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**Equilibrium and Uncertainty**

by

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**I. Equilibrium**

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**0. Introduction**

In assessing the impressive development of the Walrasian equilibrium theory from the 1950s, a number of themes were stressed by the leading researchers. To begin with, let us recall Arrow and Hahn (1971):

*"There are two basic, incompletely separable aspects of the notion of general equilibrium...: the simple notion of determinateness, that the relations describing the economic system must be sufficiently complete to determine the values of its variables, and the more specific notion that each relation represents a balance of forces. The last, usually, though not always, is taken to mean that a violation of any relation sets in motion forces tending to restore it".*

We are thus led to two questions of interest: namely, the *existence* of a Walrasian equilibrium, and its *stability*. Both of these questions were recognized in many of the earlier landmarks in economic theory. For example, in his *Value and Capital*, Hicks carefully pointed out that if the price system in the model of general equilibrium of exchange is such as to achieve the equality of demand and supply in each market, we have a "position of equilibrium. If not, some prices will be bid up or down" and went on to assert that "the determinateness...was ensured by equality between the number of equations and the number of unknowns" as shown by Walras [see Hicks (1939, p. 89) and the mathematical appendix on p. 314]. Samuelson (1947) provided the mathematical formulation of the Walrasian tatonnement process and its stability, and a definitive treatment of the topic came from Arrow, Hurwicz and Block (1958, 1959). A few remarks on the "stability" question will be made later. I shall begin with the question of "determinateness" and see the link between this and the celebrated "fixed point" theorems of mathematics.  Two other questions were also studied: *uniqueness* of equilibrium and comparative statics.  Efforts to identify conditions under which there is a unique Walrasian equilibrium eventually led to the recognition that the property holds only

*"under strong assumptions and...economies with multiple equilibria must be allowed for.  Such economies will seem to provide a satisfactory explanation of equilibrium as well as a satisfactory foundation for the study of stability provided that all the equilibria of the economy are locally unique.  But if the set of equilibria is compact (a common situation) local uniqueness is equivalent to finiteness.  One is thus led to investigate conditions under which an economy has a finite set of equilibria". [Debreu (1970)]*

One of the most influential contributions of Debreu was to point out that such finiteness is "typical" in a precise sense (it holds generically over a class of models).

Some of the difficulties in deriving comparative statics results are discussed in Arrow and Hahn (1971), and, are also suggested by a line of research initiated by Sonnenschein, and a remarkable theorem of Debreu (1972).  But the techniques introduced in Debreu (1970, 1972) are quite different from those we use in these lectures.

In his assessment of the development of general equilibrium analysis, Koopmans (1957, p. 60) noted:

*"Our authors have abandoned demand and supply functions as tools of analysis, even as applied to individuals.  The emphasis is entirely on the existence of some set of compatible optimizing choices.  This question can be answered without making assumptions that cause unique choices to be associated with any prevailing prices, a precondition for the definition of single-valued demand and supply functions.  The problem is no longer conceived as that of proving that a certain set of equations has a solution.  It has been reformulated as one of proving that a number*

*of maximizations of individual goals under interdependent constraints can be simultaneously carried out."*

Keeping these observations in mind, I shall introduce the celebrated Debreu-Gale-Nikaido lemma on set-valued mappings, which is proved by using the Kakutani fixed point theorem. Next, we look at the remarkable result of Uzawa (1962) which shows that this lemma implies the Brouwer fixed point theorem. A model of an exchange economy [similar to the one developed in Nikaido (1956)] is elaborated next, and the existence of a Walrasian equilibrium is proved by appealing to the Debreu-Gale Nikaido lemma.

The discussion in Section 1 and 2 emphasizes the close link between the problems of existence of a competitive equilibrium and a fixed point theorem. In Section 3 I sketch another equilibrium concept that has assumed a privileged position outside the Walrasian paradigm. The Cournot-Nash equilibrium is a foundation stone of game theory, and an elegant theorem on the existence of such an equilibrium is proved by using Kakutani's fixed point theorem.

Yet another theme in this literature was the role of prices in coordinating individual decisions based on self-interest and arriving at a socially optimal allocation of resources. While Adam Smith's invisible hand is usually regarded as a starting point, the literature on optimality of competitive equilibrium was also enriched by the remarkable contributions of Arrow and Debreu, and later provided useful insights into the problems of mechanism design and incentive compatibility studied rigorously by Leonid Hurwicz. In Section 4, I shall sketch the fundamental theorems of new welfare economics. In this context, let us note the following remarks of Koopmans:

*"The new tools allow us to shed new light on older and perhaps more fundamental problems.*

*The emphasis is shifted to the specification of conditions under which decentralization of economic decisions through a price system is compatible with efficient utilization of resources"*.

The "new" tools that Koopmans alluded to included the famous separation theorems for convex sets. The problem of characterizing Pareto optimality and linking it to competitive equilibrium was attacked by Samuelson, Lange and others by using the techniques of calculus. It is still useful to keep in mind the "first order conditions" characterizing Pareto optimality. But the set theoretic arguments are often more general.

**0.1 Notation**

In what follows, for any two vectors $a = (a_k)$, $b = (b_k)$ in $\mathbb{R}^\ell$, we write

$$a.b \equiv \sum_{k=1}^{\ell} a_k b_k \tag{0.1}$$

For any vector $a = (a_k)$, write

$$|a| = \sum_{k=1}^{\ell} |a_k| \tag{0.2}$$

and

$$||a|| = \left[\sum_{k=1}^{\ell} a_k^2\right]^{1/2} \tag{0.3}$$

Also, a vector $a = (a_k)$ is *non-negative* (written $a \geqq 0$) if $a_k \geqq 0$ for all k; a is *semipositive* (written $a \geq 0$) if $a_k \geq 0$ for all k, and $a_k > 0$ for some k; a is *strictly positive* (written $a >> 0$) if $a_k > 0$ for all k. Write $\Omega = \{x \in R^{\ell} \quad x \geqq 0\}$.

## 1. A Model with Excess Demand Functions

As a first step, let us recall the celebrated:

**Theorem 1.1. Brouwer's Fixed Point Theorem.**

Let X be a non-empty compact convex set in $R^n$, and $f : X \rightarrow X$ be a continuous mapping. Then f has a fixed point $\hat{x}$ (satisfying $\hat{x} = f(\hat{x})$).

