CAE Working Paper #02-07

Comparing Sunspot Equilibrium and Lottery Equilibrium Allocations: The Finite Case

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July 2002

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JULY 9, 2002

Abstract

Sunspot equilibrium and lottery equilibrium are two stochastic solution concepts for nonstochastic economies. Recent work by Garratt, Keister, Qin, and Shell (in press) and Kehoe, Levine, and Prescott (in press) on nonconvex exchange economies has shown that when the randomizing device is continuous, applying the two concepts to the same fundamental economy yields the same set of equilibrium allocations. In the present paper, we examine economies based on a discrete randomizing device. We extend the lottery model so that it can constrain the randomization possibilities available to agents in the same way that the sunspots model can. Every equilibrium allocation of our generalized lottery model has a corresponding sunspot equilibrium allocation. For almost all discrete randomizing devices, the converse is also true. There are exceptions, however: for some randomizing devices, there exist sunspot equilibrium allocations with no lottery equilibrium counterpart.

* We thank Jess Benhabib, Alberto Bisin, Lucca Bossi, David Easley, and especially Jim Peck for helpful comments and discussions. We also thank seminar participants at Cornell, ITAM, Ohio State, the 2000 NBER General Equilibrium Conference, and the Extrinsic Uncertainty Workshop at NYU in May 2000. Part of this work was completed while Keister was visiting the University of Texas at Austin, whose hospitality and support are gratefully acknowledged.

1 Introduction

In nonconvex economies, agents often prefer stochastic allocations to nonstochastic ones. When a good is indivisible, for example, even risk averse consumers can benefit from the ability to purchase a contract that delivers the good with some probability, instead of having to choose between buying it either with certainty or not at all. In such a situation, it is natural to use a stochastic equilibrium concept, even when the fundamentals of the economy are nonstochastic. That is, it is natural to introduce uncertainty that is *extrinsic* (*i.e.*, does not affect endowments, technologies, or preferences) and to allow agents to trade in contracts whose payoffs depend on the outcome of this uncertainty. This is precisely the approach taken in two well-known general equilibrium concepts: sunspot equilibrium, as introduced in Shell (1977) and Cass and Shell (1983), and lottery equilibrium, as introduced in Prescott and Townsend (1984a,b). Since the two models bring different approaches to bear on the same problem, it is natural to ask how their predictions compare. What is the relationship between the set of sunspot equilibrium allocations and the set of lottery equilibrium allocations for the same fundamental economy? We address this question for exchange economies where the number of consumers, the number of commodities, and the randomization possibilities are all finite.

The two equilibrium concepts have very different histories. Sunspot equilibrium was introduced as an explanation of "excess" market volatility within the rational-expectations framework. The original sunspots models were based on standard, convex overlapping-generations economies. Equilibria in which sunspots affect allocations in a strictly convex environment are inefficient; when consumers are risk averse, an allocation with "excess" volatility can always be Pareto dominated by a feasible allocation without such volatility. A large literature has extended the sunspots approach to a wide variety of models, with the focus remaining almost exclusively on suboptimal equilibria in convex environments. Lottery equilibrium, on the other hand, was designed as a method for decentralizing allocations in economies with adverse selection and moral hazard. Such informational asymmetries create noncovexities in consumers' opportunity sets, which makes addressing basic questions such as the existence of equilibrium very difficult. Trade in lotteries was introduced largely to "convexify" the economy and thereby permit standard general-equilibrium analysis to be performed in environments with asymmetric information. Hence the focus in lottery models has been on (constrained) optimal allocations in economies with nonconvexities.

Despite these differences, both models of trade can be applied to the general problem of a nonconvex economy where agents would like to purchase stochastic allocations. We assume that there are complete markets, no restrictions to participation on those markets, and symmetric information. Our focus in this paper is on economies with consumption nonconvexities: some goods are indivisible and consumers may be risk-loving. This represents the minimum departure from the standard Walrasian setting needed in order for stochastic trade to be useful. The fundamental economy is comprised of the set of consumers, together with the endowments and preferences, a (common) consumption set, and an extrinsic randomizing device. Both models can be applied to the same fundamental economy, but they differ considerably in the way trade is organized. In the sunspots model, extrinsic uncertainty is represented by a set of states of nature, and agents trade in state-contingent claims, such a "1 automobile to be delivered if state α occurs." Agents construct stochastic consumptions by purchasing different bundles to be delivered in different extrinsic states. In contrast, agents in the lottery model trade directly in probabilities, using assets such as "1 automobile to be delivered with probability π." In this way, agents directly purchase a probability distribution over their consumption set. No reference to a "state of nature" is made.

Since the two models bring different approaches to bear on the same problem, it is natural to ask how their predictions compare. Shell and Wright (1993) show how the equilibrium employment lotteries of Rogerson (1988) can be implemented as sunspot equilibria, indicating that there is a close connection between the two equilibrium concepts. In addition, they show how sunspots can provide the necessary coordination to allocate indivisible goods among a finite number of consumers.[1] Garratt (1995) shows how the lottery model can be extended to economies with a finite number of consumers by coordinating the individual lotteries. He then compares the equilibrium allocations of the lottery model with those generated by any sunspot variable with a finite number of states. He finds that every lottery equilibrium allocation has a corresponding sunspot equilibrium allocation, but some sunspot equilibrium allocations have no lottery equilibrium counterpart. The ability to constrain choice sets inherent in the sunspots model can lead to the existence of an equilibrium that is not present when the choice is unconstrained.[2] In a recent paper, Garratt, Keister, Qin, and Shell (in press, hereafter GKQS) show that if sunspot activity is represented by a continuous random variable, the two models generate exactly the same set of equilibrium allocations. The result is proven in a standard general equilibrium model with a finite number of consumers and a (possibly) nonconvex consumption set. Kehoe, Levine, and Prescott (in press) show that this same result holds in a moral hazard economy with a continuum of consumers. In addition to its theoretical importance, the equivalence result has practical implications, as problems that are difficult to solve in one model may be more easily addressed in the other. For example, Garratt and Keister (in press) show how an outstanding question regarding when sunspot equilibria are robust to refinements in the sunspot variable is easily solved by look-

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[1] Prescott and Townsend (1984a,b), Rogerson (1988) and others study lottery economies with a continuum of consumers, where a law of large numbers implies that no coordination of individual lotteries is necessary.

[2] This fact is also evident in Goenka and Shell (1997), which introduces the concept of *robustness* of sunspot equilibria to refinements in the randomizing device. They show that, in nonconvex economies, not all sunspot equilibria are robust to refinements, and hence sunspot equilibria can be destroyed by giving consumers additional randomization possibilities.

ing at the lottery formulation of the problem. In addition, when the consumption set has a finite number of elements, finding lottery equilibria reduces to solving a collection of linear programming problems, which can be computationally easier than solving the (nonlinear) sunspots model.

Having a continuous sunspot variable is the "right" comparison for obtaining equivalence because the lottery model gives consumers an unconstrained choice of probability distributions. Only with a continuous random variable do consumers in the sunspots model have the same opportunities. However, in many situations the randomization possibilities available to consumers are best modeled as being constrained in some way. For example, the government may place legal restrictions on the types of trading allowed (such as regulating risk classes in insurance) in order to achieve a preferred outcome. As another example, transactions costs may prevent consumers from trading in a continuum of markets. Such situations are better represented by the sunspots model with a finite number of states of nature.

In this paper, we extend the analysis of the relationship between sunspot equilibrium and lottery equilibrium to the case of constrained randomization possibilities. Doing so requires that we substantially extend the lottery model so that it can constrain consumer choice in the same way that the sunspot model does. We introduce the concept of *constrained lotteries*, under which only certain types of aggregate lotteries are possible and therefore only certain individual lotteries are available to consumers. We work in a completely finite environment – both the number of consumers and the number of possible lotteries is finite. We introduce lottery-producing firms that both generate individual lotteries and coordinate them to ensure feasibility. We present this extended model in section 2 below. In section 3, we show that the feasible allocations in our (generalized) lottery model are the same as those in the corresponding finite-state sunspots model. This is our first main point: The lottery model can be extended to purely finite economies.

Our goal is to compare the sets of equilibrium allocations generated by the two models in the finite environment. They are potentially different because of a fundamental difference in the two pricing systems. The lottery model directly assigns prices to probability distributions over the consumption set. This means that purchasing, say, a particular consumption bundle with probability one-half (and nothing otherwise) has a posted price. In the sunspots model, prices are assigned to states of nature. When there are many states, there may be many different combinations of state-contingent consumption plans that generate the same lottery, and these different combinations may have different costs. In this way the price system in the sunspots model is more flexible. However, the sunspots model also places certain restrictions on prices that are not present in the lottery model. To see this, suppose that there are three equally-likely states of nature and that the sunspots model assigns the same price vector to each of these states. Then the cost of receiving a particular bundle with probability two-thirds is twice the cost of receiving it with probability one-third, because the way a consumer constructs the two-thirds probability is by purchasing the same bundle in two states of nature. In the lottery model, however, there is no such restriction. The posted price of the bundle with two-thirds probability can be either more or less than twice the price of the same bundle with one-third probability. The price system in the lottery model is more flexible in this way. In other words, the two models define different objects to be the "basic" commodity of trade to which the law of one price applies. As a result, the relative valuations generated by a price system in one model may not have a representation in the other model. If an equilibrium allocation in one model is supported only by prices that cannot be represented in the other model, the two sets of equilibrium allocations are likely to be different.

