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Should We Be Concerned About the Distribution of Literacy Across Households? An Axiomatic Investigation

by

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Should We Be Concerned About the Distribution of Literacy Across Households? An Axiomatic Investigation*

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Abstract

This paper proposes a class of literacy measures that takes into account the externality generated by the presence of literates in the household. It is claimed that such externality is increasing in the number of literates in the household, has characteristics of rivalry in consumption, and therefore is a function of the distribution of literates and illiterates in the household. The measure is given a full axiomatic characterization, and it is shown that its use may reverse the ranking of geographical areas obtained by using other literacy measures.

Keywords: Literacy, Measuring Instrument, Externality, Developing Countries.
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1 Introduction

The importance of literacy in the process of development is now widely accepted. Firstly, the ability to read and write is valuable *per se* as it yields benefits that have great impact on everyday life. Secondly, literacy influences several other aspects of human welfare: a number of empirical works record the impact of literacy on fertility, child health and child mortality.[1] Finally, literacy is an important indicator of development: measures of literacy are used on their own, or together with other social indicators to construct general indices of development, such as the Human Development Index, the Human Poverty Index and the Capability Failure Ratio.

Given this vast importance of literacy, it is a bit of an anomaly that the *measurement* of literacy has not received the scrutiny and attention that has been directed to equality, poverty, and income. In fact, the standard measure used in the literature for measuring literacy is the *Literacy Rate*, computed as the percentage of literates among adults.

Among the few critiques of this measure is the paper by Basu and Foster (1998), which focuses on the fact that the Literacy Rate ignores that the presence of a literate person in the household generates a positive externality that the illiterate can benefit from.[2] The reasoning is that living with a literate can

[1] Among others, Stycos (1982) documents that literacy levels account for difference in fertility rates. Murthi et al. (1995) show that literacy significantly reduces fertility levels, child mortality, and the gender bias in child mortality. Thomas et al. (1991) and Sandiford et al. (1995) provide evidence that literacy improves the health of children.

[2] Basu and Foster point out that literacy may generate a negative as well as a positive externality. In fact, becoming literate may alter the household bargaining power structure, and be harmful for those who remain illiterate. If these effects coexist, the positive externality

be of great help to the illiterate; for instance, the former can read and write letters on behalf of the latter, or can fill job applications and read instructions and prescriptions for him. To capture the fact that the proximate illiterate (one who lives with at least one literate) has an advantage with respect to the isolated illiterate (one who lives in a household with no literates), Basu and Foster assign a positive externality α – a fixed number in the interval $(0, 1)$ – to the former. The idea is that each proximate illiterate counts for α literates, while each isolated illiterate counts for 0 literates. Therefore, Basu and Foster derive the *Effective Literacy Rate* as the proportion of literates among adults, augmented by the percentage of proximate illiterates multiplied by the externality α.[3]

It is however arguable that while the Basu-Foster measure is motivated by an important concern, the actual measure they derive is inadequate. In their specification, the externality does not depend on the number of illiterates and literates in the household. There are two points to be raised here. Firstly, in the Basu-Foster framework the externality does not depend on the number of illiterates as there is no rivalry in its consumption – the externality is regarded as a sort of pure public good. It can be claimed, though, that if there are too many illiterates in the household, the externality that each one can benefit from is smaller. Secondly, the externality is assumed not to vary with the number of literates. It can be contended, however, that the externality is increasing in the number of literates for at least two reasons. For one, the presence of additional literates implies greater availability of time on their part to provide literacy ser-

may be considered as representing the net effect.

[3]The hypothesis of an external effect of literacy seems to be borne out by the empirical evidence: Gibson (2001) documents a strong effect of adult proximate illiteracy on children's anthropometric measures; Basu, Narayan and Ravallion (1999) find evidence that living with at least one literate largely influences the illiterates' earnings.

vices to the illiterates. Additionally, each literate may embody distinct forms of knowledge in conjunction with the basic literacy skills, and essentially provide a broader range of services to the illiterate. Indeed, Basu and Foster do suggest that, in certain cases, the extent of the externality could depend on the percentage of literates in the household.

Subramanian (2001) refers to this point and modifies the effective literacy rate by setting the externality equal to the household literacy rate. The suggested measure is appealing because it captures the idea of literacy being characterized by positive returns and rivalry in consumption, but it has the disadvantage of not being supported by an axiomatic characterization. Building on the idea that an unequal distribution of literacy across households determines an efficiency loss, Subramanian puts forth a second index, constructed as the product of the literacy rate and the percentage of people who are not isolated illiterates. The advantage of this second measure is that, since its formulation does not directly involve the externality, the index does not require a quantification of α, either empirically or by some assumption. On the other hand, since the measure is constructed starting with the Basu-Foster formulation, it assumes that the externality is fixed and independent of the numbers of literates and illiterates in the household and is therefore subject to the same critique.

In this paper I present the *Distribution Sensitive Literacy Measure*, a class of literacy indices where the externality is a function of the ratio of literates to illiterates in the household. This formulation captures the characteristics of positive returns and rivalry in consumption of the externality and, therefore, it deals with the criticism of the Effective Literacy Rate discussed above. The main contribution of this work is to show the equivalence between the proposed measure and a set of properties that are desirable for a literacy index. I believe that an axiomatic characterization is essential to understand the priors that underlie the measure, and hence crucial for choosing among different indices. To show that there is need for caution in the choice of the literacy measure, I give an illustration of how different indices of literacy provide different rankings of the nine South African provinces. This simple application confirms that different indices may yield distinct evaluations of literacy in different areas.

