)^n$ such that $\forall (y, \theta_1, \ldots, \theta_n)$ in this neighborhood, $F_{\theta_1}F_{\theta_2}\cdots F_{\theta_n}y \in J$. This implies that for every inital state $y \in (y_1, y_2) = I_x$, say, the probability that in n steps the Markov process will reach J is at least $c_1^n(\theta_{12} - \theta_{11})\cdots(\theta_{n2} - \theta_{n1})$ where $c_1 := \inf\{h(\theta) : \theta \in [c,d]\} = \varepsilon_n(x)$, say, with $\varepsilon_n(x) > 0$. Now choose x such that it belongs to the support of π. Then $\pi(I_x) > 0$ and, with $n = n(x)$ as above,

$$\begin{aligned}\pi(J) = \int p^{(n)}(y,J)\pi(dy) &\geq \int_{I_x} p^{(n)}(y,J)\pi(dy) \\ &\geq \varepsilon_n(x)\pi(I_x) > 0.\end{aligned} \tag{3.10}$$

Also, by Lemma 3.3,

$$\pi(B) \geq \int_J p^{(m)}(x,B)\pi(dx) \geq \delta\lambda(B)\pi(J) \quad \forall \text{ Borel } B \subset J. \tag{3.11}$$

In particular, π is absolutely continuous on J w.r.t. Lebesgue measure, with a density bounded below by $\delta\pi(J) > 0$. Since the same argument would apply to every invariant ergodic probability π_1, and two distinct extremal invariant measures are mutually singular, it follows that π is the unique invariant probability.

Since we have argued above that for every $x \in S = (0,1)$ there exists $n = n(x)$ such that $p^{(n)}(x,J) > 0$, we get, using (3.4),

$$\begin{aligned}p^{(n+m)}(x,B) \geq \int_J p^{(m)}(z,B)p^{(n)}(x,dz) &\geq \delta\lambda(B)p^{(n)}(x,J) \\ &> 0 \qquad \forall x \in S \equiv (0,1), \\ &\qquad \forall\ B \subset J, \lambda(B) > 0.\end{aligned} \tag{3.12}$$

Hence the Markov process is *irreducible* with respect to the measure $\phi(B) := \lambda(B \cap J), B$ *Borel* $\subset (0,1)$. From the standard theory for Harris processes (see, e.g., Meyn and Tweedie (1993), Theorem 10.1.1, p.231) it now follows that the Markov process (3.1) is positive Harris recurrent. Part (ii) of Theorem 3.1 is a consequence of this fact (See, e.g., [1], [21], [24], or [25]). □

Remark. We do not know if the conclusion of Theorem 3.1 remains valid under the assumption " the distribution Q (of ε_1) on (0,4) has a nonzero absolutely continuous component with respect to Lebesgue measure on $(0, 4)$", in addition to (3.3). Note that such a Q may assign its entire mass on the set of all θ for which F_θ is chaotic.

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