There are $\ell$-commodities in the economy. The set of admissible price vectors is given by

$$P \equiv \{p \in R^{\ell} : p \geq 0, \sum_{k=1}^{\ell} p_k = 1\} \tag{1.1}$$

A continuous function $z : P \rightarrow R^{\ell}$ is an *excess demand function* if

$$p.z(p) = 0 \tag{WL}$$

The relation (WL) is the well-known Walras Law which is often *derived* from other assumptions when one introduces economic agents explicitly (see, the discussion below in the context of exchange economy, and Debreu (1959) for a more general model with consumers and producers).

An element $p^* \in P$ is an equilibrium price vector if

$$z(p^*) \leqq 0 \tag{1.2}$$

Take a continuous function M(p) on P with $\mathbb{R}^\ell$ such that

$$M_k(p) > 0 \quad \text{if and only if } z_k(p) > 0 \tag{1.3}$$

$$M_k(p) = 0 \quad \text{if } z_k(p) = 0 \tag{1.4}$$

$$p_k + M_k(p) \geqq 0 \quad \text{for all } k. \tag{1.5}$$

(example: $M_k(p) = \max(-p_k, z_k(p))$)

Now define a mapping T from P into P as follows:

$$T(p) = [p + M(p)]/|p + M(p)| \tag{1.6}$$

where [see (0.2)] $|p + M(p)| \equiv \sum_{k=1}^{\ell} [p_k + M_k(p)]$.

First, verify that T is well-defined by noting that for any $p \in P$, $|p + M(p)| > 0$. If $|\overline{p} + M(\overline{p})| \equiv \sum_{k=1}^{\ell} [\overline{p}_k + M_k(\overline{p})] = 0$ for some $\overline{p}$ in P, then by (1.5), $\overline{p}_k + M_k(\overline{p}) = 0$ for all k = 1,2,...,$\ell$. But $\overline{p} \in P$ implies that $\overline{p}_{k'} > 0$ for some k'. For each k' with $\overline{p}_{k'} > 0$, $\overline{p}_{k'} = -M_{k'}(\overline{p})$, so that $M_{k'}(\overline{p}) < 0$ for all such k'. This means that $z_{k'}(\overline{p}) < 0$ for all such k'. Hence, $\sum_{k=1}^{\ell} \overline{p}_k z_k(\overline{p}) < 0$. This violates (WL).

Clearly the mapping T defined in (1.6) is continuous. By Brouwer's fixed point theorem there is some $p^*$ in P such that $T(p^*) = p^*$, or,

$$p^* = \frac{p^* + M(p^*)}{|p^* + M(p^*)|} \tag{1.7}$$

or,

$$p^*\lambda = M(p^*) \tag{1.8}$$

where $\lambda = \sum_{k=1}^{\ell} [p_k^* + M_k(p^*)] - 1$.

Now, from (WL) and (1.8)

$$0 = z(p^*) \cdot p^*\lambda = z(p^*)M(p^*) \tag{1.9}$$

But $z_k(p^*)M_k(p^*) \geq 0$ for all k [use (1.3) - (1.5)]. Hence $0 = \sum_{k=1}^{\ell} z_k(p^*)M_k(p^*)$ implies

$$z_k(p^*)M_k(p^*) = 0 \quad \text{for all } k=1,\ldots,\ell.$$

This implies that

$$z_k(p^*) \leq 0 \quad \text{for all } k=1,\ldots,\ell \tag{1.10}$$

To summarize our discussion, let us state formally

**Theorem 1.2. Existence of an Equilibrium**

Assume that the excess demand function z is continuous on P and satisfies (WL). Then there is some $p^*$ such that $z(p^*) \leqq 0$.

Continuity of z on the compact set P implies that there is some constant M' such that $|z(p)| \leq M'$ for all $p \in P$. In particular, even when the price of some commodity k equals zero, the excess demand for each commodity remains bounded. This assumption is problematic if more of some commodity is always preferred to less, and the consumers attempt to maximize utility. A more satisfactory approach is to define z to be continuous at any p in P such that p >> 0, and to impose an appropriate boundary condition [see Arrow-Hahn (1971, Chapter 2) or Debreu (1970) for such results].

## 2. A Model of an Exchange Economy

In a number of contexts, the Debreu-Gale-Nikaido Lemma is the key step in proving the existence of a Walrasian equilibrium.

**Lemma 2.1. The Debreu-Gale-Nikaido Lemma**.

Let Z be a compact subset of $R^{\ell}$. If $\zeta$ is an upper semicontinuous correspondence from P into Z such that for every p in P, the set $\zeta(p)$ is (nonempty) convex and satisfies $p\zeta(P) \leqq 0$, then there is $p^*$ in P such that $\zeta(p^*) \cap (-\Omega)$ is nonempty.

**Proof**. It is easy to verify that P is (non-empty) compact and convex. Replace Z by a compact, convex subset Z' of $R^{\ell}$ which contains it. As P is nonempty, so clearly is Z, hence Z'.

Given z in Z', let $\mu(z)$ be the set of p in P which maximize p.z on P. Since P is non-empty, compact, $\mu(z)$ is nonempty, and the correspondence $\mu$ from Z' to P is upper semicontinuous on Z' (by the maximum theorem). Since P is convex, so is $\mu(z)$ for either (i) z = 0, and then $\mu(z) = P$ or (ii), $z \neq 0$ and then $\mu(z)$ is the intersection of two convex sets: P and the set $\{p \in R^{\ell} : p.z = \text{Max } P.z\}$.

Consider now the correspondence $\phi$ from P × Z' into itself defined by

$$\phi(p,z) = \mu(z) \times \zeta(p) \tag{2.1}$$

The set P × Z', a subset of $R^{2\ell}$, is non-empty, compact, convex since P and Z' are. The

correspondence $\phi$ is upper semicontinuous since $\mu$ and $\zeta$ are. Finally, for all (p,z) in P × Z', the set $\phi$(p,z) is (non-empty and) convex, since both $\mu(z)$ and $\zeta(p)$. Hence, all the conditions of Kakutani's fixed point theorem are satisfied, and $\phi$ has a fixed point $(p^*, z^*)$, i.e.,
$(p^*,z^*) \in \mu(z^*) \times \zeta(p^*)$, which is equivalent to

$$p^* \in \mu(z^*) \quad \text{and} \quad z^* \in \zeta(p^*) \tag{2.2}$$

The first relation in (2.2) implies that for every p in P, one has $p^* z^* \geqq pz^*$. The second implies that $p^* z^* \leqq 0$. Hence, for every p in P, one has

$$pz^* \leqq 0 \tag{2.3}$$

Taking the point p of P defined by $p_K = 1$, $p_{K'} = 0$ for $k' \neq k$, one obtains (from (2.3)) $z_k^* \leqq 0$. Hence, $z^* \in -\Omega$. This, with $z^* \in \zeta(p^*)$, proves that $p^*$ has the desired property.