In section 4, we examine what restrictions must be satisfied by equilibrium prices in each model. For the sunspots model, we build on the results of GKQS (in press) and show that any equilibrium allocation can be supported by a price system in which states with equal probability share the same (contingent-commodity) price vector. This eliminates some, but not all, of the additional flexibility of prices in the sunspots model. For the lottery model, we show that the absence of arbitrage opportunities for lottery-producing firms requires that prices be linear in commodities and additive in the available randomization opportunities. This eliminates *all* of the additional flexibility of prices in the lottery model. These pricing results combine to generate the following results on the sets of equilibrium allocations, which we present in section 5. For every lottery equilibrium allocation, there is a corresponding sunspot equilibrium allocation. The converse of this statement is true *unless* the sunspot equilibrium allocation relies on the additional flexibility of sunspot prices, that is, unless the support prices cannot be translated into the lottery model. For almost all discrete randomizing devices, this cannot happen and hence the two models lead to the same set of equilibrium allocations. However, for some randomizing devices the extra generality in the sunspot price system *does* matter and there exist sunspot equilibrium allocations with no lottery equilibrium counterpart. We present one such example in Section 5.2. The fact that the set of randomizing devices for which nonequivalence can occur has Lebesgue measure zero does not, of course, imply that such cases are unimportant in an economic sense. Hence, we summarize our results as follows: The sets of sunspot equilibrium and lottery equilibrium allocations are often, but not always, equivalent.

2 The Two Models

We begin by describing the fundamental elements of the economy that are common to both models of trade. We then describe each model in detail.

2.1 The Environment

There is a finite set H of consumers (indexed by $h = 1, \ldots, H$). There are L indivisible consumption goods, each of which is only available in integer amounts. There is a finite upper bound b_j on the amount of good j that may be consumed by any one consumer. These bounds allow us, for example, to study the case of $\{0, 1\}$ goods that has received so much attention in the literature on labor-market lotteries (i.e., Rogerson 1988, Hansen 1985, and Shell and Wright 1993). We also assume that there is a single divisible good (which we label good zero), so that the consumption set is given by

$$\mathcal{C} = C \times \mathbb{R}_+,$$

where C is a finite set with $K \equiv \prod_\ell b_\ell$ elements. We can think of the divisible good as "money" or "income spent on divisible goods." The literature on lottery equilibrium often assumes that the consumption set has only a finite number of points, like our set C, because this simplifies notation, proofs, and computations (see Prescott and Townsend 1984a, Garratt 1995, and Prescott and Townsend 2000). However, in the case of finite randomization possibilities this would imply that consumers are locally satiated at every consumption bundle and would create equilibria where consumers do not spend all of their income. Such equilibria are of limited interest, since they are not robust to changes in the environment such as the introduction of a divisible good.[3] By adding a divisible good, we eliminate these equilibria while maintaining much of the notational convenience of the finite-set approach. Adding more divisible goods would not change the analysis in any way.

Each consumer has preferences represented by a Bernoulli utility function $U_h : \mathcal{C} \to \mathbb{R}$. To simplify the analysis in what follows, we assume that this function is additively separable in the divisible good, so that utility can be written as the sum of two functions u_h and v_h, with

$$u_h : C \to \mathbb{R} \qquad \text{and} \qquad v_h : \mathbb{R}_+ \to \mathbb{R}.$$

We assume that v_h is strictly concave for all h, so that consumers are risk averse in the divisible good. This implies that Pareto optimal allocations can never involve randomization in the assignment of the divisible good. Our focus in this paper is on how the two different models of trade generate stochastic allocations of the indivisible goods. The purpose of the divisible good is to provide consumers with a productive use for any "left over" income. Consumer h also has endowments $e_h \in \mathbb{R}_+^L$ of the indivisible goods and an endowment $e_{0h} \in \mathbb{R}_+$ of the divisible good.

[3] GKQS (in press) examine sunspots economies with finite randomization possibilities and provide results that apply only to equilibria satisfying certain conditions. The conditions deal with exactly this issue – they rule out equilibria that rely on satiation.

The set of *fundamentals* that is common to both models is the list $\left(\mathcal{C}, \{U_h, e_h, e_{0h}\}_{h \in H}\right)$ *plus* a randomizing device, which represents the set of stochastic trades that agents can make. We find it helpful to think of this device as a roulette wheel. The wheel has on it a finite number M of slots, and the probability that the ball will fall into slot m is given by π_m. The wheel can be spun only one time, so that a single spin represents all of the randomization possibilities available in the economy.[4] As we discussed above, there are practical reasons why the randomization possibilities available to consumers may be constrained. To keep things simple, however, we interpret the constraints as being technological in nature. The roulette wheel is, in our framework, the only way in which stochastic allocations can be generated.

To further simplify the notation, we do not allow stochastic allocations of the divisible good. This is without any loss of generality, because risk-aversion implies that even if we allowed the allocation of the divisible good to be based on the realization of the randomizing device, it would never occur in equilibrium in either model. This assumption actually complicates the specification of the sunspots model slightly, but it greatly simplifies the presentation of the lottery model, as we show below.

Let F denote the set of feasible pure (nonstochastic) allocations of the endowments of the indivisible goods. Using $a = (a_h)_{h \in H}$ to denote a pure allocation with $a_h \in C$ for every h, we then have

$$F = \left\{ a \in C^H : \sum_h a_h \leq \sum_h e_h \right\}.$$

Both models generate equilibria that consist of a pure allocation of the divisible good paired with a probability distribution over F. The difference between them is the way in which stochastic trade is organized. We now describe the two models in detail.

2.2 The Sunspots Model

In the sunspots model, each slot on the roulette wheel is marked with a number and called a "state of nature." Consumers then trade in state-contingent claims on every good. Formally, the sunspot variable is represented by a probability space (S, Σ, π). Here S is a finite set with M elements and Σ is the set of all subsets of S. The probability of each state s is denoted $\pi(s)$ and the probability of any subset A of S is denoted $\pi(A)$.

Let X be the set of functions $x_h : S \to C$, that is, the set of allowable, stochastic individual consumption plans for the indivisible goods. Prices for the indivisible goods are given by a function $p : S \to \mathbb{R}^L_+$.[5] We

[4] If the wheel could be spun more than one time, we could always redefine the wheel so that each slot on the new wheel represents a sequence of realizations from the multiple spins of the old wheel. In this sense, allowing only a single spin is without loss of generality.

[5] In constrast to the notation in GKQS (in press), the prices p here are the actual contingent-commodity prices. They are *not* probability-adjusted prices.

take the divisible good to be the numeraire, and we use x_{0h} to denote consumer h's (certain) consumption of this good. Consumer h chooses her consumption plan to solve

$$\max_{x_h, x_{0h}} \sum_s \pi(s)\, u_h(x_h(s)) + v_h(x_{0h}), \tag{1}$$

$$\text{subject to } \sum_s p(s) \cdot x_h(s) + x_{0h} \leq \sum_s p(s) \cdot e_h + e_{0h},$$
$$x_h \in X, \quad x_{0h} \in \mathbb{R}_+.$$

Let X^H be the set of functions $x : S \to \mathbb{R}^{LH}$ such that $x_h^* \in X$ for all h. The definition of equilibrium for the sunspots economy is as follows.

Definition 1 A *sunspot equilibrium* consists of a price function $p^* : S \to \mathbb{R}_+^L$ and an allocation $(x^*, x_o^*) \in X^H \times \mathbb{R}_+^H$ such that

(i) Given p^*, (x_h^*, x_{0h}^*) solves the consumer's problem (1) for each $h \in H$,

and

(ii) (x^*, x_0^*) is feasible, *i.e.*, we have $x^*(s) \in F$ for all $s \in S$ and $\sum_h x_{0h} \leq \sum_h e_{0h}$.

Using condition (ii), we see that an equilibrium allocation x^* of the indivisible goods generates a probability distribution over the set F that is given by $\pi \circ (x^*)^{-1}$. In other words, to every allocation $a \in F$, assign the probability of the set of states s such that $x^*(s) = a$. This distribution, together with the allocation of the divisible good, summarizes an equilibrium allocation of the sunspots model.

This specification of the sunspots model is fairly standard. We now turn our attention to the lottery model, which we modify to allow for constraints on the randomization possibilities.

2.3 The Lottery Model

Following Prescott and Townsend (1984a,b), we treat each consumption bundle in C as a separate commodity, so that there are K commodities (plus the divisible good). A quantity of commodity k corresponds to the probability of receiving bundle c_k. In this way, consumers in the lottery model directly choose probability distributions over C, which are called lotteries. However, in the constrained lottery model consumers are not able to purchase an arbitrary probability distribution over C. Rather, they are only able to choose lotteries that can be generated by the pre-specified randomizing device. As in the sunspots model, it is natural to think of a roulette wheel. However, in the lottery model consumers are not able to specify how their lottery is arranged on the wheel. As an example, suppose the wheel has two equally-likely slots (black and red) and that a consumer purchases a lottery that delivers a particular bundle with probability one-half. The consumer is not able to specify whether she will receive that bundle when the black slot is realized or when the red slot is realized. If she could, she would be buying state-contingent consumption and we would be back in the sunspots model. Instead, the task of arranging the demanded lotteries on the wheel so that they meet feasibility requirements is left to the wheel's operator; we discuss this in detail below. This is the fundamental difference between the two models: commodities in the sunspots model are indexed by states of nature, while in the lottery model they are indexed by probabilities.

