CAE Working Paper #02-17

A Characterization of the Turnpike Property of Optimal Paths in the Aggregative Model of Intertemporal Allocation

by

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November 2002

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A Characterization of the Turnpike Property of Optimal Paths in the Aggregative Model of Intertemporal Allocation*

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Abstract

The paper provides a complete characterization of the turnpike property of optimal paths in the (reduced form) aggrgative model of intertemporal allocation. The characterization allows one to identify precisely the bifurcation point between globally stable and cyclical long-run optimal behavior. The complete characterization result is used to evaluate several sufficient conditions for global asymptotic stability of optimal paths that have been proposed in the literature. It is also used to examine sufficient conditions for the emergence of competitive equilibrium cycles in two-sector models.

Journal of Economic Literature Classification Numbers: C61, D90, O41.

Key Words: turnpike property, competitive cycles, bifurcation.

*An earlier version of this paper was presented at a conference on "New perspectives on instability: the role of heterogeneity" in June, 2001, at Marseille, France. Comments received from participants at the conference are gratefully acknowledged. The current version has benefited from conversations with Cuong Le Van and Gerhard Sorger.

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1 Introduction

This paper provides a *complete* characterization of the "turnpike property" (global asymptotic stability of a non-trivial stationary optimal stock) in the aggregative (reduced-form) model of optimal intertemporal allocation[1]. This characterization result is used to evaluate various *sufficient* conditions for global asymptotic stability that have been suggested in the literature. It is also used to examine conditions for the existence of competitive equilibrium cycles.

The dynamic optimization model that we are concerned with is described by a triple (Ω, u, δ), where Ω is a *transition possibility set* in X^2 (here $X = [0,1]$ will be taken to be the *state space*) , u is a *reduced-form utility function* from Ω to $\mathbb{R}$, and $\delta \in (0,1)$ is a *discount factor*. The dynamic optimization problem we seek to solve is written as:

$$\left.\begin{array}{lll} Maximize & \sum_{t=0}^{\infty} \delta^t u(x(t), x(t+1)) & \\ subject\ \ to & (x(t), x(t+1)) \in \Omega \quad for \quad t \geq 0 \\ and & x(0) = x & \end{array}\right\} \tag{1.1}$$

where x is an arbitrary point in the state space, X. Under standard convexity and continuity assumptions, the solution to the above problem can be completely described by a pair of functions (V, h), where V is the (optimal) *value function* from X to $\mathbb{R}$, and h is the (optimal) *policy function* from X to X. It turns out, under the standard assumptions, that h is continuous on X.

The study of the turnpike property of optimal programs thus reduces to a study of the asymptotic properties of the (one-dimensional) dynamical system, (X, h). Clearly, if every trajectory of this dynamical system is to be convergent, it is necessary that period-two cycles be absent. The absence of period-two cycles also turns out to be *sufficient* for every trajectory to converge. This result was first proved by Coppel (1955), and it was rediscovered

[1] It should be clarified that it is only in the sense of "global asymptotic stability" that the term "turnpike property" is used in this paper. In particular, the result of McKenzie (1982), known as the "neighborhood turnpike theorem" is not discussed in this paper. In McKenzie's theorem, given the size of a neighborhood, there is a bound on the discount factor, such that for all higher discount factors, optimal paths starting from arbitrary initial stocks are eventually confined to the specified neighborhood of the stationary optimal stock. It is not required in this result that the optimal paths actually converge to the stationary optimal stock.

independently by Sarkovskii (1960).[2]

Using Coppel's result, a characterization of the turnpike property (when there is a unique non-trivial stationary optimal stock) is reduced to a characterization of the conditions under which period-two cycles *cannot* occur. This is essentially a simpler problem to solve because, instead of dealing with the infinite-horizon dynamic optimization problem described above, we can now concentrate on a *two-period* dynamic optimization problem.