2 Distribution Sensitive Literacy

This section introduces some basic notation, reviews the existing literacy indices, and presents the Distribution Sensitive Literacy Measure.

Let household h be composed by n_h adults. Define its *household literacy profile* as the vector $x^h = (x_1^h, \ldots, x_{n_h}^h)$, where $x_j^h = 1$ if the j^{th} member is literate, and $x_j^h = 0$ if the j^{th} member is illiterate. With a little abuse of notation, at times I will refer to x^h as the household having literacy profile x^h. Let society x be a collection of m_x households. Denote society x by the vector of household literacy profiles $\mathsf{x} = (x^1, \ldots, x^{m_\mathsf{x}})$. For instance, if society x is composed by three households of two, four and five people respectively, and the literacy profiles of the three households are given by $(0,0)$, $(1,1,0,0)$ and $(1,0,0,1,1)$, then society x is defined as $\mathsf{x} = ((0,0), (1,1,0,0), (1,0,0,1,1))$. *Society* x*'s literacy profile* is denoted by x^* and is defined as the vector obtained by concatenating the household literacy profiles. In the above example, society x's literacy profile is: $x^* = (0,0,1,1,0,0,1,0,0,1,1)$. Note that the society literacy profile hides the information on household structure: isolated illiterates and proximate illiterates are not distinguishable.

Let Δ be the set of all societies and define a *measure of literacy* to be a mapping $L : \Delta \rightarrow \mathsf{R}$ from the set of all societies to the set of real numbers.

The Literacy Rate is a measure of literacy, R, which can be calculated as follows: for all $\mathsf{x} \in \Delta$,

$$R(\mathsf{x}) \equiv \frac{\sum_i x_i^*}{n_{\mathsf{x}}}.$$

The Literacy Rate is the simple percentage of literates among adults; it ignores the household structure and therefore neglects the externality that the proximate illiterate can benefit from. In order to capture this externality, Basu and Foster (1998) introduce *household h's effective literacy profile* and define it as the vector $\tilde{x}^h = (\tilde{x}_1^h, \ldots, \tilde{x}_{n_h}^h)$, where

$$\tilde{x}_j^h \equiv \begin{cases} 1 & \text{if } x_j^h = 1 \\ \alpha & \text{if } x_j^h = 0 \text{ and } x_i^h = 1 \text{ for some } i \\ 0 & \text{if } x_i^h = 0 \text{ for every } i \end{cases}$$

and $0 < \alpha < 1$ represents the externality that the proximate illiterate receives when he lives with at least one literate. If household h's literacy profile is $x^h = (1, 1, 0, 0)$, then its effective literacy profile is given by $\tilde{x}^h = (1, 1, \alpha, \alpha)$. *Society* x*'s effective literacy profile* is denoted by $\tilde{x}^*$ and defined as the vector obtained by concatenating the household's effective literacy profiles; in the above example, $\tilde{x}^* = (0, 0, 1, 1, \alpha, \alpha, 1, \alpha, \alpha, 1, 1)$. Observe that the society effective literacy profile conveys information on household structure; although it is formed by the concatenation of household literacy profiles, it allows us to distinguish between isolated illiterates and proximate illiterates.

The Effective Literacy Rate is a measure of literacy, L^*, defined as follows: for all $\mathsf{x} \in \Delta$,

$$L^*(\mathsf{x}) \equiv \frac{\sum_i \tilde{x}_i^*}{n_{\mathsf{x}}}.$$

In this formulation, the proximate illiterate benefits from the externality as long as he lives with one literate person in the household. Additional literates and illiterates in the household have no effect on the magnitude of the externality. In contrast, this paper suggests that the importance of the externality crucially depends on the numbers of literates and illiterates in the household. To capture the relationship between the externality and the distribution of literacy, define *household h's distribution sensitive literacy profile* as the vector $\hat{x}^h = (\hat{x}_1^h, \ldots, \hat{x}_{n_h}^h)$, where

$$\hat{x}_j^h \equiv \begin{cases} 1 & \text{if } x_j^h = 1 \\ \alpha\left(\frac{r_h}{s_h}\right) & \text{if } x_j^h = 0 \end{cases}$$

and $r_h = r(x^h) = \sum_j x_j^h$ and $s_h = s(x^h) = n_h - r_h$ are the numbers of literates and illiterates in household h.[4] Observe that, unlike in Basu and Foster's work, α now denotes a function.

Let N denote the set of natural numbers, let $\overline{\mathsf{N}}$ be equal to $\mathsf{N} \cup \{0\}$, and denote by Q_+ the set of non negative rational numbers: $\mathsf{Q}_+ \equiv \left\{\frac{r}{s} : r \in \overline{\mathsf{N}}; s \in \mathsf{N}\right\}$. The function α is assumed to have the following properties:

(i) $\alpha : \mathsf{Q}_+ \rightarrow [0,1)$; $\alpha(0) = 0$;

(ii) $\forall p, p' \in \mathsf{Q}_+$, if $p' > p$, then $\alpha(p') > \alpha(p)$;

(iii) $\forall p, p' \in \mathsf{Q}_+$ with $p \neq p'$, and $\forall \lambda \in \mathsf{Q}_+ \cap (0,1)$, the following holds: $\alpha(\lambda p + (1-\lambda) p') > \lambda \alpha(p) + (1-\lambda)\alpha(p')$.