2.3.1 Constrained Lotteries and Consumer Choice

An individual lottery is a probability distribution δ_h over C. Because C is a finite set, this distribution is a vector $(\delta_h(c_1), \ldots \delta_h(c_K)) \in \mathbb{R}_+^K$ satisfying

$$\sum_{k=1}^{K} \delta(c_k) = 1,$$

where $\delta(c_k)$ is the probability assigned to the consumption bundle c_k. An individual consumption plan is a pair (δ_h, c_{0h}) specifying a lottery over C and a (certain) amount of the divisible good.

Instead of allowing agents to assign arbitrary probabilities to each bundle, we restrict trade to lotteries that can be generated by the given randomizing device. Let $\Delta(C)$ denote the set of all probability distributions over the bundles of indivisible goods; this is equivalent to the $(K-1)$-dimensional unit simplex in $\mathbb{R}^K$. Only a fraction of these lotteries can actually be generated by the roulette wheel. Consider a function g_h that assigns an element of C to each of the M slots on the wheel. Abusing notation somewhat, let M also represent the set of slots on the wheel, so that we have

$$g_h : M \to C.$$

Let G be the set of all such functions (note that since M and C are finite sets, G is also a finite set). Then the set of probability distributions over C that can be generated by a particular randomizing device is given by

$$\Gamma(C) = \left\{ \delta_h \in \Delta(C) : \exists\, g_h \in G \text{ such that } \left(\delta_h(c_k) = \sum_{m: g_h(m) = c_k} \pi_m \right) \text{ holds for all } k \right\}. \qquad (2)$$

As an example, suppose that $K = 3$ and that there are 3 equally-likely slots on the wheel. Then $\Delta(C)$ is the triangular simplex shown in figure 1. The set $\Gamma(C)$ contains those lotteries in which the probability placed on each consumption bundle is a multiple of one-third; these are the ten dots in the figure. Each consumer must choose one of these lotteries. In this way, the consumer is constrained to choose a lottery that is "available in the market." It is important to recognize that this is exactly the set of probability distributions

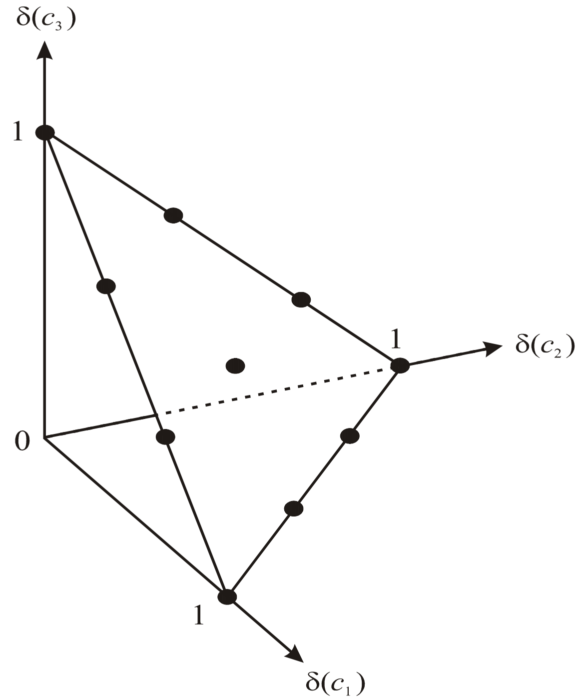

Figure 1: The set of possible individual lotteries

over C that a consumer in the sunspots model can construct using state-contingent commodities when there are three equally-likely states. This is what we mean when we say that our (generalized) lottery model constrains consumer choice in the same way the finite-state sunspot model does.

We again take the divisible good to be the numeraire. A general formulation of lottery prices is then a function

$$\phi : \Gamma(C) \rightarrow \mathbb{R}_{+},$$

that is, a function that assigns a price (in units of the divisible good) to each possible lottery over the indivisible goods. Using this formulation, we can write the consumer's lottery-choice problem as

$$\max_{\delta_h, c_{0h}} \sum_{C} \delta_h(c_k)\, u_h(c_k) + v_h(c_{0h}), \tag{3}$$

$$\text{subject to } \phi(\delta_h) + c_{0h} \leq \phi(e_h) + e_{0h},$$

$$\delta_h \in \Gamma(C), \quad c_{0h} \in \mathbb{R}_{+},$$

where e_h here represents the degenerate lottery that gives consumer h's endowment of the indivisible goods with probability one. Notice that the consumer chooses one and only one lottery. We do not allow the consumer to purchase multiple lotteries and combine them because the correlation structure would not be known to her (and hence the expected utility of such purchases would not be well defined). Consumers cannot engage in arbitrage activity in this model. Such activity is undertaken solely by firms, as we now discuss.

2.3.2 Lottery-Producing Firms

Lotteries are produced by a representative, competitive firm that has access to the randomization technology represented by the roulette wheel. The firm operates by buying and selling "lottery tickets," where each ticket entitles the holder to a particular lottery. Let j index the lotteries that the firm can produce, so that δ^j is a typical element of $\Gamma(C)$. Let $y\left(\delta^j\right)$ denote the firm's net sales of tickets promising the distribution δ^j. This number may be positive or negative, but must be an integer. Thus, a production plan for the firm is a function

$$y : \Gamma(C) \to \mathbb{Z},$$

where $\mathbb{Z}$ is the set of integers. The firm must choose a feasible production plan. That is, it must be able to arrange the lotteries that it buys and sells on the roulette wheel in such a way that, for all realizations, it gives away no more resources than it takes in. Let n index the individual tickets of lottery j bought or sold by the firm, so that we have $n = 1, ..., \left|y\left(\delta^j\right)\right|$. Each ticket is therefore identified by a pair (j, n), indicating the type of lottery it delivers (j) and the number of the ticket within that type (n). We need to count identical lottery tickets individually here because they may need to be generated by different slots on the roulette wheel. A function $g_{j,n}$ that assigns the distribution promised by lottery ticket (j, n) to spaces on the lottery wheel has the form $g_{j,n} : M \to C$ with

$$\delta^j(c_k) = \sum_{m:g_{j,n}(m)=c_k} \pi_m \quad \text{for all } k. \tag{4}$$

For all (j, n) define

$$I_{j,n} = \begin{cases} -1 & \text{if } \ y\left(\delta^j\right) < 0 \\ 0 & \text{if } \ y\left(\delta^j\right) = 0 \\ 1 & \text{if } \ y\left(\delta^j\right) > 0 \end{cases}$$

Feasibility then requires that we have

$$\sum_{j,n} I_{j,n} g_{j,n}(m) \leq 0 \qquad \text{for all } m. \tag{5}$$

For each slot m, we add up the net resources the firm must deliver on all lottery tickets when m is realized; this number must be non-positive. Conditions (4) and (5) together are equivalent to saying that each of the individual lottery tickets must be a marginal distribution of some common joint lottery.[6] The production set of the firm is

$$Y = \left\{ y : \Gamma(C) \to \mathbb{Z} : \exists \ \{g_{j,n}\}_{j,n} \ \text{ such that (4) holds for each } \ (j,n) \ \text{ and (5) is satisfied.} \right\}.$$

The firm's problem is given by

$$\max_{y \in Y} \sum_{\delta \in \Gamma(C)} \phi(\delta)\, y(\delta). \tag{6}$$

Before proceeding, it may be helpful to look at a simple example that illustrates how our firms differ from those used in the previous literature (see especially Rogerson 1988).

Example: $C = \{0, 1\}$. There is a single indivisible good that can only be consumed in either one unit or not at all. Suppose that all consumers are identical and that prices are such that everyone demands the lottery that gives one unit of the good with probability two-thirds and nothing with probability one-third. First consider the case where there is a continuum of consumers and randomization is unconstrained. Then the firm buys two-thirds of a unit of the good per consumer, and sells the demanded lottery. Each consumer comes to the firm and a weighted coin is tossed to see if the consumer receives the good or not. Since the coin tosses are independent across consumers,[7] two-thirds of the consumers will receive the good and hence this plan is feasible. This is the "traditional" approach to lottery-producing firms.

Next, let's look at how the firm we described above operates in the finite case. Suppose that there are three consumers and the roulette wheel has three equally-likely slots. The firm faces the above demand conditions – all consumers want to receive the good with probability two-thirds. If the firm buys two units of the indivisible good (or, more precisely, two units of the degenerate lottery that delivers one unit of the good with probability one), it can offer three units of the demanded lottery. Consider the joint lottery

	Roulette wheel slot:	red	black	green
	ticket-holder 1:	1	1	0
Payoff to	ticket-holder 2:	1	0	1
	ticket-holder 3:	0	1	1

Looking at the columns shows that, regardless of which slot is realized, the firm pays out two units of the good (exactly equal to the resources it purchased). Looking at the rows shows that each ticket-holder receives the good with probability two-thirds, as desired.

Two comments are in order here. First, the plan of the finite-case firm only works if the number of

[6] The idea of using aggregate or joint lotteries to ensure coordination when there is a finite number of consumers was introduced in Garratt (1995), where coordination is provided by the auctioneer as a part of the market-clearing process.

[7] This statement ignores problems associated with integrating over a continuum of i.i.d. random variables.

tickets it sells is a multiple of three. There is no way it can sell four of these lotteries, for example, because there is no joint lottery that would generate four such marginal distributions. (See Shubik 1971 on this phenomenon.) Outside of this restriction, the finite-case firm behaves very much like the continuous-case one: it purchases two-thirds of a unit of the good per customer and delivers the good with probability two-thirds to each customer. Second, suppose that the roulette wheel has only two slots, with probabilities one-third and two-thirds. Now the plan of the finite-case firm is infeasible, since there is no longer a joint lottery that gives the same marginal distribution to every ticket-holder. That such seemingly small changes in the randomizing device can have important effects is a recurrent theme in the finite case.