Consider the artificial two-period optimization problem, given $x \in X$, described by:

$$\left.\begin{array}{lll} Maximize & u(x,y)+\delta u(y,x) \\ subject\ \ to & (x,y)\in\Omega,(y,x)\in\Omega \\ and & x\in X \end{array}\right\} \tag{1.2}$$

Let us denote by W the value function, and by g the policy function, associated with this problem. Denote by Y the open interval $(0,1)$, and define the function, $f : Y \rightarrow \mathbb{R}$, by:

$$f(x) = u_2(g(x),x) + \delta u_1(x, g(x)) \tag{1.3}$$

If $\hat{x} \in Y$ is a stationary optimal stock, so that $h(\hat{x}) = \hat{x}$, then we must also have $g(\hat{x}) = \hat{x}$, and by the Ramsey-Euler equation, $\hat{x}$ must be a zero of the function, f. On the other hand, a zero of the function f on the set Y could be either a stationary optimal stock, or a stock from which there is a period-two optimal cycle. Thus (if the structure of our model rules out the possibility of boundary optimal cycles, then) the condition that there is a unique non-trivial stationary optimal stock, and that there are no period-two optimal cycles is equivalent to asserting that the function f has a unique zero on the set Y (since our standard end-point conditions will always ensure the existence of a stationary optimal stock $\hat{x}$ in Y). Thus, our characterization of global asymptotic *stability* is in terms of a *uniqueness* condition.[This central part of the paper is developed in Sections 4, 5 and 6].

In Section 7, the characterization result is used to obtain several sufficient conditions for global asymptotic stability of optimal paths that have been proposed in the literature, including those by Brock-Scheinkman, Cass-Shell, Rockafellar and Araujo-Scheinkman.

[2]These contributions pre-date much of the recent developments in one-dimensional dynamics, following the discovery of the famous Sarkovskii order of occurence of cycles. A nice treatment of Coppel's result, using the Sarkovskii order, can be found in Block and Coppel [1992].

Our characterization of the turnpike property has a feature worth noting . Because the characterization result is stated in terms of a solution to a two-period dynamic optimization problem, which can be solved, the relevant condition can in fact be checked for concrete examples of (Ω, u, δ).[3] We use this feature in Section 8 to evaluate the relative merits of these sufficient conditions, by obtaining the precise *bifurcation point* in the relevant parameter space between global asymptotic stability of a stationary optimal stock and the existence of optimal cycles in the context of the well-known Weitzman-Samuelson example.

In Section 9, the complete characterization result is used to examine sufficient conditions for the emergence of cycles. These conditions are compared to those proposed by Benhabib and Nishimura (1985). In the final section (Section 10) the conditions for period-two cycles are related to the capital-intensity condition in two-sector models.

2 Preliminaries

2.1 Dynamical Systems

Let $X = [0, 1]$ and g a map from X to X. We refer to X as the *state space*, and to g as the *law of motion* of the state variable $x \in X$. The pair (X, g) is called a *dynamical system*. Thus, if $x(t) \in X$ is the state of the system in period t, (where $t = 0, 1, 2, ...$) then $x(t + 1) = g(x(t)) \in X$ is the state of the system in time period $(t + 1)$.

We write $g^0(x) = x$ and for any integer $t \geq 1$, $g^t(x) = g[g^{t-1}(x)]$. If $x \in X$, the sequence $\tau(x) = (g^t(x))_0^\infty$ is called the *trajectory* from (the initial condition) x. The *orbit* from x is the set $\gamma(x) = \{y : y = g^t(x)$ for some $t \geq 0\}$. The asymptotic behavior of a trajectory from x is described by the *limit set*, $\omega(x)$, which is defined as the set of all limit points of $\tau(x)$.

[3] One might note, in this connection, that some sufficient conditions for global asymptotic stability that have been proposed in the literature are rather hard to check in concrete examples. Typically, if the sufficient condition has to be checked *along the optimal path*, it is not very restrictive, but difficult to verify without knowing the optimal path itself. On the other hand, if the condition has to hold for all points in, say, the transition possibility set (or the interior of this set), then the condition is easy to verify, but might turn out to be too strong a restriction, especially for points which are not close to the stationary optimal stock.