Q.E.D.

We shall now prove the striking result of Uzawa (1962) that links the Debreu-Gale-Nikaido Lemma to the fixed point theorem of Brouwer.

**Theorem 2.1 Uzawa's Theorem**.

The Debreu-Gale-Nikaido Lemma implies Brouwer's fixed point theorem.

**Proof**. It suffices to prove that Debreu-Gale-Nikaido Lemma implies that any continuous function f from P into itself has a fixed point [see, e.g., Nikaido (1968, Chapter 1, Theorems 2.7 and 4.3)]. Write $f(p) = (f_k(p))$ for $p \in P$. Then,

$$f_k(p) \geq 0 \quad (k = 1,\ldots,\ell), \ \sum_{k=1}^{\ell} f_k(p) = 1 \tag{2.4}$$

Now, $||p||^2 > 0$ for all p in P. Noting this fact, define for p in P,

$$\lambda(p) = \frac{f(p) \bullet p}{||p||^2} \tag{2.5}$$

and define $\ell$ single-valued functions

$$\zeta_k(p) = f_k(p) - \lambda(p)p_k \quad (k=1,\ldots,\ell) \tag{2.6}$$

Writing $\zeta(p) = (\zeta_k(p))$ we obtain a continuous mapping from P into P. Since P is compact, there is a compact subset Z in $R^\ell$ that contains $\zeta(p)$ for all p in P. Since $\zeta(p)$ consists of a single point, it is surely true that (viewed as a correspondence) $\zeta(p)$ is convex. Continuity of the function $\zeta(p)$ means that - again viewed as a correspondence - $\zeta(p)$ is an upper semicontinuous correspondence. Now,

$$\begin{aligned} p \bullet \zeta(p) &= \sum_{k=1}^{\ell} p_k[f_k(p) - \lambda(p)p_k] \\ &= \sum_{k=1}^{\ell} p_k f_k(p) - \lambda(p)\left(\sum_{k=1}^{\ell} p_k^2\right) = 0 \end{aligned} \tag{2.7}$$

Hence, by using the Debreu-Gale-Nikaido Lemma we get the existence of some $p^*$ in P such that

$$\zeta(p^*) \leqq 0. \tag{2.8}$$

This means that

$$f_k(p^*) - \lambda(p^*)p_k^* \leqq 0 \quad \text{for all } k$$

or

$$f_k(p^*) \leqq \lambda(p^*)p_k^* \quad \text{for all } k \tag{2.9}$$

The last step in the proof is to show that $\lambda(p^*) = 1$ and that equality holds in (2.9). Note first that the validity of (2.7) implies that (2.9) must hold with equality for any k such that $p_k^* > 0$. On the other hand, (2.9) implies that $f_k(p^*) \leqq 0$, if $p_k^* = 0$. Using (2.4) we get $f_k(p^*) = 0$ if $p_k^* = 0$. Hence, equality holds in (2.9) for all $k = 1,...,\ell$, i.e.,

$$f_k(p^*) = \lambda(p^*)p_k^*, \quad k=1,...,\ell \tag{2.10}$$

Now, summing over k and using (2.4) we get $\lambda(p^*) = 1$. Hence,

$$p_k^* = f_k(*p^*) \qquad (k=1,...,\ell)$$

so that $p^*$ is a fixed point of the mapping f.

Q.E.D.

## 2.2. A Decentralized Exchange Economy

To get an example of the application of the Debreu-Gale-Nikaido lemma, let us quickly review the model of a decentralized exchange economy similar to that of Nikaido (1956). This pioneering work has been interpreted as a rigorous presentation of a substantial literature on the "neo-classical" theory of international trade. In his masterly *survey*, Chipman (1965) introduced this theme as follows:

*"What is generally considered to be the "neo-classical" theory of international values actually consists of at least two separate strands that have been gradually woven together. One is the Marshallian apparatus of the reciprocal demand curve (or "offer curve" as it is now usually called). The other strand consists of what appears to be a spontaneous development on the part of different writers writing (in many cases) independently of one another in the early 1930's.... The diagrammatic technique introduced by these writers was finally perfected by Meade, and the model was given mathematical rigor by Nikaido (1956, 1957)."*

Consider a model of an exchange economy with m agents (indexed by i) and $\ell$ commodities (indexed by k). I shall refer to the agents as consumers, but they can be interpreted as countries. Each agent i is characterized by its preferences $\succsim_i$ and its endowment vector $\omega_i$. We assume that:

(A.1) $\omega_i >> 0$ for all i

Note that this assumption is particularly problematic when the agent is identified as a country. It can be weakened, however, at the cost of considerable technical difficulties (see Nikaido (1957)).

(A.2) $\succsim_i$ defined on $\Omega$ is reflexive, transitive and complete.

(A.3) For any $y \in \Omega$, and for each i=1,2,...,m, the sets $\{x \in \Omega : x \succsim_i y\}$ and $\{x \in \Omega : y \succsim_i x\}$ are closed.

Given (A.2) and (A.3), for each agent i, there is a continuous *utility function* $u_i : \Omega \to R$ representing the preferences $\succsim_i$, i.e., there is a continuous function $u_i : \Omega \to R$ such that

$$u_i(x) \geqq u_i(y) \qquad \text{if and only if } x \succsim_i y$$

I shall now introduce convexity properties of preferences. In the following statements, $x^2$ and $x^1$ are different points of $\Omega$, and $\lambda$ is a real number in (0,1).

(A.4) *weak convexity*: if $x^2 \succsim_i x^1$, then $\lambda x^2 + (1-\lambda)x^1 \succsim_i x^1$.

This property (P.1) is equivalent to:

(A.4') For every $x'$ in $\Omega$, the set $\{x \in \Omega : x \succsim_i x'\}$ is convex.

(A.4") For every $x'$ in $\Omega$, the set $\{x \in \Omega : x \succ_i x'\}$ is convex.