[4]Note that $\alpha\left(\frac{r_h}{s_h}\right)$ is a meaningful expression for $s \neq 0$ only. When the household is composed by literates only, $\alpha\left(\frac{r_h}{s_h}\right)$ need not be defined.

If household h's literacy profile is $x^h = (1, 1, 0, 0)$, its distribution sensitive literacy profile is given by $\hat{x}^h = (1, 1, \alpha(1), \alpha(1))$. By the same token, if society x is given by $\mathsf{x} = ((0, 0), (1, 1, 0, 0), (1, 0, 0, 1, 1))$, its *society distribution sensitive literacy profile* is represented by the vector $\hat{x}^* = \left(0, 0, 1, 1, \alpha(1), \alpha(1), 1, \alpha\left(\frac{3}{2}\right), \alpha\left(\frac{3}{2}\right), 1, 1, \right)$. Notice that the society distribution sensitive literacy profile conveys even more information on the household structure: it is possible to identify isolated and proximate illiterates, and also observe the distribution of literates and illiterates across households.

The Distribution Sensitive Literacy Rate is a measure of literacy, $\hat{L}$, which is defined as follows: for any society $\mathsf{x} \in \Delta$,

$$\hat{L}(\mathsf{x}) \equiv \frac{\sum_i \hat{x}_i^*}{n_{\mathsf{x}}}.$$

Let H_1 be the set of all households composed of literates only, and H_2 be the set of all households where there is at least one illiterate. By definition of $\hat{x}_i^*$, it follows that the Distribution Sensitive Literacy Rate may be written as:

$$\hat{L}(\mathsf{x}) = \frac{\sum_{h \in H_1} r_h + \sum_{h \in H_2} \left(r_h + \alpha\left(\frac{r_h}{s_h}\right) s_h \right)}{n_{\mathsf{x}}}.$$

The above expression is equivalent to

$$\hat{L}(\mathsf{x}) = \frac{\sum_h \left(r_h + \hat{\alpha}(r_h, s_h) s_h\right)}{n_{\mathsf{x}}},$$

where $\hat{\alpha}(r_h, s_h) \equiv \begin{cases} \alpha\left(\frac{r_h}{s_h}\right) & \text{if } s \neq 0 \\ 0 & \text{if } s = 0 \end{cases}$

Since the externality depends positively on the number of literates and negatively to the number of illiterates, this formulation captures the characteristics of positive returns and rivalry in consumption of the externality.

3 Axiomatic Characterization of the Distribution Sensitive Literacy Measure

In choosing among different indices, knowing which properties each one satisfies is of considerable value, because it helps us understand the acceptability of each measure. Economists have identified a number of appealing axioms for measures of poverty and inequality. Some of these properties are also attractive for measuring literacy, and will be discussed here. This section presents a set of axioms desirable for measures of literacy, and shows that the Distribution Sensitive Literacy Measure is the only class of literacy indices which satisfies them all.

The first property relates literacy in society x to literacy in its *subsocieties*. Societies $\mathsf{y}, \mathsf{z} \in \Delta$ are said to be subsocieties of $\mathsf{x} \in \Delta$ if $y^h = x^h$ for $0 \leq h \leq m_{\mathsf{y}}$, and $z^h = x^{m_{\mathsf{y}}+h}$ for $0 \leq h \leq m_{\mathsf{z}}$, with $m_{\mathsf{y}} + m_{\mathsf{z}} = m_{\mathsf{x}}$. That is, y and z are subsocieties of x if they are obtained by splitting society x in two, maintaining the household structures unchanged. The following axiom requires the overall literacy to be a weighted sum of literacy in its subsocieties, with the weights being their population shares. Such choice of weights implies that, in constructing the index, each individual counts the same in contrast, for instance, to each household counting the same which would imply weights equal to $\frac{m_{\mathsf{y}}}{m_{\mathsf{x}}}$ and $\frac{m_{\mathsf{z}}}{m_{\mathsf{x}}}$.

D (Decomposition): *If* $\mathsf{y}, \mathsf{z} \in \Delta$ *are subsocieties of* $\mathsf{x} \in \Delta$*, then* $L(\mathsf{x}) = \frac{n_{\mathsf{y}}}{n_{\mathsf{x}}} L(\mathsf{y}) + \frac{n_{\mathsf{z}}}{n_{\mathsf{x}}} L(\mathsf{z})$.

Because any subsociety is a society itself, it can be decomposed further, as long as it is composed of more than one household. Since the smallest subsocieties are the single households, applying axiom D repeatedly implies that the overall literacy is a weighted sum of literacy in each household, with the weights being the population shares of each household.