A point $x \in X$ is a *fixed point* of g if $g(x) = x$. A point $x \in X$ is called *periodic* if there is $t \geq 1$ such that $g^t(x) = x$. The smallest such t is called the *period* of x.

Note that if $x \in X$ is a periodic point, then $\omega(g^t(x)) = \gamma(x)$ for every $t = 0, 1, 2, \ldots$. A periodic point $\hat{x} \in X$ is *locally stable* if there is an open interval U (in X) containing $\hat{x}$, such that $\omega(x) = \gamma(\hat{x})$ for all $x \in U$. In this case, the periodic orbit $\gamma(\hat{x})$ is also called locally stable.

If g is continuously differentiable on X , and $\hat{x}$ is a periodic point of period t, then a *sufficient condition* for $\hat{x}$ to be locally stable is that $|Dg^t(\hat{x})| < 1$. If $|Dg^t(\hat{x})| > 1$, then $\hat{x}$ is not locally stable.

The principal result from the theory of one-dimensional dynamical systems that we will use is the following one, due to Coppel(1955).

Proposition 1 *Let $X = [0, 1]$, and g a continuous map from X to X. A necessary and sufficient condition for the trajectory $\tau(x)$ to converge for every $x \in X$, is that the equation $g^2(x) = x$ have no roots except the roots of the equation $g(x) = x$.*

2.2 The Model

We consider $X = [0, 1]$ to be the *state space*; the interior of the state space, $(0, 1)$, is denoted by Y. The model is described by a triple (Ω, u, δ), where $\Omega \subset X \times X$ is the *transition possibility set*, u is the (period) *utility function* defined on Ω, and δ is the *discount factor*.

The following assumptions on (Ω, u, δ) will be maintained throughout the paper:

(A.1) Ω is a convex and compact subset of $X \times X$, which contains (x, x) for all $x \in X$.

(A.2) $u : \Omega \to \mathbb{R}$ is a continuous function.

(A.3) u is concave on Ω, and if (x, z) and (x, z') belong to Ω, with $u(x, z) \neq u(x, z')$, then for every $\lambda \in (0, 1)$, we have $(x, \lambda z + (1 - \lambda)z') \in \Omega$, and $u(x, \lambda z + (1 - \lambda)z') > \lambda u(x, z) + (1 - \lambda)u(x, z')$.

(A.4) If $(x, z) \in \Omega$, and $x \leq x' \leq 1$, $0 \leq z' \leq z$, then $(x', z') \in \Omega$, and $u(x', z') \geq u(x, z)$. Also,

$$M \equiv \max_{(x,z)\in\Omega} u(x, z) > \min_{(x,z)\in\Omega} u(x, z) \equiv m$$

Further, defining $\Pi = \{(x, z) \in \Omega : u(x, z) > m\}$, (i) if $(x, z) \in \Pi$, and $x < x' \leq 1$, then $u(x', z) > u(x, z)$; (ii) if $(x, z) \in \Pi$, and $0 \leq z' < z$, then

$u(x, z') > u(x, z)$.
(A.5) $0 < \delta < 1$.

Assumptions (A.1), (A.2) and (A.5) are fairly standard. Assumption (A.3) ensures concavity of u, and a weaker form of strict concavity (in the second argument) than is commonly used. Similarly, assumption (A.4) ensures monotonicity of u, and strict monotonicity when the minimum utility is not attained. These weaker forms allow us to consider some standard examples as special cases of our aggregative reduced form model, which would otherwise be excluded. However, these weaker forms entail some extra work in ensuring that an optimal policy *function* will exist in our framework.