It is useful to take a minute and look at the implications. First, we prove that (A.4') implies (A.4). Let $x^2 \succsim_i x^1$, then $x^2 \in \{x \in \Omega : x \succsim_i x^1\}$. Also, $x^1 \succsim_i x^1$; hence, $x^1 \in \{x \in \Omega : x^1 \succsim_i x^1\}$. By (A.4'), for any $\lambda \in (0,1)$, $\lambda x^2 + (1-\lambda)x^1 \succsim_i x^1$. Next, (A.4) implies (A.4"). Let $x^2$ and $x^1 \in \{x \in \Omega : x \succ_i x'\}$. Suppose $x^2 \succsim_i x^1$. Then by (A.4), for $\lambda \in [0,1]$, $\lambda x^2 + (1-\lambda)x^1 \succsim_i x^1 \succ_i x'$. This establishes the convexity property (A.4"). A similar argument applies if $x^1 \succsim_i x^2$. Finally, we show that (A.4") implies (A.4'). Let $x^1, x^2 \in \{x \in \Omega : x \succsim_i x'\}$. If (A.4') is not valid, then there is some $\bar{\lambda} \in (0,1)$ such that $x' \succ_i \bar{\lambda} x^1 + (1-\bar{\lambda})x^2$. But, then, by transitivity, $x^1 \succ_i \bar{\lambda} x^1 + (1-\bar{\lambda})x^2$ and $x^2 \succ_i \bar{\lambda} x^1 + (1-\bar{\lambda})x^2$. By (A.4"), $\bar{\lambda} x^1 + (1-\bar{\lambda})x^2 \succ_i \bar{\lambda} x^1 + (1-\bar{\lambda})x^2$, a contradiction.

(A.5) *Convexity*: if $x^2 \succ_i x^1$, then $\lambda x^2 + (1-\lambda)x^1 \succ_i x^1$.

When $\succsim_i$ satisfies (A.2), i.e., the preferences are continuous, (A.5) implies (A.4). We note this formally. Suppose (A.2) - (A.3) hold, and let $x^2 \succsim_i x^1$. For convenience, write

$[x^2, x^1]$ to indicate the line joining $x^2$ to $x^1$ i.e. $[x^2, x^1] = \{x \in \Omega : x = \lambda x^2 + (1-\lambda)x^1, \lambda \in [0,1]\}$. We want to conclude that if (A.5) holds, the set

$$\{x \in [x^2, x^1] : x^1 \succ_i x\} \tag{2.11}$$

is empty. It cannot consist of a single point, since its *complement* in $[x^2, x^1]$ is the set $\{x \in [x^2, x^1] : x \succsim_i x^1\}$ which is closed. Therefore, if the set (2.11) is nonempty, it must contain two distinct points say $x', x''$.

$$\overline{x^1 \quad x' \quad x'' \quad x^2}$$

However, $x^1 \succ_i x''$ implies that [by (A.5)] $x' \succ_i x''$. Moreover, $x^2 \succsim_i x^1 \succ_i x'$ implies that $x'' \succ_i x'$. This leads to a contradiction.

(A.4) is consistent with "thick" indifference curves; (A.5) rules this out.

A *Walrasian equilibrium* consists of commodity bundles $\{x_1^*, \ldots x_m^*\}$ and a price system p in P such that

(α) $\sum_{i=1}^{m} x_i^* \leqq \sum_{i=1}^{m} \omega_i$.

(β) For each agent i, $x_i^*$ is a solution to the following optimization problem:

$$\begin{array}{ll} \text{"maximize} & u_i(c) \\ \text{subject to} & p * c \leqq p^* \omega_i \\ & c \in \Omega\text{"} \end{array}$$

The first condition (α) requires that for each commodity, excess demand is non-positive. The second condition (β) requires that for all i, $x_i^*$ maximizes the utility of agent i on its budget set

$\{c \in \Omega \;\; p^* \;\; c \leqq p^* \;\; \omega_i\}$ determined by the price system $p^*$.

The competitive system is regarded as a canonical model of a decentralized resource allocation mechanism, and the equilibrium price system coordinates individual decisions made in the pursuit of individual interest. In equilibrium, one can think of the following *verification scenario* [paraphrased from Hurwicz (1986)] as an interpretation of the model: the agents are presented (say, on a display board) with a proposed message (consisting of an allocation $(x_i)$ satisfying $\sum_i x_i \leqq \sum \omega_i$ and a price system $p^*$). The ith agent says "yes" if and only if $x_i$ is an equilibrium *for* him or her (i.e., $x_i$ maximizes $u_i$ on the budget set $\{c \in \Omega : pc \leqq p\omega_i\}$ determined by the proposed p). Note that this "yes" is based on calculations involving the characteristics $(\omega_i, u_i)$ of the i-th agent alone, and the price system p that is common knowledge). If all the agents say "yes" the message is an equilibrium message for the organization. If there is any "no", an alternative message must be proposed.

The difficulty of formally designing a system of proposing alternative messages if the initial message is not an equilibrium has turned out to be formidable, particularly when one demands 'decentralization' in some sense. Even when the initial proposal is an equilibrium, one is entitled to ask how the redistribution of the initial endowment pattern $(\omega_1,...,\omega_m)$ is achieved in a decentralized manner. Exploring this direction leads us to a better appreciation of the role of 'money' as a medium of exchange.

To prove the existence of equilibrium, we show that when (A.5) holds, under the condition $(\alpha)$, the condition $(\beta)$ can be replaced by

$(\beta')$ For each agent i, $x_i^*$ is a solution to the following optimization problem:

$$\text{"maximize} \quad u_i(c)$$
$$\text{subject to} \quad p^*c \leqq p\omega_i,$$
$$0 \leqq c \leqq \mathbf{C}, \quad \text{where } \mathbf{C} >> \sum_{i=1}^{m} \omega_i\text{"}.$$

Thus, in $(\beta')$ we introduce an (ad hoc) constraint in terms of an upper bound on the choice of c.

To show that [under ($\alpha$)], the conditions ($\beta$) and $(\beta')$ are equivalent, it is trivial to see that $x_i^*$ satisfying ($\beta$) necessarily satisfies $(\beta')$. To go in the other direction, write $\underset{\sim}{E} = \{c \in \Omega : 0 \leqq c \leqq C\}$. Suppose that $x_i^*$ satisfies $(\beta')$ but ($\beta$) does NOT hold (for some agent i). Then there is $c^* \in \Omega$, $p^*c^* \leqq p^*\omega_i$ and $u_i(c^*) > u_i(x_i^*)$. Surely, $c^* \notin \underset{\sim}{E}$. Now, $x_i^* \leqq \sum_{i=1}^{m} \omega_i << C$. For any $\lambda \in [0,1]$, we have:

$$p^*[\lambda x_i^* + (1-\lambda)c^*] \leqq \lambda p^*x_i^* + (1-\lambda)p^*c^* \leqq p^*\omega_i \tag{2.12}$$

and

$$\lambda x_i^* + (1-\lambda)c^* \in \Omega \tag{2.13}$$

By the convexity assumption (A.5), for all $\lambda \in [0,1)$

$$u_i(\lambda x_i^* + (1-\lambda)c^*) > u_i(x_i^*) \tag{2.14}$$

Hence, for $\lambda$ sufficiently close to 1, $\lambda x_i^* + (1 - \lambda)c^* << \omega$, and this contradicts $(\beta')$.