2.3.3 Equilibrium

The definition of equilibrium in our generalized lottery economy is the following.

Definition 2 A *lottery equilibrium* consists of a price function $\phi^* : \Gamma(C) \rightarrow \mathbb{R}_+$ and an allocation (δ^*, c_0^*, y^*) such that

(i') Given ϕ^*, (δ_h^*, c_{0h}^*) solves the consumer's lottery problem (3) for each $h \in H$,

(ii') Given ϕ^*, y^* solves the firm's problem (6),

and

(iii') We have

$$y^*(\delta) = \sum_h \left(I_{(\delta_h^* = \delta)} - I_{(e_h = \delta)} \right) \text{ for every } \delta \in \Gamma(C)$$

and

$$\sum_h c_{0h}^* \leq \sum_h e_{0h}.$$

Condition (iii') is the market-clearing constraint. It simply requires that the number of units of each lottery that the firm produces be equal to the net demand for the lottery by households (here I is the indicator function), and that the market for the divisible good clears.

Recall that a sunspot equilibrium allocation generates a probability distribution over the set F of (pure) feasible allocations of the indivisible goods. The same is true in the lottery model. Let g_h denote the arrangement function for the lottery δ_h^*.[8] Define g to be the vector-valued function comprised of the functions

[8] This notation is slightly different from our earlier use of subscripts to the function g. Instead of indexing by type of lottery and number of ticket, we are now indexing by the purchaser of the ticker. Of course, this is not important; it only serves to simplify the argument here.

g_h, so that we have

$$g : M \to C^H.$$

The feasibility condition (5) guarantees that for every realization of the wheel, the assignment of resources is feasible. Therefore we have $g(m) \in F$ for all m. Then $\pi \circ g^{-1}$ is the probability distribution over F (or, the joint lottery) associated with the lottery equilibrium allocation. This relates the market-clearing conditions here to those in Garratt (1995) and GKQS (in press). In those papers, market-clearing is stated directly in terms of the joint lottery. This is because the joint lottery is arranged by the auctioneer (and hence is naturally considered part of the market-clearing process). Here the joint lottery is arranged by the firm and hence is determined by the firm's equilibrium production plan.

3 Comparing the Models

While the two models are stated in very different terms, the literature (beginning with Shell and Wright 1993) has shown that they are actually quite similar. Our introduction of constrained lotteries has, for the finite case, brought them closer still. In this section, we compare the sets of feasible allocations and of possible prices in the two models. We show that the sets of feasible allocations are identical. In this sense, our definition of constrained lotteries places the "right" restrictions on stochastic allocations in the lottery model. The price systems, on the other hand, are fundamentally different.

3.1 Corresponding Allocations

Because an individual consumption plan is a function in the sunspots model and a probability distribution in the lottery model, we need to be precise about how we compare these objects. At the individual level, a sunspot consumption plan (x_h, x_{0h}) induces a lottery consumption plan (δ_h, c_{0h}) through the equations

$$\delta_h(c_k) = \pi \circ x_h^{-1}(c_k) \qquad \text{for all } k \tag{7}$$

$$c_{0h} = x_{0h}. \tag{8}$$

In other words, the probability assigned by the individual lottery δ_h to the bundle of indivisible goods c_k is equal to the probability assigned by π to the set of states in which x_h delivers c_k. Note that $x_h \in X$ holds if and only if $\delta_h \in \Gamma(C)$ holds, that is, the lottery δ_h defined in (7) is individually feasible in the lottery model if and only if the plan x_h generating it is individually feasible in the sunspots model. At the aggregate level, a sunspot allocation (x, x_0) induces a lottery allocation (δ, c_0, y) in two steps. First, the individual sunspot consumption plans induce the individual lotteries through (7). The production plan y is then given by

$$y(\delta) = \sum_h \left(I_{(\delta_h = \delta)} - I_{(e_h = \delta)} \right) \tag{9}$$

for all $\delta \in \Gamma(C)$. In other words, y is the unique production plan that makes the consumption allocation feasible. We now show that, through this relationship, feasible sunspot allocations always correspond to feasible lottery allocations and *vice versa*.

Proposition 1 *A sunspot allocation (x, x_0) is feasible if and only if the corresponding lottery allocation (δ, c_0, y) given by (7), (8), and (9) is feasible.*

Proof: (a) : Suppose that (x^*, x_0^*) is feasible, so that we have $x^*(s) \in F$ for all s and $\sum_h x_{0h} \leq \sum_h e_{0h}$. To show that the corresponding lottery allocation is also feasible, we need to show that (1) $\sum_h c_{0h} \leq \sum_h e_{0h}$ holds and (2) $y \in Y$ holds. (That the individual lotteries being consumed are equal to those being produced is guaranteed by the definition of y in (9)). The first of these is immediate from the definition of c_{0h} in (8).

For the second, we need to construct the assignment functions that distribute each individual lottery on the roulette wheel. From (9), we see that the firm is buying one degenerate lottery from each consumer (her endowment) and selling one possibly-nondegenerate lottery to each consumer (her consumption). Arranging the degenerate lotteries is trivial (the same bundle is assigned to every slot). For the possibly-nondegenerate lotteries, let $f : M \to S$ be the function that maps slots on the wheel to their "names" in the sunspots model. We then define the assignment function for individual lottery δ_h by

$$g_h(m) = x_h(f(m)) \quad \text{for all } m \in M. \tag{10}$$

In other words, have the lottery-producing firm arrange the stochastic allocations on the wheel in exactly the same way that they are arranged in the sunspots model. Then (4) must clearly hold for each h. Furthermore, we can write (5) as follows

$$\begin{aligned} \sum_{j,n} I_{j,n} g_{j,n}(m) &= \sum_h (-e_h + g_h(m)) \\ &= \sum_h (-e_h + x_h(f(m))) \\ &= \sum_h (-e_h + x_h(s)) \leq 0 \quad \text{for all } s. \end{aligned}$$

In this way, feasibility of the sunspot allocation guarantees feasibility of the lottery allocation.

(b) : The proof of the converse is essentially the same argument in reverse. We start with a feasible lottery allocation and a sunspot allocation that induces it. Feasibility of the lottery allocation implies the existence of assignment functions g_h, which correspond to the sunspot consumption plans as in (10) above.[9] The lottery feasibility condition (5) then implies that we have $x(s) \in F$ for all s, and therefore the sunspot allocation is also feasible. ■

3.2 Corresponding Prices

Although both models generate the same set of feasible allocations, there is an important difference in the way they assign prices. This can been seen in the context of a simple example. Suppose the roulette wheel has two equally-likely slots. Pick an arbitrary consumption bundle c_k and ask: What is the price of receiving that bundle with probability one-half and nothing otherwise? In the lottery model, there is a unique answer: $\phi(\delta)$, where δ represents the specified lottery. In the sunspots model, however, the answer is either $[p(\alpha) \cdot c_k]$ or $[p(\beta) \cdot c_k]$, depending on the state (α or β) in which the bundle is purchased. Hence the sunspots model has the ability to assign different prices to something that the lottery model considers a single commodity. Now ask: What is the relationship between the cost of buying a bundle with probability one-half (and nothing otherwise) and the cost of buying it with probability one? In the sunspots model, there is a unique answer: the cost of buying c_k with probability one is given by $[p(\alpha) + p(\beta)] \cdot c_k$. In the lottery model, however, no such relationship need hold between $\phi(\delta)$ and $\phi(\delta')$, where δ' represents the (degenerate) lottery giving c_k with probability one. In this way, the lottery model has more flexibility in the assignment of prices across different probabilities.

The critical issue is under what conditions the prices in one model can be translated into corresponding prices in the other model. We first introduce an additional piece of notation. Let Γ denote the image of $\Gamma(C)$ in the interval $[0,1]$. Then Γ is the set of probabilities that can be assigned to consumption bundles in a lottery generated by the roulette wheel. As an example, if there are three equally-likely slots, then we have $\Gamma = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$. The same is true if there are two slots with probabilities one-third and two-thirds; different randomizing devices can lead to the same set Γ. Note that we have the relationship

$$\Gamma = \left\{\theta \in [0,1] : \theta = \sum_{s \in A} \pi(s) \text{ for some } A \subseteq S\right\}. \tag{11}$$

In other words, a consumption bundle in the lottery model can be purchased with probability θ if and only if it can be purchased in the sunspots model in a set of states whose total probability is θ.

9 There is an additional complication in this direction: different sunspot allocations may correspond to the same lottery allocation. Therefore matching the functions g_h with the functions x_h may first require "relabelling" the slots on the wheel (or, equivalently, changing the function f) in such a way that the probability of receiving each consumption bundle is preserved. See GKQS (in press) for an extensive discussion of this many-to-one relationship.

Suppose a sunspot price function p has the following property: for any two disjoint subsets A and B of S, we have

$$\sum_{s\in A}\pi\left(s\right)=\sum_{s\in B}\pi\left(s\right)\quad\text{implies}\quad\sum_{s\in A}p\left(s\right)=\sum_{s\in B}p(s). \tag{12}$$

In other words, suppose that whenever there are different ways of combining states together to get the same probability, the total cost of purchasing any consumption bundle in either of the two sets of states is the same. When this is true, the sunspots model assigns a unique price vector to each level of probability in Γ, and therefore the sunspot prices can be transformed into lottery prices. Let $\widehat{P}$ be the set of functions $p : S \to \mathbb{R}^L_+$ satisfying condition (12).