It is worth noting that the Decomposition axiom implies household anonymity: even if society x is split in more than two groups, and even if the households ordering is not maintained, its overall literacy is still the weighted sum of literacy in the groups. In fact, both literacy in the society as a whole and the weighted mean of literacy in the groups in which the society is split can be shown to be equal to the weighted sum of literacy in each household. For instance, consider society $\mathsf{x} = ((1,1,0,0,0),(1,0),(0,1),(1,0,1,1),(1,1,1))$ and split it in the following way: $\mathsf{y} = ((1,1,0,0,0),(1,0,1,1))$, $\mathsf{z} = ((0,1),(1,0))$, and $\mathsf{w} = ((1,1,1))$. Applying axiom D repeatedly to society x yields $L(\mathsf{x}) = \frac{n_{\mathsf{x}}^1}{n_{\mathsf{x}}} L((1,1,0,0,0)) + \frac{n_{\mathsf{x}}^2}{n_{\mathsf{x}}} L((1,0)) + \frac{n_{\mathsf{x}}^3}{n_{\mathsf{x}}} L((0,1)) + \frac{n_{\mathsf{x}}^4}{n_{\mathsf{x}}} L((1,0,1,1)) + \frac{n_{\mathsf{x}}^5}{n_{\mathsf{x}}} L((1,1,1))$. On the other hand, $L(\mathsf{y}) = \frac{n_{\mathsf{y}}^1}{n_{\mathsf{y}}} L((1,1,0,0,0)) + \frac{n_{\mathsf{y}}^2}{n_{\mathsf{y}}} L((1,0,1,1))$, $L(\mathsf{z}) = \frac{n_{\mathsf{z}}^1}{n_{\mathsf{z}}} L((0,1)) + \frac{n_{\mathsf{z}}^2}{n_{\mathsf{z}}} L((1,0))$, and $L(\mathsf{w}) = L((1,1,1))$. Since $n_{\mathsf{x}}^1 = n_{\mathsf{y}}^1$, $n_{\mathsf{x}}^2 = n_{\mathsf{z}}^2$, $n_{\mathsf{x}}^3 = n_{\mathsf{z}}^1$, $n_{\mathsf{x}}^4 = n_{\mathsf{y}}^2$, and $n_{\mathsf{x}}^5 = n_{\mathsf{w}}$, we have that $L(\mathsf{x}) = \frac{n_{\mathsf{y}}}{n_{\mathsf{x}}} L(\mathsf{y}) + \frac{n_{\mathsf{z}}}{n_{\mathsf{x}}} L(\mathsf{z}) + \frac{n_{\mathsf{w}}}{n_{\mathsf{w}}} L(\mathsf{w})$.

The Decomposition axiom rules out the existence of inter-household externalities. There are surely cases where literates generate an externality to the illiterates living in a proximate household, but this can be easily accommodated in the same framework, carrying the analysis in terms of units larger than the household.

The second axiom requires the index to increase as one illiterate person becomes literate, while the literacy of the others and the household structures are unaffected. Society $\mathsf{x} \in \Delta$ is obtained from society $\mathsf{y} \in \Delta$ by a *simple increment* if $x_j^h = 1$ and $y_j^h = 0$, while $x_{j^\theta}^{h^\theta} = y_{j^\theta}^{h^\theta}$ for all $(h', j') \neq (h, j)$.

M (Monotonicity): *If* $\mathsf{x} \in \Delta$ *is obtained from* $\mathsf{y} \in \Delta$ *by a simple increment, then* $L(\mathsf{x}) > L(\mathsf{y})$*.*

The third axiom normalizes the measure, so that it is bounded by 0 and 1. Society $\mathsf{x} \in \Delta$ is *completely literate* if everyone is literate, i.e. if $x_j^h = 1$ for all (h, j); it is *completely illiterate* if everyone is literate, i.e. if $x_j^h = 0$ for all (h, j).

N (Normalization): *If* $\mathsf{x} \in \Delta$ *is completely literate, then* $L(\mathsf{x}) = 1$*; if* $\mathsf{x} \in \Delta$ *is completely illiterate, then* $L(\mathsf{x}) = 0$*.*

It is worth noting that both the Literacy Rate and the Effective Literacy Rate satisfy the above three properties. The axiom which is crucial to the formulation of the Distribution Sensitive Literacy Measure is the following one. It is concerned with the distribution of literacy across households, and requires the measure to penalize societies with unequal distribution of literacy.

Society $\mathsf{x} \in \Delta$ is *uniform* if every household has the same number of literates and every household has the same number of illiterates. Society $\mathsf{y} \in \Delta$ has *perfect literacy equality* if either y is uniform or it can be formed by starting with a uniform society and merging some of its households.

Q (Equality): *Suppose that societies* $\mathsf{x}, \mathsf{y} \in \Delta$ *have the same number of literates and the same number of illiterates. If* x *has perfect literacy equality, then* $L(\mathsf{x}) \geq L(\mathsf{y})$*. If in addition* y *does not have perfect literacy equality, then* $L(\mathsf{x}) > L(\mathsf{y})$*.*

The Equality axiom introduces a distributional concern in the measure of literacy.

The Literacy Rate is indifferent to any distributional matter. In particular, according to this measure, as long as the numbers of literates and illiterates are the same, a society with perfectly equitable distribution of literacy across households and a society with polarization of literates and illiterates are equally desirable.