$\underset{\sim}{E}$ is a (nonempty) compact, convex set. Define, for each agent i, the correspondence $B_i(p)$ as follows:

$$B_i(p) = \{c \in \underset{\sim}{E} : pc \leqq p\omega_i\} \tag{2.15}$$

Clearly $B_i(p)$ is a (nonempty) compact, convex subset of $\underset{\sim}{E}$. We want to prove that

$$B_i : P \rightarrow \underset{\sim}{E} \text{ is a continuous correspondence} \tag{2.16}$$

Let $p^n \in P$ converge to some $p \in P$, and $c^n \in B_i(p^n)$ converge to c, then $c \in \underset{\sim}{E}$; and,

$$p^n c^n \leqq p^n \omega_i \tag{2.17}$$

implies, in the limit, that

$$pc \leqq p\omega_i \tag{2.18}$$

Hence, $c \in B_i(p)$. This establishes the upper-semicontinuity of the correspondence $B_i$. To establish the lower semicontinuity of $B_i$, let $p^n \in P$ converge to $p \in P$ and $c \in B_i(p)$. One must construct a sequence $c^n \in B_i(p^n)$ such that $c^n$ converges to c.

Two cases need to be considered.

Case I. Suppose that $pc < p\omega_i$. Then there is some $n_0$ such that for all $n \geq n_0$, $p^n c < p\omega_i$. Hence, $c \in B_i(p^n)$ for all $n \geq n_0$. Now, choose the sequence $c^n$ as follows: for all $n < n_0$, $c^n$ is an arbitrary element of $B_i(p^n)$; for all $n \geq n_0$, $c^n = c$. Clearly, $c^n$ converges to c, and for all n, $c^n \in B_i(p^n)$.

Case II. Suppose that $pc = p\omega_i$. Since $\omega_i >> 0$, $p\omega_i > 0$. Hence there is some $c' \in \Omega$, such that

$$pc' < p\omega_i$$

Hence, there is some $n_0$ such that for all $n \geq n_0$,

$$p^n c' < p^n \omega_i, \quad p^n c' < p^n c$$

Consider the point $a^n$ where the line joining $c'$ to c intersects $\{z : p^n z = p^n \omega_i\}$. For all $n \geq n_0$, $a^n$ exists, is unique and tends to c. The $c^n$ is chosen as follows: for $n < n_0$, $c^n$ is an arbitrary element of $B_i(p^n)$; for $n \geq n_0$, choose $c^n = a^n$. This establishes the lower semicontinuity of $B_i(p)$, and completes the proof of (2.16).

For any $p \in P$, consider now the following optimization problem for agent i:

$$\begin{array}{ll} \text{"maximize} & u_i(c) \\ \text{subject to} & c \in B_i(p)\text{"} \end{array} \tag{2.19}$$

Since $u_i$ is assumed to be continuous, Weierstrass' theorem ensures the existence of a solution to (2.19). Let $\phi_i(p)$ be the set of solutions to (2.19). By (A.5), $\phi_i(p)$ is convex. By the maximum theorem, $\phi_i(p)$ is an upper semicontiuous correspondence on P. Define

$$\zeta_i(p) = \phi_i(p) - \{\omega_i\}$$

Then $\zeta_i(p)$ is the excess demand correspondence of agent i which is upper semicontinuous on P and is convex-valued. Also, if $x \in \zeta_i(p)$, then $x = y - \omega_i$ where $y \in \phi_i(p)$. Hence $py \leqq p\omega_i$, so that $px \leqq 0$. Define

$$\zeta(p) = \sum_{i=1}^{m} \zeta_i(p).$$

Then, $\zeta(p)$ is an upper semicontinuous correspondence on P with values in the set $\{c \in \Omega : 0 \leqq c \leqq mC\}$; $\zeta(p)$ is convex for each p and if $z \in \zeta(p)$, $pz \leqq 0$. By using the Debreu-Gale-Nikaido lemma, there is some $p^* \in P$, and $z^* \leqq 0$ such that

$$z^* \in \zeta(p^*)$$

This implies that $z^* = \sum_{i=1}^{m} z_i^*$ where $z_i^* \in \zeta_i(p^*)$; or, there is $x_i^* \in \phi_i(p^*)$ such that

$$z_i^* = x_i^* - \omega_i$$

Hence, we get $\sum_{i=1}^{m} x_i^* \leqq \sum_{i=1}^{m} \omega_i$, and $x_i^*$ satisfies $(\beta')$, hence $(\beta)$. To summarize: under the assumptions (A.1), (A.2), (A.3) and (A.5), there exists an equilibrium in the exchange economy.

## 3. Cournot-Nash Equilibrium

While the Walrasian equilibrium focuses on the role of prices in coordinating self-seeking actions of a 'large' number of 'small' agents, the Cournot-Nash equilibrium concept captures the possibility of direct interaction among a 'small' number of agents. As in the case of the Debreu-Gale-Nikaido lemma, a proof of the existence of a Cournot-Nash equilibrium can be obtained by the use of Kakutani's fixed point theorem.

I shall sketch a general model due to Debreu (1952). Consider an abstract *social system* $(A_i\ \phi_i, u_i)_{i \in M}$ described as follows: there are m agents; write $M = \{1,2,...,i,...m\}$ and $M\text{-}i = \{1,2, i\text{-}1, i+1,...,m\}$. The i-th agent must choose an element $a_i$ in the $A_i$ of his a priori available *actions*. The sets $A_i$ are assumed to be nonempty, compact, convex sets of Euclidean spaces. When the agents other than the i-th choose actions $a_{m-1} \equiv (a,...,a_{i-1}, a_{i+1},...,a_m)$, the choice of the i-th agent is restricted to a nonempty subset of $A_i$, depending on $a_{M-1}$. Formally, we define a correspondence $\phi_i$ from $A = \mathop{X}\limits_{i=1}^{m} A_i$ to $A_i$ that associataes with the generic element a = $(a_1,...,a_m)$ of A, the nonempty subset $\phi_i(a)$ of $A_i$ to which the choice of agent i must be restricted. The set $\phi_i(a)$ is actually independent of the i-th component of a, but it is more convenient (from the technical point of view) to define it formally on A, rather than on $\mathop{X}\limits_{j\neq i} A_j$. The correspondence $\phi_i$ is assumed to be continuous and convex-valued.