Definition 4 Given a sunspot price function $p \in \widehat{P}$, the *corresponding lottery price function* $\phi : \Gamma(C) \to \mathbb{R}^L_+$ is defined in two steps. First, define $q : \Gamma \to \mathbb{R}^L_+$ by

$$q\left(\theta\right)=\sum_{s\in A}p\left(s\right) \tag{13}$$

for any $A \subseteq S$ with $\sum_{s\in A}\pi(s) = \theta$, for all $\theta \in \Gamma$. Then the price function ϕ is given by

$$\phi\left(\delta\right)=\sum_{C}q\left(\delta\left(c_k\right)\right)\cdot c_k. \tag{14}$$

The function q assigns a price vector to each probability in Γ. Any consumption bundles that are purchased with probability θ are then priced according to $q(\theta)$. When a lottery price function ϕ can be represented by a function q satisfying (14), lottery prices are linear in commodities. Prices in the sunspots model are necessarily linear in commodities, and therefore any lottery prices that correspond to some sunspot prices must have this property. Only when sunspot prices satisfy (12) will the function q, and hence the lottery prices, be well defined. If the sunspot price function does not satisfy (12), we say that it has *no corresponding lottery prices*. If, for a given lottery price function ϕ, there does not exist p and q satisfying (13) and (14), we say that ϕ has *no corresponding sunspot prices*.

To see what restrictions must be placed on prices in order for them to translate from one model to the other, consider the case of three equally-likely slots on the wheel. In order for a sunspot price function p to have corresponding lottery prices in this case, it must assign a unique price to receiving any consumption bundle with probability two-thirds. This implies that we must have

$$p\left(1\right)+p\left(2\right)=p\left(1\right)+p\left(3\right)=p\left(2\right)+p\left(3\right)$$

or

$$p(1) = p(2) = p(3).$$

In this case, sunspot prices would need to be constant across states in order to have corresponding lottery prices. Conversely, any lottery price function ϕ must be linear in quantities and have a price vector q that satisfies

$$q\left(\frac{2}{3}\right) = 2q\left(\frac{1}{3}\right) \qquad \text{and} \qquad q(1) = 3q\left(\frac{1}{3}\right)$$

in order to correspond to some sunspot prices. If, however, there are only two sunspot states, with $\pi(1) = 1/3$ and $\pi(2) = 2/3$, then there is no restriction on the sunspot price function p. The only restriction on the lottery price function in this case is

$$q(1) = q\left(\frac{1}{3}\right) + q\left(\frac{2}{3}\right).$$

Hence the strength of the restrictions required for prices to translate from one model to the other depend on the details of the randomizing device. Many restrictions are needed when slots are equally likely, while fewer are required when each slot has a distinct probability.

4 Equilibrium Prices

In order for an equilibrium allocation in one model to also be part of an equilibrium in the other model, at least one price function supports the allocation must translate to the other model. Hence knowing what restrictions prices must satisfy in equilibrium in the two models is crucial for determining the relationship between the two sets of equilibrium allocations. In this section, we investigate what restrictions can be placed on equilibrium prices.

4.1 Sunspot Equilibrium Prices

We first prove a result for the sunspots model. GKQS (in press, Theorem 2) show that with a finite number of equally-likely states, any equilibrium allocation can be supported by prices that are constant across states. We extend this result to the case where only some states are equally likely. We show that any equilibrium allocation can be supported by prices in which $\pi(s) = \pi(s')$ implies $p(s) = p(s')$ for any states s and s'. As shown in the previous section, such a result is critical for supporting the corresponding allocation as an equilibrium of the lottery model, since the lottery model necessarily assigns a single price to receiving a bundle with probability $\theta = \pi(s)$.

We begin by defining sets of individual consumption plans that are in some sense equivalent. Suppose that we have a subset $A \subseteq S$ of states, each of which has the same probability. Let N_A be the number of states in A, and suppose (without any loss of generality) that these states are consecutively numbered.

Definition 5 For any $x_h \in X$ and any set A of equi-probable states, the *A-shift class* of x_h, denoted $T_A(x_h)$, is the set of plans x_h^t such that

$$x_h^t = \left\{ \begin{array}{lcc} x_h(s+t) & \text{for} & s \in A \\ x_h(s) & & s \notin A \end{array} \right\}$$

holds for some $t \in \{0, 1, \ldots N_A\}$, where the addition is modulo N_A.

The idea here is simply to "shift" the consumption bundles across the equally-probable states of nature. If these states were not consecutively numbered, the idea would be exactly the same (but the notation would be more complicated).

The A-shift class is a set of plans among which a consumer with von Neumann-Morgenstern preferences would clearly be indifferent.[10] We next show that this fact has important implications for the form of equilibrium prices. For any price vector p and any equi-probable set A, define another price vector $\overline{p}_A$ by

$$\overline{p}_A = \left\{ \begin{array}{lcc} \frac{1}{N_A}\sum_A p(s) & \text{for} & s \in A \\ p(s) & & s \notin A \end{array} \right\}.$$

The price vector $\overline{p}_A$ replaces prices of states in A with the average price across those states. We want to show that if p^* supports some allocation (x^*, x_0^*) as a sunspot equilibrium, then that same allocation is also supported as an equilibrium by $\overline{p}_A^*$. We begin with some lemmas relating the two price functions.

Lemma 1 *For any equi-probable set A, if x_h satisfies*

$$\sum_s p(s) \cdot x_h(s) \leq \sum_s p(s) \cdot x_h^t(s)$$

for $t = 1, \ldots, N_A$, then we have

$$\sum_s p(s) \cdot x_h(s) \leq \sum_s \overline{p}_A(s) \cdot x_h(s).$$

[10] Our results actually apply to a broader class of preferences, including those defined by Balasko (1983).

Proof: The hypothesis of the lemma implies that we have

$$\begin{aligned}\sum_s p(s)\cdot x_h(s) &\leq \frac{1}{N_A}\sum_t\sum_s p(s)\cdot x_h^t(s)\\ &= \sum_s p(s)\cdot\sum_t \frac{x_h^t(s)}{N_A}.\end{aligned}$$

The right-hand side of this inequality replaces consumption in state s with the average consumption over all states in A. Because this average is the same for all s in A, we can replace price vector p with $\overline{p}_A$ without changing the value of the inner product,

$$\sum_s p(s)\cdot\sum_t \frac{x_h^t(s)}{N_A} = \sum_s \overline{p}_A(s)\cdot\sum_t \frac{x_h^t(s)}{N_A}.$$

Note that since $\overline{p}_A$ takes on the same values for all s in A, we can "undo" the averaging of the allocation. In other words, for states in A, it makes no difference if we multiply the average price by the average consumption in each state or if we multiply the average price by the actual consumption in each state,

$$\sum_s \overline{p}_A(s)\cdot\sum_t \frac{x_h^t(s)}{N_A} = \sum_s \overline{p}_A(s)\cdot x_h(s).$$

This establishes the desired result. ■

We next show that we can make a stronger statement about equilibrium prices.

Lemma 2 *Suppose (p^*, x^*) is a sunspot equilibrium. Then for any equi-probable set A,*

$$\sum_s p^*(s)\cdot x_h(s) = \sum_s \overline{p}_A^*(s)\cdot x_h(s)$$

must hold for all h.

Proof: Given local nonsatiation, individual optimization implies that an equilibrium allocation x_h^* must be the minimal cost element of any A-shift class $T_A(x_h^*)$. Therefore, by Lemma 1, we have

$$\sum_s p^*(s)\cdot x_h^*(s) \leq \sum_s \overline{p}_A^*(s)\cdot x_h^*(s)$$

for all h. Suppose that this holds with strict inequality for some h. Then summing this inequality across all consumers and using the fact that each consumer's budget constraint must hold with equality (again due to local nonsatiation) yields

$$\sum_s p^* (s) \cdot \sum_h e_h < \sum_s \overline{p}_A^* (s) \cdot \sum_h x_h^* (s) .$$

But market clearing (or the feasibility of x^*) requires

$$\sum_h x_h^* (s) \leq \sum_h e_h$$

for every state s, we have

$$\sum_s p^* (s) \cdot \sum_h e_h < \sum_s \overline{p}_A^* (s) \cdot \sum_h e_h.$$

Because $\sum_s p^* (s) = \sum_s \overline{p}_A^* (s)$, this is a contradiction. ■

With this information in hand, we are ready to prove a result on the form of sunspot equilibrium prices: any sunspot equilibrium allocation can be supported by prices that are constant across equiprobable sets of states.