The Effective Literacy Rate is not concerned with an equal distribution of literacy *per se*, but it favors an egalitarian distribution since it maximizes the benefit from the externality. However, this is only partially true: when there is at least one literate person in each household, improving the distribution of literacy across households has no effect.

The Distribution Sensitive Literacy Rate maintains that the distribution of literates across households matters, but pushes this further by claiming that spreading literates and illiterates across households maximizes the benefit from the externality, even when each household already has one literate member. The Equality axiom captures the idea that increasing the number of literates has a positive effect on the externality, while increasing the number of illiterates has a negative effect. It should be noted that aiming to an equal distribution of literacy across household stems from an efficiency concern: an egalitarian distribution is seen to maximize the benefit from the externality, and is therefore desirable even if we were not concerned with equity in itself.

It will now be shown that the Distribution Sensitive Literacy Measure is the only class of literacy indices that satisfies the above four properties.

Lemma 1: *Consider two single household societies $x^h, y^h \in \Delta$. If a measure of literacy L satisfies* Q*, then $L\left(x^h\right) = L\left(y^h\right)$ whenever $r(x^h) = r(y^h)$ and $s(x^h) = s(y^h)$.*

Proof. Suppose $r(x^h) = r(y^h)$ and $s(x^h) = s(y^h)$. Since x^h and y^h have the same numbers of literates and illiterates, and they both have perfect equality, axiom Q implies that $L\left(x^h\right) \geq L\left(y^h\right)$ and $L\left(y^h\right) \geq L\left(x^h\right)$. Hence, $L\left(x^h\right) = L\left(y^h\right)$. ■

Lemma 1 shows that the Equality axiom entails individual anonymity within the household: what matters for the household literacy measure is the number of literates and illiterates, while the order in which the individuals are listed in the household is irrelevant. For instance, household $x^h = (1, 0, 0, 1)$ and household $y^h = (0, 1, 1, 0)$ have the same literacy measure, even though the members who are literate are not the same in the two households.

Lemma 2: *Consider two single household societies $x^h, y^h \in \Delta$. If a measure of literacy L satisfies* D *and* Q*, then $L\left(x^h\right) = L\left(y^h\right)$ whenever $r(x^h) = kr(y^h)$ and $s(x^h) = ks(y^h)$.*

Proof. Suppose $r(x^h) = kr(y^h)$ and $s(x^h) = ks(y^h)$. Consider society $\mathsf{y} \in \Delta$ composed of k households which are identical to y^h. Since x^h and y have the same numbers of literates and illiterates, and they both have perfect equality, axiom Q implies that $L\left(x^h\right) \geq L(\mathsf{y})$ and $L(\mathsf{y}) \geq L\left(x^h\right)$. Hence, $L\left(x^h\right) = L(\mathsf{y})$. By axiom D, it follows that $L(\mathsf{y}) = \sum_{h=1}^{k} \frac{r+s}{k(r+s)} L(y^h) = L(y^h)$. Thus, $L(x^h) = L(y^h)$. ■

Lemma 2 makes evident that the Equality axiom involves population independence within the household: replicating the same number of times all the individuals in the household has no effect on its literacy measure. For instance, households $x^h = (1, 0, 0, 1)$ and $y^h = (1, 1, 0, 0, 0, 0, 1, 1)$ have the same literacy measure.

The Literacy Rate and the Basu-Foster measure both satisfy the Anonymity and the Population Independence axioms.

The central result of this paper may now be stated:

Theorem: *A measure of literacy L satisfies axioms* D, M, N, *and* Q *if and only if, for any society* $\mathsf{x} \in \Delta$, *L can be written as* $\hat{L}(\mathsf{x}) = \frac{\sum_{\mathsf{h}} (r_{\mathsf{h}} + \hat{\alpha}(r_{\mathsf{h}}, s_{\mathsf{h}}) s_{\mathsf{h}})}{n_{\mathsf{x}}}$, *where* $\hat{\alpha}(r_h, s_h) = 0$ *if* $s_h = 0$, $\hat{\alpha}(r_h, s_h) = \alpha\left(\frac{r_{\mathsf{h}}}{s_{\mathsf{h}}}\right)$ *if* $s_h \neq 0$, *and* α *is a function which has the properties (i)-(iii) listed above.*

Proof. Let L be a measure of literacy which satisfies axioms D, M, N, and Q. Consider any society $\mathsf{x} \in \Delta$ of size n_{x}, and let $r = r_h$ and $s = s_h$ be the numbers of literates and illiterates in household h. First, I will show that $L(\mathsf{x})$ can be written as $\frac{\sum_{\mathsf{h}} (r_{\mathsf{h}} + \hat{\alpha}(r_{\mathsf{h}}, s_{\mathsf{h}}) s_{\mathsf{h}})}{n_{\mathsf{x}}}$. Next, I will prove that $\alpha\left(\frac{r_{\mathsf{h}}}{s_{\mathsf{h}}}\right)$ has the desired properties.