The *utility function* (or the return function) $u_i : A \rightarrow R$ specifying the utility to agent i resulting from $a = (a_1,...,a_i,...,a_m)$ (the m-tuple of actions) is assumed to be *continuous* and *quasi-concave* in $a_i$.

Let $a_{M-i}$ be the (m-1)-tuple of actions of the set M-i of agents (excepting agent i); the i-th agent chooses $a_i$ so as to maximize $u_i(\bullet,a_{M-i})$ on $\phi_i(a)$ (again, it should be stressed that $\phi_i(a)$ depends only on $a_{M-i}$). Thus, the i-th agent chooses an element of the set

$$\mu_i(a) = \{x \in \phi_i(a) : u_i(x,a_{M-i}) = \max_{b \varepsilon \phi_i(a)} u_i(b,a_{M-i})\}. \tag{3.1}$$

By the Weierstrass' theorem, $\mu_i(a)$ is nonempty, and by quasi-concavity of $u_i$, $\mu_i(a)$ is convex-valued.

An element $a^*$ of A is an *equilibrium* if for every i $\varepsilon$ M, $a_i^*$ maximizes $u_i(\bullet, a_{M-i}^*)$ on $\phi_i(a^*)$, i.e., if for every i $\varepsilon$ M, $a_i^* \ \varepsilon \ \mu_i(a^*)$.

Thus, if the correspondence $\mu$ is defined from A into A by

$$\mu(a) = \mathop{X}\limits_{i \varepsilon M} \mu_i(a),$$

the element $a^*$ of A is an equilibrium if and only if $a^* \varepsilon \mu(a^*)$, i.e., if and only if $a^*$ is a fixed point of the correspondence $\mu$.

The basic existence theorem on Cournot-Nash equilibrium is now stated:

**Theorem 3.1. Cournot-Nash Equilibrium**

*If for each* i $A_i$ *is a nonempty, compact, convex subset of a Euclidean space,* $\mu_i$ *is a continuous, real-valued function on* $A = \underset{i \in M}{X} A_i$ *which is quasi-concave in the i-th variable and* $\phi_i$ *is a continuous, convex-valued correspondence from* A *to* $A_i$, *then the social system* $(A_i, u_i, \phi_i)_{i \in M}$ *has a* Cournot-Nash equilibrium.

**Proof:**

First, by using the maximum theorem, show that $\mu_i$ is upper-semi-continuous. Moreover, for every a in A, $\mu_i(a)$ is convex, since it is the intersection of two convex sets $\phi_i(a)$ and $\{x \varepsilon A_i : u_i(x, a_{M-i}) \geqq \max_{b \varepsilon \phi_i(a)} u(b, a_{M-i})\}$. The set A is nonempty, compact, convex. The correspondence $\mu$ is (nonempty) upper-semicontinuous and convex valued, from A into A. Hence, by Katutani's theorem, $\mu$ has a fixed point. Q.E.D.

## 4. Pareto Optimality

Let us go back to an economy without production and without any specification of ownerships of endowments: an economy E now consists of m individuals, each characterized by a preference preordering $\succsim$, and a total resource vector $\omega \in R^{\ell}_{++}$. An *allocation* [or, a redistribution] $\underset{\sim}{\mathbf{x}} = (x_i)$ of $\omega$ consists of m-nonnegative $\ell$-vectors

$(x_1,\ldots,x_m)$ such that

$$\sum_{i=1}^{m} x_i = \omega \tag{4.1}$$

An allocation $\underset{\sim}{x} = (x_i)$ is said to *dominate* another allocation $\underset{\sim}{x}' = (x_i')$ *strongly* if

$$x_i \succ_i x_i' \quad \text{for all } i = 1,\ldots,m \tag{4.2}$$

We shall say that $\underset{\sim}{x} = (x_i)$ *dominates* $\underset{\sim}{x}' = (x_i')$ *weakly* if

$$x_i \succsim_i x_i' \quad \text{for all } i \tag{4.3}$$

and

$$x_i \succ_i x_i' \quad \text{for some } i$$

It is clear that if $\underset{\sim}{x}$ dominates $\underset{\sim}{x}'$ strongly, $\underset{\sim}{x}$ also dominates $\underset{\sim}{x}'$ weakly. It is an exercise to identify monotonicity and continuity properties of the preferences $(\succsim_i)$ such that weak domination implies strong domination.

An allocation $\underset{\sim}{x} = (x_i)$ is said to be *Pareto optimal* if there is no other allocation $\underset{\sim}{x}' = (x_i')$ such that $\underset{\sim}{x}'$ dominates $\underset{\sim}{x}$ weakly. An allocation $\underset{\sim}{x}$ is said to be *weakly Pareto optimal* if there is no other allocation $\underset{\sim}{x}' = (x_i')$ such that $\underset{\sim}{x}'$ dominates $\underset{\sim}{x}$ strongly.

We now introduce the concept of a *valuation equilibrium* relative to a price system $p^* > 0$ in this economy. An allocation $\underset{\sim}{x}^* = (x_i^*)$ is a valuation equilibrium relative to a price system

$p^* > 0$ if for each i,

$$"p^* x \leqq p^* x_i^*" \text{ implies } "x_i^* \succsim_i x" \tag{4.4}$$

In other words, an allocation $\underset{\sim}{x}^* = (x_i*)$ [satisfying (4.1)] is a valuation equilibrium $p^*$ if any consumption vector x that the agent i prefers to $x_i^*$ costs more at the price system $p^*$. We can now state and prove:

**Theorem 4.1**.

If $\underset{\sim}{x} = (x_i^*)$ is a valuation equilibrium at the price system $p^* > 0$, it is weakly Pareto optimal.