Proposition 2 *Suppose* (p^*, x^*, x_0^*) *is a sunspot equilibrium. Then for any set of equi-probable states* A, $(\overline{p}_A^*, x^*, x_0^*)$ *is also a sunspot equilibrium.*

Proof: It suffices to show that (x_h^*, x_{0h}^*) is still an optimal choice for consumer h when prices are given by $\overline{p}_A^*$. Lemma 2 shows that (x_h^*, x_{0h}^*) is still affordable at these prices. Suppose it is not optimal for some consumer h. Then there exists some other plan $(\widetilde{x}_h, \widetilde{x}_{0h})$ that is affordable at prices $\overline{p}_A^*$ and is strictly preferred to (x_h^*, x_{0h}^*) . Thus we would have

$$\sum_s \overline{p}_A^* (s) \cdot \widetilde{x}_h (s) + \widetilde{x}_{0h} \leq \sum_s \overline{p}_A^* (s) \cdot e_h + e_{0h} = \sum_s p^* (s) \cdot e_h + e_{0h}.$$

Let $\widetilde{y}_h$ denote the minimum cost element of $T_A (\widetilde{x}_h)$ at prices p^*. Then $(\widetilde{y}_h, \widetilde{x}_{0h})$ is also strictly preferred to (x_h^*, x_{0h}^*) and $\widetilde{y}_h$ costs exactly the same as $\widetilde{x}_h$ at prices $\overline{p}_A^*$. Therefore we have

$$\sum_s \overline{p}_A^* (s) \cdot \widetilde{y}_h (s) + \widetilde{x}_{0h} \leq \sum_s p^* (s) \cdot e_h + e_{0h}.$$

Because $\widetilde{y}_h$ is the minimum cost element of its A-shift class, Lemma 1 implies that we have

$$\sum_s p^* (s) \cdot \widetilde{y}_h (s) \leq \sum_s \overline{p}_A^* (s) \cdot \widetilde{y}_h (s) ,$$

meaning that $(\widetilde{y}_h, \widetilde{x}_{0h})$ was affordable at prices p^*. This contradicts the optimality of x_h^* at prices p^*. ■

This result can be used to give sufficient conditions on the randomizing device to guarantee that sunspot prices have corresponding lottery prices. As an example, suppose that there are three states of nature, with $\pi(1) = \pi(2) = 1/6$ and $\pi(3) = 2/3$. Then Proposition 2 says that any equilibrium allocation can be supported by a price function that assigns a unique cost to receiving a particular bundle with probability one-sixth. The other possible probabilities (one-third, two-thirds, five-sixths, and one) then have unique costs as well. Therefore condition (12) is satisfied, and the corresponding lottery prices are given by (13) and (14). In addition, when there are multiple sets of equi-probable states, the proposition applies to each of them. Suppose there are four states with probabilities $\pi(1) = \pi(2) = 1/10$ and $\pi(3) = \pi(4) = 2/5$. Then Proposition 2 says that we can find support prices where $p(1) = p(2)$ and $p(3) = p(4)$. From this it follows that each of the other possible probabilities (there are a total of seven in this case) has a unique cost, and therefore (12) is again satisfied.

Notice, however, that the conclusion in Proposition 2 applies to equally-likely states, not to sets of states that add to the same total probability. This distinction will be critical in our example in section 5.2 of a sunspot equilibrium allocation with no lottery equilibrium counterpart.

4.2 No-Arbitrage Lottery Prices

In the lottery model, arbitrage arguments based on the constant-returns-to-scale nature of the firm's production technology can be used to place restrictions on the set of prices that could appear in equilibrium. Garratt (1995) demonstrates that in a model where the auctioneer coordinates individual lotteries, the absence of arbitrage requires that lottery prices be linear in the underlying goods. This is also true under our specification of the lottery-producing firm.

Proposition 3 *If there exists a solution to the firm's problem (6), then there exists a function $q : \Gamma \to \mathbb{R}^L_+$ such that (14) holds for every $\delta \in \Gamma(C)$.*

Proof: Let $e^\ell \in C$ denote the commodity bundle that has one unit of good ℓ and zero units of every other good, for $\ell = 1, ..., L$, and let $\mathbf{0} \in C$ denote the zero vector; i.e., the commodity bundle that contains zero units of every indivisible good. Let $\delta^{(\theta,\ell)}$ denote the lottery that delivers the commodity bundle e^ℓ with probability θ and the commodity bundle $\mathbf{0}$ with probability $(1-\theta)$. Note that this lottery is in $\Gamma(C)$ if (and only if) θ is in Γ. Define the new price function $q : \Gamma \to \mathbb{R}^L_+$ by $q_\ell(\theta) = \phi\left(\delta^{(\theta,\ell)}\right)$ for every $\theta \in \Gamma$ and for $\ell = 1, ..., L$.

Consider an arbitrary lottery $\delta^j \in \Gamma(C)$ and let g_j be a function that distributes δ^j on the lottery wheel, i.e., we have $\delta^j(c_k) = \sum_{m:g_j(m)=c_k} \pi_m$ for all k. The lottery δ^j is a K-dimensional vector

$$\delta^j = \left(\delta^j(c_1), \delta^j(c_2), \ldots, \delta^j(c_K)\right)$$

whose elements sum to unity. Suppose that $\phi(\delta^j) > \sum_C q(\delta^j(c_k)) \cdot c_k$ held, and consider the following production plan. Set $y(\delta^j) = 1$, so that the firm is buying one unit of lottery δ^j. Partition M into sets $\{M_{\theta_1}, ..., M_{\theta_i}, ..., M_{\theta_{|\Gamma|}}\}$ where M_{θ_i} denotes the sections of the wheel that occur with probability $\theta_i \in \Gamma$. Some of the sets in the partition may be empty. For each probability $\theta_i \in \Gamma$ and for each good ℓ, set $y\left(\delta^{(\theta_i,\ell)}\right) = -\sum_{m\in M_{\theta_i}} [g_j(m)]^\ell$, where $[g_j(m)]^\ell$ denotes the ℓth component of the vector $g_j(m)$. Set all other values of y equal to zero. This plan is feasible by construction. To see this, note that $y(\delta^{(\theta_i,\ell)})$ gives the total number of lottery tickets of type (θ_i, ℓ) that the firm agrees to sell according to the production plan. For each type of lottery $\delta^{(\theta_j,\ell)}$ place each of the n tickets on the wheel in the following way: For each $m' \in M_{\theta_j}$ and $k = 1, ..., [g_j(m')]^\ell$,

$$g_j(m) = \begin{cases} e^\ell & \text{if } m = m' \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Then (4) is clearly satisfied. Moreover, $\sum_{(\theta_j,\ell)} I_{(\theta_j,\ell),n} g_{\theta_i,\ell}(m) = 0$ and hence (5) is satisfied. Thus we have that the specified production plan is feasible and generates positive profits, which violates the no arbitrage requirement. If $\phi(\hat{\delta}) < \sum_C q(\hat{\delta}(c_k)) \cdot c_k$ the firm can earn positive profits by performing the above production plan in reverse. Therefore, if (14) does not hold, by replicating either the specified plan or its negative the firm can earn arbitrarily large profits and (6) has no solution. ■

The previous literature has assumed that lottery prices are also linear in probabilities, that is, the price of probability θ on any bundle c_k is given by $\theta\psi \cdot c_k$ for some $\psi \in \mathbb{R}^\ell_+$. This is a much stronger restriction on prices than that given in Proposition 3. Neither prices in the sunspots model nor those in the lottery model need be linear in probabilities. However, lottery prices must be "additive" in a certain sense in order to prevent firms from having an arbitrage opportunity. The precise restriction is the following.

Definition 6 For a given randomizing device, a lottery price function $q : \Gamma \to \mathbb{R}^L_+$ is *additive* if for any set $A \subseteq M$ of slots on the wheel, we have

$$q\left(\sum_{m\in A} \theta_m\right) = \sum_{m\in A} q(\theta_m),$$

where $\theta_m = \pi(m)$ for each slot m.

Notice that this does *not* imply that prices are linear in probabilities. As an example, suppose that $L = 1$, and $M = 2$ with $\pi(1) = \frac{1}{3}$ and $\pi(2) = \frac{2}{3}$. Then the following price function is additive:

$$q\left(\frac{1}{3}\right) = \frac{1}{4}, \quad q\left(\frac{2}{3}\right) = \frac{3}{4}, \quad q(1) = 1.$$

In this example, $q(1)$ gives the price of receiving the good with certainty, which can be generated by adding together the probabilities of the two individual slots. Additivity therefore requires that we have

$$q(1) = q\left(\frac{1}{3}\right) + q\left(\frac{2}{3}\right),$$

but does not impose any particular relationship between $q\left(\frac{1}{3}\right)$ and $q\left(\frac{2}{3}\right)$. Our next result is that no-arbitrage prices must have this type of additivity.

Proposition 4 *If there exists a solution to the firm's problem (6), then lottery prices are additive.*

Proof: Suppose prices are not additive. Using Proposition 3, this would imply that there exists a set $A \subseteq M$ of slots such that $q_\ell\left(\sum_{m\in A}\theta_m\right) \neq \sum_{m\in A} q_\ell(\theta_m)$ for at least one good ℓ. Suppose that

$$q_\ell\left(\sum_{m\in A}\theta_m\right) > \sum_{m\in A} q_\ell(\theta_m) \tag{15}$$

holds. (The reverse case is symmetric.) Consider the following production plan. For each slot $m \in A$, the firm buys one unit of the lottery that delivers one unit of good ℓ with probability θ_m and nothing with probability $(1-\theta_m)$. The firm sells one unit of the lottery that delivers one unit of good ℓ with probability $\left(\sum_{m\in A}\theta_m\right)$ and nothing with probability $\left(1-\sum_{m\in A}\theta_m\right)$. This plan is feasible by construction – the firm takes in and gives out one unit of good ℓ if one of the slots in A is realized, and does nothing otherwise. Under (15), this plan yields a strictly positive profit. Since the firm could replicate this plan on an arbitrarily large scale, the problem (6) has no solution. ■

For the case where all slots on the wheel are equally likely, every probability in Γ is a multiple of $\frac{1}{M}$. Therefore, for any $\theta \in \Gamma$, additivity implies that we have

$$q(\theta) = q\left(\alpha\frac{1}{M}\right) = \alpha q\left(\frac{1}{M}\right)$$

for some integer α. In other words, in this case prices must be linear in probabilities. We state this as a corollary.