Applying axiom D repeatedly yields $L(\mathsf{x}) = \sum_h \frac{n_{\mathsf{h}}}{n_{\mathsf{x}}} L(x^h)$. Therefore, in order to determine the form of $L(\mathsf{x})$, there is need to study the form of $L(x^h)$ only. In fact, if it is possible to show that $L(x^h)$ can be written as $\frac{r_{\mathsf{h}} + \hat{\alpha}(r_{\mathsf{h}}, s_{\mathsf{h}}) s_{\mathsf{h}}}{n_{\mathsf{h}}}$, then it follows that $L(\mathsf{x}) = \sum_h \frac{n_{\mathsf{h}}}{n_{\mathsf{x}}} \frac{r_{\mathsf{h}} + \hat{\alpha}(r_{\mathsf{h}}, s_{\mathsf{h}}) s_{\mathsf{h}}}{n_{\mathsf{h}}} = \sum_h \frac{r_{\mathsf{h}} + \hat{\alpha}(r_{\mathsf{h}}, s_{\mathsf{h}}) s_{\mathsf{h}}}{n_{\mathsf{x}}} = \hat{L}(\mathsf{x})$. Define $f : \overline{\mathsf{N}} \times \overline{\mathsf{N}} \to \mathsf{R}$ and $\tilde{\alpha} : \overline{\mathsf{N}} \times \mathsf{N} \to \mathsf{R}$ as follows: $f(r, s) \equiv (r + s) L(x^h)$ and $\tilde{\alpha}(r, s) = \frac{f(r,s) - r}{s}$. Since Lemma 2 implies that $f(kr, ks) = kf(r, s)$, it follows that $\tilde{\alpha}(kr, ks) = \tilde{\alpha}(r, s)$. Thus, it is possible to write $\tilde{\alpha}(r, s) = \alpha\left(\frac{r}{s}\right)$. Now, define $\hat{\alpha} : \overline{\mathsf{N}} \times \overline{\mathsf{N}} \to \mathsf{R}$

$$\hat{\alpha}(r_h, s_h) \equiv \begin{cases} \alpha\left(\frac{r}{s}\right) & \text{if } s \neq 0 \\ 0 & \text{if } s = 0 \end{cases}$$

For all households, we have that $L(x^h) = \frac{f(r,s)}{n_{\mathsf{h}}}$. For $s \neq 0$, $f(r, s) = r_h + \alpha\left(\frac{r_{\mathsf{h}}}{s_{\mathsf{h}}}\right)$. For $s = 0$, axiom N implies $L\left(x^h\right) = 1$, and therefore $f(r, s) = r_h$. Hence, for all households the following holds:

$$L\left(x^h\right) = \frac{r_h + \hat{\alpha}(r_h, s_h) s_h}{n_h}.$$

It will now be proved that: (i) $\alpha : \mathsf{Q}_+ \to [0, 1)$; $\alpha(0) = 0$; (ii) $\forall p', p \in \mathsf{Q}_+$, if $p' > p$, then $\alpha(p') > \alpha(p)$; (iii) $\forall p, p' \in \mathsf{Q}_+$ with $p \neq p'$, and $\forall \lambda \in \mathsf{Q}_+ \cap (0,1)$, the following holds: $\alpha(\lambda p + (1-\lambda)p') > \lambda\alpha(p) + (1-\lambda)\alpha(p')$.

(i)$\forall p \in \mathsf{Q}_+$, $\exists r \in \overline{\mathsf{N}}$ and $s \in \mathsf{N}$ s.t. $p = \frac{r}{s}$. If $p = 0$, then $\alpha(0) = \frac{f(0,s)}{s}$ by definition. Since $f(0,s) = 0$ because of axiom N, we directly obtain $\alpha(0) = 0$. If $p > 0$, then $\alpha(p) = \frac{f(r,s)-r}{s}$. Axioms Q and D imply $f(r,s) > f(r,0) + f(0,s)$. Next, applying axiom N, we get $f(r,s) > r$ which implies $\alpha(p) = \frac{f(r,s)-r}{s} > 0$. Axiom M yields $f(r,s) < f(r+1, s-1)$ and, by induction, $f(r,s) < f(r+s, 0)$. Applying axiom N, we get $f(r,s) < r+s$ and therefore $\alpha(p) = \frac{f(r,s)-r}{s} < 1$. Thus, $0 \leq \alpha(p) < 1$ as desired.

(ii)$\forall p, p' \in \mathsf{Q}_+$, $\exists r, r' \in \overline{\mathsf{N}}$ and $s, s' \in \mathsf{N}$ s.t. $p = \frac{r}{s}$ and $p' = \frac{r'}{s'}$. Without loss of generality, suppose $p' > p$. If $p = 0$, $\alpha\left(\frac{r'}{s'}\right) > \alpha(0) = 0$ directly follows from (i). If $p > 0$, $\frac{r'}{s'} > \frac{r}{s}$ implies $r's - rs' > 0$. Consider a society x composed of one household with $r's$ literates and $s's$ illiterates, and a society y composed by two households: one having rs' literates and $s's$ illiterates, and the other having $r's - rs'$ literates only. By axioms Q and D, $f(r's, s's) > f(rs', s's) + f(r's - rs', 0)$ and applying axiom N we get $f(r's, s's) > f(rs', s's) + r's - rs'$. With a little manipulation, this yields $\alpha\left(\frac{r'}{s'}\right) > \alpha\left(\frac{r}{s}\right)$.