**Proof**. Suppose that $\underset{\sim}{x}^*$ is *not* weakly Pareto optimal; then there is some allocation $\underset{\sim}{x}'$ that Pareto dominates $\underset{\sim}{x}^*$ strongly. This means that, for all i,

$$x_i' \succ_i x_i^* \tag{4.5}$$

Since $\underset{\sim}{x}^*$ is a valuation equilibrium relative to $p^*$, we have:

$$p^* x_i' > p^* x_i^* \tag{4.6}$$

Now, summing over all i,

$$p^* \left( \sum_{i=1}^{m} x_i' \right) > p^* \left( \sum_i x_i^* \right) \tag{4.7}$$

But since $\underset{\sim}{x}'$ and $\underset{\sim}{x}^*$ are both allocations,

$$\sum_{i=1}^{m} x_i' = \sum_{i=1}^{m} x_i^* = \omega \tag{4.8}$$

This leads to

$$p^*\left(\sum_{i=1}^{n} x_i'\right) = p^*\left(\sum_{i=1}^{n} x_i^*\right)[= p^*\omega] \tag{4.9}$$

We have a contradiction between (4.7) and (4.9).

The argument above needs a minor extension to arrive at a stronger conclusion with an additional assumption. A consumption vector $x \in R_+^\ell$ is *locally nonsatiated* for a consumer i, if every neighborhood $N_{x,\varepsilon}$, ($\varepsilon > 0$)) of x contains some y such that $y \succ_i x$. Here, for any $\varepsilon > 0$

$$N_{x,\varepsilon} = [y : ||y - x|| \leq \varepsilon\} \tag{4.10}$$

We now have:

**Theorem 4.2 The First Fundamental Theorem of Welfare Economics**

Suppose $\underset{\sim}{x}^*$ is a valuation equilibrium relative to $p^* > 0$, and, for every i, $x_i^*$ is locally non-satiated. Then $\underset{\sim}{x}^*$ is Pareto optimal.

**Proof**. Suppose not. Then there is some allocation $\underset{\sim}{x} = (x_i)$ such that

$$x_i \succsim_i x_i^* \quad \text{for all } i \text{ and} \tag{4.11}$$

$$x_i \succ_i x_i^* \quad \text{for some } i \tag{4.12}$$

Consider, first, the nonempty set of agents for whom (4.12) is satisfied. We follow the arguments of Theorem 4.1 to establish that for *each* such agent i,

$$p^*x_i > p^*x_i^* \tag{4.13}$$

Now, consider the (possibly empty) set of agents for whom (4.11) hold, but (4.12) does not. This means that for all these agents

$$x_i \sim_i x_i^* \tag{4.14}$$

We shall show that for each of these agents

$$p^*x_i \geqq p^*x_i^* \tag{4.15}$$

If not, suppose that $p^*x_i < p^*x_i^*$. Then, since $x_i^*$ is locally nonsatiated, there is some y such that $y \succ x_i^*$ *and* $p^*y < p^*x_i^*$ (write out the details) and this contradicts (4.4).

Now, using (4.13) and (4.15) and summing over all i we still get

$$p^*\left(\sum_i x_i\right) > p^*\left(\sum_{i=1}^m x_i^*\right). \tag{4.16}$$

But

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} x_i^* = \omega \tag{4.17}$$

again leads to

$$p^*\left(\sum_{i=1}^{m} x_i\right) = p\left(\sum_{i=1}^{m} x_i^*\right) = p^*\omega \tag{4.18}$$

and we have a contradiction.

It should be stressed that this ‘first’ fundamental theorem does not appeal to convexity (this statement continues to hold even when we introduce production). The weaker version (Theorem 4.1) does not impose any non-satiation or monotonicity condition on preferences either.

We now turn to the second fundamental theorem. A commodity vector $y \in \Omega$ is not a *satiation consumption* for agent i if there is some $z \in \Omega$ such that $z \succ_i y$.

**Theorem 4.3. The Second Fundamental Theorem**

Suppose that the preferences $\succsim_i$ of all agents satisfy (A.2), (A.3) and (A.4). Furthermore, there is some agent $i'$ who satisfies the monotonicity condition:

$$\text{"for any } c \geqq 0,\ \underset{\sim}{u} >> 0,\ c + \underset{\sim}{u} \succ_{i'} c\text{"} \tag{4.19}$$

Let $\underset{\sim}{x}^* = (x_1^*, \ldots. x_m^*)$ be a Pareto optimal allocation. Then there is $p^* > 0$ such that for *all* i,

$$\text{"}x \in \Omega,\ x \succsim_i x_i^*\text{"} \text{ implies } p^*x \geqq p^*x_i^* \tag{4.20}$$

**Proof**: Name the agents so that the first agent satisfies the monotonicity property (4.19). Write

$$M_1^* = \{x \in \Omega : x \succ_1 x_1^*\}$$

$$M_i = \{x \in \Omega : x \succsim_i x_i^*\} \qquad i=1,2,...,m$$

Note that by (A.4), all the sets $M_1^*, M_i$ are convex. Of course, $M_1^*$ is nonempty. Define

$$S = \{\omega\} - M_1^* - \sum_{i=2}^{M} M_i \tag{4.21}$$

Clearly, S is convex. Next, we verify that S cannot contain any $\underset{\sim}{u} >> 0$. Suppose it does. Then there are bundles $(x_i)$, i=1,...,m, $x_1 \in M_1^*$, $x_i \in M_i$ (i=2,...,m) such that

$$\underset{\sim}{u} = \omega - \sum_{i=1}^{m} x_i$$

Now, define $y_1 = x_1 + \underset{\sim}{u} \succ_1 x_1$ and $y_i = x_i =$ (i=2,...,m). Then we have:

$$\sum_i y_i = \omega$$

and, $y_1 \succ_1 x_1 \succ_1 x_1^*$, $y_i \succsim_i x_i^*$ for i=2,...,m. Thus, $\underset{\sim}{y} = (y_i)$ is an allocation that Pareto dominates $\underset{\sim}{x}^* = (x_i^*)$ weakly, a contradiction.

By a separation theorem (see, Appendix), there is $p^* \geq 0$ such that

$$p^*z \leqq 0 \qquad \text{for all } x \text{ in } S \tag{4.22}$$

This means that for any $x_1 \in M_1^*$, $x_i \in M_i$ $(i=2,...,m)$ one has

$$p^*(\omega - x_1 - \sum_{i=2}^{m} x_i) \leqq 0 \tag{4.23}$$

Now, if $x_1 \in M_1$, note that $x_1 + \underset{\sim}{u} \in M_1^*$ for all $\underset{\sim}{u} >> 0$. Hence, for *any* $(x_1,...,x_m)$ such that $x_i \in M_i$, $i=1,...,m$, one continues to have

$$p^*(\omega - \sum_{i=1}^{m} x_i) \leqq 0 \tag{4.24}$$

But $\omega = \sum_{i=1}^{m} x_i^*$, so that, for any $(x_1,...,x_m)$ with $x_i \in M_i$,

$$p^*(\omega - \sum_{i=1}^{m} x_i) \leqq 0 = p^*(\omega - \sum_{i=1}^{m} x_i^*) \tag{4.25}$$

or, for any $(x_1,...,x_m)$ such that $x_i \in M_i$,

$$p^*(\sum_{i=1}^{m} x_i) \geqq p^*(\sum_{i=1}^{m} x_i^*) \tag{4.26}$$

Now consider any particular agent i, and let $x \succsim_i x_i^*$, $x \in \Omega$, i.e., $x \in M_i$. Set $x_j = x_j^*$ for all $j \neq i$, and use (4.26) to conclude

$$p^*x \geqq p^* x_i^* \tag{4.27}$$

This establishes (4.20) Q.E.D.