Corollary 1 *Suppose the randomizing device has M equally-likely slots. Then if the firm's problem (6)* *has a solution, lottery prices must be linear in probabilities, i.e., the price of lottery δ must be given by*

$$q(\delta) = q(1) \cdot \sum_{k=1}^{K} \delta(c_k)\, c_k.$$

The most important implication of the results in this section is that *every* no-arbitrage lottery price function has a corresponding sunspot price function. While in principle there are many lottery prices that cannot be represented in the sunspots model, none of them can ever support an equilibrium allocation in the lottery model. To see why this is true, recall that a lottery price function has a corresponding sunspot price function if there exist functions p and q such that (13) and (14) hold. The q function satisfying (14) is given by Proposition 3, which tells us that no-arbitrage lottery prices must be linear in commodities. The function p is constructed in the following way. For every state s in the sunspots model, set $p(s) = q(\theta)$, where $\theta = \pi(s)$. This completely defines the sunspot price function p. What remains is to show that p does in fact generate q through (13), that is, that we have

$$q(\theta) = \sum_{s \in A} p(s)$$

for every $A \subseteq S$ with $\sum_{s \in A} \pi(s) = \theta$, for all $\theta \in \Gamma$. This follows from the fact that q is additive with respect to the given randomizing device. Each $p(s)$ has already been set equal to $q(\theta_m)$ for some slot m. Therefore we have

$$\sum_{s \in A} p(s) = \sum_{m \in A} q(\theta_m),$$

which is in turn equal to $q\left(\sum_{m \in A} \theta_m\right)$ by additivity. We state this important result as our second corollary.

Corollary 2 *Every no-arbitrage lottery price function ϕ has a corresponding sunspot price function p.*

5 Comparing Equilibrium Allocations

We are now in a position to address our main question: Under what conditions do the two models generate the same sets of equilibrium allocations? We begin by showing that this occurs whenever the prices supporting the allocation as an equilibrium in one model can be translated into the other model.

5.1 Conditions for Equivalence

Corollary 2 states that any no-arbitrage lottery price function has a corresponding sunspot price function. We begin by restricting sunspot prices to be such that they have corresponding lottery prices. We show that, under this restriction, the two models generate exactly the same equilibrium allocations. Any difference in the sets of equilibrium allocations must therefore result from the more general form of the sunspot pricing function and not from the way commodities are defined or traded. Using Proposition 3, we can rewrite the consumer's lottery-choice problem (3) as

$$\max_{\delta_h, c_{0h}} \sum_C \delta_h(c_k)\, u_h(c_k) + v_h(c_{0h}), \tag{16}$$

$$\text{subject to } \sum_C q(\delta(c_k)) \cdot c_k + c_{0h} \le q(1) \cdot e_h + e_{0h},$$

$$\delta_h \in \Gamma(C), \quad c_{0h} \in \mathbb{R}_+.$$

We now present and prove the (restricted) equivalence result. Let (x^*, x_0^*) be a sunspot allocation and let (δ^*, c_0^*, y^*) be the corresponding lottery allocation generated by equations (7), (8), and (9). Let Q denote the set of all functions $q : \Gamma \to \mathbb{R}_+^L$, and recall that $\widehat{P}$ is the set of sunspot price functions satisfying (12). We then have the following.

Proposition 5 *There exists a $p^* \in \widehat{P}$ such that (p^*, x^*, x_0^*) is a sunspot equilibrium if and only if there exists a $q^* \in Q$ such that $(q^*, \delta^*, c_0^*, y^*)$ is a lottery equilibrium.*

Proof: (a) : First, suppose that (p^*, x^*, x_0^*) is a sunspot equilibrium with $p^* \in \widehat{P}$. Then we know that at prices p^*, (x_h^*, x_{0h}^*) solves problem (1) for every h. By applying a change of variables to replace the summation across states with a summation across the consumption set, we can rewrite this problem as

$$\max_{x_h} \sum_C u_h(c)\pi \circ x_h^{-1}(c) + v_h(x_{0h}),$$

$$\text{subject to } \sum_C \left(\sum_{s \in x_h^{-1}(c)} p^*(s) \cdot c \right) + x_{0h} \le \sum_S p^*(s) \cdot e_h + e_{0h},$$

$$x_h \in X, \; x_{0h} \in \mathbb{R}_+.$$

Here, $x_h^{-1}(c)$ is the set of states in which the consumer buys the bundle c. Since $p^* \in \widehat{P}$ holds, we can use (13) together with the definitions $\delta_h^* = \pi \circ x_h^{*-1}$ and $c_{0h} = x_{0h}$ to write the problem as

$$\max_{\delta_h} \sum_C \delta_h(c) u_h(c) + v_h(c_{0h}),$$

$$\text{subject to } \sum_C (q^*(\delta_h(c)) \cdot c) \le q^*(1) \cdot e_h,$$

$$\delta_h \in \Gamma(C), \; c_{0h} \in \mathbb{R}_+,$$

which is exactly the consumer's lottery problem as given in (16). Hence there exists a $q^* \in Q$ at which (δ_h^*, c_{0h}^*) solves problem (16) for all h, and condition (i') in the definition of lottery equilibrium is satisfied.

Condition (ii') requires that y^* be an optimal choice for the firm at prices q^*. It is easy to verify that y^* yields zero profits at these prices. Because the prices are of the no-arbitrage form (see Section 4.2), positive profits are not possible and therefore the firm is indeed optimizing. Condition (iii') follows directly from Proposition 1. Therefore we have a $q^* \in Q$ such that $(q^*, \delta^*, c_{0h}^*, y^*)$ is a lottery equilibrium.

(b) : Now suppose that $(q^*, \delta^*, c_{0h}^*, y^*)$ is a lottery equilibrium with $q^* \in Q$. Because equilibrium prices must be of the no-arbitrage form, reversing the argument above shows that at the unique price function p^* corresponding to q^* through (13), (x_h^*, x_{0h}^*) is optimal for each consumer h. Therefore condition (i) in the definition of a sunspot equilibrium is satisfied. Note that this p^* is in the set $\widehat{P}$ by definition. Condition (ii) in this definition follows directly from Proposition 1. Therefore we have a $p^* \in \widehat{P}$ such that (p^*, x^*, x_0^*) is a sunspot equilibrium. ■

This result establishes that when the price systems in the two models are comparable, the equilibrium allocations are identical and the differences in the trading stories do not matter. Combined with Corollary 2, it implies that every lottery equilibrium allocation has a corresponding sunspot equilibrium allocation. This provides a partial answer to our main question.

Corollary 3 *Every lottery equilibrium allocation has a corresponding sunspot equilibrium allocation.*

The remaining question is under what conditions sunspot equilibrium prices necessarily have corresponding lottery prices, so that full equivalence between the sets of equilibrium allocations obtains. GKQS (in press) and Kehoe, Levine and Prescott (in press) show that this is always the case when the sunspot variable is continuous and lottery choice is unconstrained. When the randomization possibilities are finite, it is not difficult to construct examples where sunspot equilibrium prices have this form (both Shell and Wright (1993) and GKQS, in press, contain such examples). We now present two results that give sufficient conditions for this to be the case. First, Proposition 2 implies that when all slots on the wheel are equally likely, equilibrium prices in the sunspots model are linear in probabilities (Corollary 1 states that the same is true for the lottery model). Hence in this case the sets of equilibrium allocations coincide.

Corollary 4 *When the randomizing device is such that all slots on the wheel are equally likely, the set of sunspot equilibrium allocations is equivalent to the set of lottery equilibrium allocations.*

Next, we point out that a sunspot price function with no corresponding lottery prices can exist only if the randomizing device is such that there exist two disjoint subsets of slots, both of which have the same total probability. If this is not the case, then all sunspot prices automatically satisfy (12) and therefore have corresponding lottery prices. As our final proposition, we show that the set of discrete randomizing devices with this property is small – it has Lebesgue measure zero. This implies that the equivalence of sunspot and lottery equilibrium allocations obtains for almost all discrete randomizing devices.