(iii)$\forall p, p' \in \mathsf{Q}_+$ and $\forall \lambda \in \mathsf{Q}_+ \cap (0,1)$, $\exists r, r' \in \overline{\mathsf{N}}$ and $s, s', u, w \in \mathsf{N}$ s.t. $p = \frac{r}{s}$, $p' = \frac{r'}{s'}$, and $\lambda = \frac{u}{w}$. Therefore, it is possible to write

$$\begin{aligned}
\alpha(\lambda p + (1-\lambda)p') &= \alpha\left(\frac{u}{w}\frac{r}{s} + \left(1 - \frac{u}{w}\right)\frac{r'}{s'}\right) \\
&= \alpha\left(\frac{urs' + wr's - ur's}{wss'}\right) = \\
&= \frac{f(urs' + wr's - ur's, wss') - urs' - wr's + ur's}{wss'}.
\end{aligned}$$

Since by axioms Q and D we have that $f(urs' + wr's - ur's, wss') > f(urs', uss') +$

$f\left(wr's - ur's, wss' - uss'\right)$, we obtain

$$\begin{aligned}\alpha\left(\lambda p+(1-\lambda)p'\right) &> \frac{f\left(urs', uss'\right)-urs'}{wss'}+\frac{f\left(wr's-ur's, wss'-uss'\right)-wr's+ur's}{wss'}= \\ &= \frac{u}{w}\alpha\left(\frac{r}{s}\right)+\left(1-\frac{u}{w}\right)\alpha\left(\frac{r'}{s'}\right).\end{aligned}$$

Therefore, $\alpha\left(\lambda p+(1-\lambda)p'\right) > \lambda\alpha(p)+(1-\lambda)\alpha(p')$ as desired.

To complete the proof, one can check that $\hat{L}$ satisfies axioms D, M, and N. To see that $\hat{L}$ satisfies Q, consider a society $\mathsf{x} \in \Delta$ composed of m identical households, each one having r literates and s illiterates, and a society $\mathsf{y} \in \Delta$ composed of mr literates and ms illiterates belonging to n households. Since the function α is concave, $\alpha\left(\sum_h \frac{s_h}{ms}\frac{r_h}{s_h}\right) = \alpha\left(\sum_h \frac{r_h}{ms}\right) \geq \sum_h \frac{s_h}{ms}\alpha\left(\frac{r_h}{s_h}\right)$, with strict inequality if $\exists\ h', h''$ s.t. $\frac{r_{h'}}{s_{h'}} \neq \frac{r_{h''}}{s_{h''}}$ that is, if society $\mathsf{y} \in \Delta$ does not have perfect literacy equality. Since $\sum_h r_h = mr$, the above inequality implies that $\alpha\left(\frac{r}{s}\right) \geq \frac{1}{ms}\sum_h s_h\alpha\left(\frac{r_h}{s_h}\right)$. A little manipulation yields to $mr + ms\alpha\left(\frac{r}{s}\right) \geq mr + \sum_h s_h\alpha\left(\frac{r_h}{s_h}\right)$, i.e. $\hat{L}(\mathsf{x}) \geq \hat{L}(\mathsf{y})$, with strict inequality if $\mathsf{y} \in \Delta$ does not have perfect literacy equality. ■

The index obtained by setting the externality equal to the household literacy rate belongs to the class of distribution sensitive literacy indices. In fact: (i) $\frac{r}{n} \in [0,1]$; (ii) $\frac{r}{n} = \frac{\frac{r}{s}}{\frac{r}{s}+1}$ is increasing in $\frac{r}{s}$; (iii) $\frac{r}{n} = \frac{\frac{r}{s}}{\frac{r}{s}+1}$ is a concave function of $\frac{r}{s}$.

Other functions of $\frac{r}{s}$ that satisfy the desired properties are the following:

- $\left(\frac{r}{n}\right)^a$, with $a < 1$
- $1-\exp(-r/s)$, the exponential cumulative distribution function.
- $2\left[\left(1+\exp\left(-\frac{r}{s}\right)\right)^{-1}-\frac{1}{2}\right]$ transformation of the logistic cdf.

The above expressions imply different concavities and, therefore, assign different penalties to unequal distributions. Since the more concave α is, the more penalized are the societies where literacy is unequally spread across households, the choice of α may be crucial in determining the ordering of societies.

As an example, consider the following societies: $\mathsf{x} = ((1,1,0), (1,1,0,0,0))$; $\mathsf{y} = ((1,0,0), (1,1,0,0,1))$. The axioms alone do not allow to tell which society has higher literacy. Now, consider the following three indices, belonging to the class of Distribution Sensitive Literacy Measure: $\hat{L}_1 = \frac{\sum_{\mathsf{h}} \left(r_{\mathsf{h}} + \left[\frac{\mathsf{r}_{\mathsf{h}}}{\mathsf{n}_{\mathsf{h}}} \right]^{\frac{1}{3}} s_{\mathsf{h}} \right)}{n_{\mathsf{x}}}$, $\hat{L}_2 = \frac{\sum_{\mathsf{h}} \left(r_{\mathsf{h}} + \left[\frac{\mathsf{r}_{\mathsf{h}}}{\mathsf{n}_{\mathsf{h}}} \right] s_{\mathsf{h}} \right)}{n_{\mathsf{x}}}$, and $\hat{L}_3 = \frac{\sum_{\mathsf{h}} \left(r_{\mathsf{h}} + \left[1 - \exp\left(-\frac{\mathsf{r}_{\mathsf{h}}}{\mathsf{s}_{\mathsf{h}}} \right) \right] s_{\mathsf{h}} \right)}{n_{\mathsf{x}}}$. According to $\hat{L}_1$, society x has higher literacy; according to $\hat{L}_2$, the two societies are equivalent; according to $\hat{L}_3$, society y has greater literacy.