It is useful to amplify (4.20) a bit and relate it to the concept of a valuation equilibrium (4.4). Note that (4.20) does NOT preclude the possibility that for some agent i, there is some $y \in \Omega$, $y \succ_i x_i^*$ AND $p^* y = p^* x_i^*$.

However, assume that at the price system $p^* > 0$ - whose existence we asserted by appealing to a separation theorem - $p^* x_i^* > 0$ for all i. This assumption clearly holds if $x_i^* >> 0$ for all i (i.e., if the Pareto optimal allocation we are considering is an 'interior' allocation).

In this situation if (4.20) holds, but (4.4) does not, we get a contradiction. If for some i, (4.4) does not hold, then there is $y \in \Omega$, $y \succ_i x_i^*$ and $p^* y = p^* x_i^*$. By (A.2), $\{z \in \Omega : z \succ_i x_i^*\}$ is open. Hence, there is some $\lambda \in (0,1)$ such that $\lambda y \succ_i x_i^*$ (choose $\lambda$ 'sufficiently close' to 1). Clearly, $p^*(\lambda y) = \lambda(p^* y) = \lambda(p^* x_i^*) < p^* x_i^*$, and we have a contradiction.

Appendix

In what follows, S is a subset $\mathfrak{S}$ of $R^m$ and T is a compact subset of $R^n$. A correspondence $\phi$ from S into T is a rule that associates with each x in S, a nonempty subset $\phi(x)$ of T. The correspondence $\phi$ is upper semicontinuous at $x^0$ if:

"for every sequence $x^n$ converging to $x^0$, and every sequence $y^n \in \phi(x^n)$ converging to $y^0$" it follows that "$y^o \in \phi(x^0)$".

The correspondence is lower semicontinuous at $x^0$ if:

"for every sequence $x^n$ converging to $x^0$, and every $y^0 \in \phi(x^0)$, there is a sequence $y^n \in \phi(x^n)$ such that $y^n$ converges to $y^0$."

The correspondence $\phi$ is continuous at $x^0$ if it is upper and lower semicontinuous at $x^0$. The correspondence $\phi$ is upper (lower) semicontinuous (continuous) on S if it is upper (lower) semicontinuous (continuous) at every $x \in S$.

Now, consider a continuous real valued function f on S × T. Let $\phi$ be a continuous correspondence from S into T. Fix $x \in S$, and consider the function f(x,y) on $\phi(x)$ [i.e. with x fixed, vary y over $\phi(x)$]. Let $M = \max_{\phi(x)} f(x,\bullet)$. Since $\phi$ is upper semicontinuous, and T is compact, one can show that $\phi(x)$ is compact. By continuity of f, M is well-defined. Write

$$\mu(x) = \{y \in \phi(x) : f(x,y) = M\}$$

**A.1 The Maximum Theorem**

The correspondence $\mu$ is upper semicontinuous.

We also have the celebrated:

**A.2 Kautani Fixed Point Theorem**

Let S be a nonempty, compact, convex subset of $R^m$, and $\phi$ be an upper semicontinuous correspondence from S to S such that for all $x \in S$, $\phi(x)$ is (nonempty) convex. Then, there is some $x \in S$ such that $x \in \phi(x)$.

**R.2. Separation Theorem**

Let X be a convex set in $R^\ell$ containing no strictly positive vector $u >> 0$. Then there is a semipositive $p \geq 0$ such that $px \leqq 0$ for all $x \in X$.

**Proof:** See Nikaido (1968, Theorem 3.5).

**References**


Arrow, K.J. and Hurwicz, L (1958): "On the Stability of the Competitive Equilibrium, I", *Econometrica*, 26, pp. 222-552.

Arrow, K.J., Block, H.D., and Hurwicz, L (1959): "On the Stability of the Competitive Equilibrium, II", *Econometrica*, 27,

Arrow, K.J. and Hahn, F. (1971): General Competitive Analysis, Holden Day, San Francisco.

Chipman, J.S. (1965): A Survey of the Theory of International Trade, Econometrica, 33, pp. 685-760.

Debreu, G. (1959): Theory of Value, Wiley, New York.

Debreu, G. (1952): "A Social Equilibrium Existence Theorem", *Proceedings of the National Academy of Sciences of the U.S.A.*, 38, pp. 886-893.

Debreu, G. (1954): "Valuation Equilibrium and Pareto Optimum", *Proceedings of the National Academy of Sciences of the U.S.A.*, 40, pp. 588-592.

Debreu, G. (1970): "Economies with a Finite Set of Equilibria", *Econometrica*, 38, pp. 387-392.

Debreu, G. (1972): Excess Demand Functions, Journal of Mathematical Economics, vol. 1,

Gale, D. (1955): "The Law of Supply and Demand", *Mathmatica Scandinavica*, 3, 155-169,

Hurwicz, (1986): "On Informational Decentralization and Efficiency in Resource Allocation Mechanisms", in Studies in Mathematical Economics (ed. S. Reiter), Mathematical Association of America.

Hicks, J. (1939): Value and Capital, Clarendon Press, Oxford.

Koopmans, T.C. (1957): Three Essays on the State of Economic Science, McGraw Hill, New York.

Nikaido, H. (1956): "On the Classical Multilateral Exchange Problem", *Metroeconomica*, 8, 135-145. [A Supplementary Note (1957)]

Nikaido, H. (1968): Convex Structures and Economic Theory, Academic Press, New York.

Samuelson, P.A. (1947): Foundations of Economic Analysis, Harvard University Press, Cambridge, Mass.

Uzawa, H. (1962): “Walras’ Existence Theorem and Brouwer’s Fixed Point Theory”, *Econ. Studies Quarterly*, 31,