Proposition 6 *Given any fundamental economy and any number M of slots, the set of randomizing devices for which the equivalence of the sets of sunspot and lottery equilibrium allocations fails has Lebesgue measure zero.*

Proof: The set of all M-slot randomizing devices can be represented as the $(M-1)$-dimensional unit simplex, which we denote by Σ_M. The possibility that there exists a sunspot price function with no lottery counterpart can only arise if there exist nonempty, disjoint sets $A, B \subset M$ with

$$\sum_{m \in A} \pi(m) = \sum_{m \in B} \pi(m). \tag{17}$$

If there are no such sets A and B, then $\widehat{P} = P$ holds and Proposition 5 establishes full equivalence. Let α_n denote the number of ways of choosing exactly n of the M slots. Let β_n be the number of ways of dividing these n elements into two nonempty, disjoint groups. For any finite M and n, both α_n and β_n are finite numbers. There are then

$$\sum_{n=2}^{M} \alpha_n \beta_n$$

ways of creating nonempty, disjoint sets A and B that might satisfy (17); this number is also finite. Each possible combination defines a linear $(M-2)$-dimensional subset of Σ_M, and therefore has Lebesgue measure zero in Σ_M. Hence the set of randomizing devices satisfying (17) is a finite union of sets of Lebesgue measure zero, and therefore itself has Lebesgue measure zero in Σ_M. ■

This is *not* to say that any randomizing device with subsets of slots satisfying (17) will lead to nonequivalence; it is only a necessary condition. For example, for nonequivalence to obtain we need at least one of the subsets to have at least two elements, so that Proposition 2 does not apply. In addition, when $M = 4$ with $\pi(1) = \pi(2) = 1/10$ and $\pi(3) = \pi(4) = 2/5$, there are several ways of creating sets A and B such that (17) holds, but, as we mentioned in Section 4.1 above, applying Proposition 2 (twice) shows that equivalence necessarily obtains for this device. Furthermore, a randomizing device where all slots are equally-likely satisfies (17), but Corollary 4 shows that equivalence always obtains with such device. What Proposition 6 shows is that the set of devices for which nonequivalence obtains is a (proper) subset of a set of measure zero, and therefore has measure zero itself.

This result has an obvious probabilistic interpretation. Suppose the randomizing device were chosen at random in the following way. First M is drawn from any distribution over the set of integers greater than one. Then a randomizing device is drawn from the set of M-slot devices using any distribution that has a density with respect to Lebesgue measure. With probability one, the chosen randomizing device will then be such that equivalence between the sets of sunspot and lottery equilibrium allocations obtains. However, we offer no theory of how the randomizing device is selected; it is certainly not clear that it should be viewed as being randomly selected in this way. A government regulating risk classes in insurance, for example, might impose a device with "critical" properties in order to achieve a desired allocation. Hence we cannot dismiss sets of Lebesgue measure zero as being unimportant. We next provide an example of a sunspot equilibrium allocation with no lottery equilibrium counterpart.

5.2 An Example of Nonequivalence

There is a single indivisible commodity, and all consumers have the consumption set $\mathcal{C} = C_h = \{0, 1, 2\}$.[11] There are 4 states of nature, with probabilities given by

$$\pi(1) = \frac{1}{3},\ \pi(2) = \frac{1}{6},\ \pi(3) = \frac{1}{5},\ \pi(4) = \frac{3}{10}.$$

Notice that $\pi(1) + \pi(2) = \pi(3) + \pi(4) = 1/2$, so that there are two different ways of adding states together to get a total probability of one-half. The first three consumers have utility functions that are linear in consumption

$$u_1(c) = u_2(c) = u_3(c) = c,$$

while the fourth consumer has quadratic utility.

$$u_4(c) = c^2.$$

The total endowment in each state is 3 units of the good, which is divided into the following private endowments

$$(e_1, e_2, e_3, e_4) = (1.15, 1.11, 0.4, 0.3).$$

[11] Note that there is no divisible good in this example. However, in the equilibrium we construct all consumers exhaust their income. Because of this, it is straightforward to add a divisible good and keep the allocation (and price) of the indivisible good the same. We present the example without the divisible good to simplify the presentation, and then we discuss below how to add the divisible good.

We normalize prices so that $\sum_s p(s) = 1$ holds. Then the following is a sunspot equilibrium:

$$\begin{aligned}
p^* &= (0.32, 0.17, 0.20, 0.31) \\
x_1^* &= (2, 0, 1, 1) \\
x_2^* &= (1, 1, 0, 2) \\
x_3^* &= (0, 0, 2, 0) \\
x_4^* &= (0, 2, 0, 0).
\end{aligned}$$

This can be verified by computing the cost and the utility level associated with each of the $3^4 = 81$ possible consumption bundles to see that, at prices p^*, x_h^* is the unique optimal choice for each consumer.

In addition, p^* is the only price vector that supports this allocation as an equilibrium. First, note that the fact that no resources are wasted in equilibrium implies that every consumer must exhaust her income in equilibrium.[12] Therefore, at any prices that support x^* as an equilibrium, the four budget constraints must hold, as must our price normalization equation. The budget constraints are not linearly independent equations – any allocation that does not waste resources and satisfies the first three will also satisfy the fourth. Therefore we drop the fourth budget constraint and, writing p and x_h^* as column vectors, arrive at the system of equations

$$p^{\mathrm{T}} \left[\begin{array}{cccc} x_1^* & x_2^* & x_3^* & 1 \end{array} \right] = \left[\begin{array}{c} 1.15 \\ 1.11 \\ 0.34 \\ 1 \end{array} \right].$$

There are no prices on the right-hand side of the budget constraints because of our choice of price normalization. The equilibrium allocation for this example was chosen to that the matrix of allocations (with the column of 1's appended) is full rank. As a result, there is a unique solution to this equation for the prices, which is the equilibrium prices p^*. Hence no other price vector can support x^* as an equilibrium allocation.

Notice that consumer 1 receives one unit of the good in states 3 and 4, and pays $p(3) + p(4) = 0.51$ for this. At the same time, consumer 2 receives one unit of the good in states 1 and 2, and pays $p(1) + p(2) = 0.49$ for this. In other words, two consumers are buying the same consumption bundle with the same probability and paying *different* prices for it. This is something that simply cannot happen in the lottery model. Hence the unique prices supporting this sunspot equilibrium allocation have no corresponding lottery prices, and therefore the corresponding lottery allocation is not part of a lottery equilibrium. Why is

[12] If some consumer were not spending all of her income, the value of total demand would be less than the value of total supply. This would imply that aggregate consumption is less than the aggregate endowment in some state, which is clearly not the case here.

consumer 1 willing to buy one unit of the good in the more expensive states (3 and 4) rather than switching to the cheaper states (1 and 2)? The reason is that she is able to purchase two units in state 1, which has probability one-third. The proposed switch would require that she move her consumption of two units to state 3, which only has probability one-fifth; this would make her worse off. Hence consumption in states 1 and 2 is not a perfect substitute for consumption in states 3 and 4 because it changes the available options in the remaining one-half of probability.

There is a simple trick for adding a divisible good to this example to verify that the result does not depend on the absence of local nonsatiation. Endow all consumers with zero units of the divisible good. Set the function v_h so that the marginal utility of consumption of the divisible good at zero is less than the minimum of the utility gained from purchasing the last unit of the indivisible good in each state. Then, since v_h is strictly concave, each consumer will demand zero units of the divisible good and the same consumption plan for the indivisible goods, and we have an equilibrium with the prices and allocation given above.

6 Concluding Remarks

In this paper, we have extended the analysis of the relationship between sunspot equilibrium and lottery equilibrium allocations to a class of purely finite models. Previous work based on an exchange economy with non-convexities has shown that when the randomizing device is continuous, the two sets of equilibrium allocations are equivalent. Here we have focused on the case where the randomizing device is discrete and the number of possible lotteries is finite. We have generalized the lottery model so that it can be applied to such economies. Our main finding is that equivalence between the two sets of equilibrium allocations often, but not always, obtains.

The key difference between the two models, and the source of potential nonequivalence, is in their respective price systems. We show that equivalence will hold unless prices in the sunspots model are such that buying a particular consumption bundle with a particular probability has two (or more) different costs assigned to it, depending on which states of nature the bundle is purchased in. For all randomizing devices except a set of measure zero, this cannot happen simply because each probability is generated by a unique combination of states. A different argument shows that equivalence also obtains for the "leading" case where all events are equally likely. In this case, it is the ability of consumers or firms to substitute, or "shift," their purchases from high-cost to low-cost states that is critical for the result. Hence the equivalence result proven in GKQS (in press) and Kehoe, Levine, and Prescott (in press) does not depend on having a continuous sunspot variable (or, unconstrained randomization possibilities). It naturally extends to large class of models with finite randomization possibilities.

We also present an example of nonequivalence: a sunspot equilibrium allocation with no lottery equilibrium counterpart. In the example, the slots on the wheel each have a different probability. Hence at least some of the states have "monopoly power," in the sense that consumers cannot buy the same probability using a combination of the other states. In such cases, the extra generality available in the price system in the sunspots model is important. This result is similar in spirit to that of Garratt (1995), who showed that all lottery equilibrium allocations have corresponding sunspot equilibrium allocations, but that the converse is not true. It is important to bear in mind, however, that the results here are fundamentally different because we are using a generalized lottery model that constrains the randomization opportunities available to agents. The sunspot equilibrium allocation that Garratt (1995) shows to have no lottery equilibrium counterpart *does* have a counterpart in our generalized lottery model. Our example of nonequivalence based on the additional flexibility of prices in the sunspots model is an entirely new and different phenomenon. We have redefined the lottery model to bring it as close as possible to the sunspots model. Nonetheless, there are still some sunspot equilibrium allocations that are not lottery equilibrium allocations.

As a final note, we reiterate that the environment we have studied in this analysis, as well as those studied by GKQS (in press) and Kehoe, Levine, and Prescott (in press), is special. Markets are perfect, the number of commodities in the underlying certainty economy is finite, there is no role for money, etc. Sunspot equilibrium has been applied in a much wider range of settings. It is unclear whether and how the lottery model can be extended to many of these environments. A recent step in this direction was made by Berentsen, Molico, and Wright (in press), who introduced lotteries into search-theoretic models of money. Whenever the lottery model is extended to a new environment, it is natural to ask whether or not equivalence of the two sets of equilibrium allocations obtains. We expect that the basic approach we have taken here, if not our specific results, will be useful in addressing the equivalence question whenever it arises.

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