In general, the axioms alone do not provide a complete ordering over the set of all societies. Mitra (2001) characterizes those situations in which societies are ranked the same way regardless of which index is chosen among those satisfying a set of axioms he proposes. The axioms he suggests can be shown to be slightly weaker than those proposed here. In particular, Mitra's positive externality and scale invariance axioms together are slightly weaker than the equality axiom used in the present analysis.

4 Literacy in South Africa

This section provides an application of the different measures of literacy to the nine provinces of South Africa. The data set I use is the 1999 October Household Survey (OHS). The survey gathered detailed information on approximately 140,000 people living in 30,000 households across the country. There are two questions on literacy:

1. Can ... read in at least one language?
2. Can ... write in at least one language?

I use both questions and define a literate person as one who is able to read and write.

For the nine South African provinces, I compute the standard Literacy Rate, the Effective Literacy Rate ($\alpha = 0.25$), and the Distribution Sensitive Index ($\alpha = \frac{r}{n}$). Among the functional forms suggested for the externality, $\alpha = \frac{r}{n}$ is an intermediate choice in terms of cost of inequality.

As the results in Table 1 indicate, the ranking of provinces is sensitive to the choice of the index.

Table 1

Literacy Rate		Effective Literacy Index[1]		Distribution Sensitive Index[2]	
Gauteng	0.939	Gauteng	0.947	Gauteng	0.959
Western Cape	0.927	Western Cape	0.934	Western Cape	0.952
Eastern Cape	0.872	Eastern Cape	0.892	Eastern Cape	0.924
Free State	0.866	Free State	0.888	Free State	0.921
KwaZulu-Natal	0.848	KwaZulu-Natal	0.870	KwaZulu-Natal	0.904
Northern Cape	0.834	North West	0.856	Northern Province	0.895
North West	0.831	Northern Cape	0.855	Mpumalanga	0.891
Mpumalanga	0.811	Mpumalanga	0.845	North West	0.889
Northern Province	0.809	Northern Province	0.844	Northern Cape	0.888

Source: South Africa October Household Survey 1999
[1] The Effective Literacy Rate is computed for $\alpha = 0.25$.
[2] The Distribution Sensitive Literacy Rate is computed for $\alpha = \frac{r}{n}$.

In South Africa, the Literacy Rate is higher than 80% in all provinces; where it is higher than 85%, the externality does not have a major effect, and the ranking is the same no matter which index is chosen. However, the ranking changes sharply for the 4 provinces where the Literacy Rate is the lowest. In fact, note that for the last four provinces the Distribution Sensitive Index exactly reverses the ranking of the literacy rate measure. This suggests that the use of the Distribution Sensitive Index is crucial in less developed countries, where literacy is particularly low. The data suggest that literacy is unequally spread across households in the province of Northern Cape in particular. Although the Literacy Rate for this province is the sixth highest, the Distribution Sensitive Index is the lowest.

Some of the first empirical tests of the externality thesis show that literacy becomes a better predictor of different aspects of human welfare once we allow for a positive externality as in the Basu-Foster framework, but this in turn alerts us to the possibility that we can do even better if we measure literacy by the Distribution Sensitive Index developed in this paper.

5 Conclusion

In this paper, I have built on the idea that the externality generated by the presence of literates in the household presents characteristics of positive returns and rivalry in consumption.

Recognizing that the numbers of literates and illiterates in the household have a role in determining the externality, I have suggested a measure of literacy that depends on the distribution of literacy across households. I showed that this measure – the Distribution Sensitive Literacy Rate – is fully characterized by four properties, three of which are also satisfied by the Literacy Rate and the Effective Literacy Rate.

The property that distinguishes the Distribution Sensitive Literacy Rate from other measures is the Equality axiom, which considers the issue of distribution in the measure of literacy. The Equality axiom pushes further the idea that the targeting of isolated illiterates is favorable for purposes of efficiency, supporting the view that spreading literates and illiterates across households maximizes the benefit from the externality. The targeting of isolated illiterates is still valid, but once there is at least one literate member in each household, we should aim to have as egalitarian a distribution as is feasible. It should be noted that such a recommendation stems from an efficiency concern: an egalitarian distribution is desirable even if we were not concerned with equity in itself, because it is seen to maximize the benefit from the externality.

Using data from the October Household Survey, I have shown that the ranking of literacy across South African provinces is sensitive to the choice of index. This provides a word of caution that different indices may yield varying evaluations of the literacy levels in different areas. This in turn may imply contrasting determination of priorities for policy intervention, or a different assessment of progress over time.

The choice of the externality function is crucial in determining the ordering: the more concave the externality function is, the more penalized are societies where literacy is unequally spread across households. Refraining from making any value judgment, the choice of the functional form for the externality should be determined empirically.

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