A few basic implications of our assumptions can now be noted[4]. First, we observe that

$$u(x, 0) > m \quad for\ all\ x \in (0, 1] \tag{2.1}$$

For each $x \in X$, we have $(x, 0) \in \Omega$ by (A.1) and (A.4). Thus, the set $A(x) = \{z : (x, z) \in \Omega\}$ is a non-empty subset of X; it is compact by (A.1). Thus, we can define the function, $a : X \to X$ by $a(x) = \max\{z : z \in A(x)\}$. One can think of a as a *net output* or a *production* function.

It follows from (2.1) that we have:

$$u(x, z) > m \quad for\ all\ x \in (0, 1], z \in [0, a(x)) \tag{2.2}$$

Using (2.2) and (A.4), one can extend the scope of (2.2) to accommodate changes in z :

$$u(x, z') > u(x, z) \quad for\ all\ x \in (0, 1], 0 \le z' < z \le a(x) \tag{2.3}$$

and similarly to accomodate changes in x :

$$u(x', z) > u(x, z) \quad for\ all\ 0 \le x < x' \le 1, 0 \le z < a(x) \tag{2.4}$$

2.3 Value and Policy Functions

A *path* from $x \in X$ is a sequence $(x(t))_0^\infty$ satisfying:

$$x(0) = x,\ and\ (x(t), x(t+1)) \in \Omega \quad for\ t \ge 0 \tag{2.5}$$

An *optimal path* from $x \in X$ is a path $(\bar{x}(t))_0^\infty$ from x, such that:

$$\sum_{t=0}^{\infty} \delta^t u(\bar{x}(t), \bar{x}(t+1)) \ge \sum_{t=0}^{\infty} \delta^t u(x(t), x(t+1)) \tag{2.6}$$

[4] The proofs are fairly routine and are therefore omitted.

for every path $(x(t))_0^\infty$ from x.

Under our maintained assumptions, there exists an optimal path from every $x \in X$. Thus, we can define a *value function*, $V : X \to \mathbb{R}$ by:

$$V(x) = \sum_{t=0}^{\infty} \delta^t u(\bar{x}(t), \bar{x}(t+1))$$

where $(\bar{x}(t))_0^\infty$ is an optimal path from x. Then, V is concave, non-decreasing and continuous on X.

It can be shown that for each $x \in X$, the Bellman equation:

$$V(x) = \max_{z \in X}\{u(x, z) + \delta V(z)\}$$

holds. For each $x \in X$, we denote by $h(x)$ the set of $z \in X$ which maximize $\{u(x, z) + \delta V(z)\}$ among all $z \in X$. That is, for each $x \in X$,

$$h(x) = \arg[\max_{z \in X}\{u(x, z) + \delta V(z)\}]$$

Then, a path $(x(t))_0^\infty$ from $x \in X$ is an optimal path from x if and only if it satisfies the equation: $V(x(t)) = u(x(t), x(t+1)) + \delta V(x(t+1)$ for $t \geq 0$; that is, if and only if $x(t+1) \in h(x(t))$ for $t \geq 0$.We call h the (optimal) policy correspondence.

Using the properties of u developed in the previous subsection, we can establish that the value function, V, is strictly increasing on X. This can be used in turn to demonstrate that $h(x)$ is a singleton for each $x \in X$; that is, h is an *optimal policy function*[5]. It can be shown (by an application of the maximum theorem) that h is continuous on X.

The continuity of h on X ensures the existence of a fixed point of h; that is, of a point $\bar{x} \in X$, such that $h(\bar{x}) = \bar{x}$. We refer to $\bar{x}$ as a *stationary optimal stock*, and the sequence $(\bar{x}, \bar{x}, \bar{x}, ...)$ as a *stationary optimal path*.

3 Basic Properties of Optimal Paths

In this section, we establish some basic properties of optimal paths, which will be useful in our investigation in the later sections. To this end, we strengthen our maintained set of assumptions as follows:

[5] The proofs of these statements are fairly straightforward, and are therefore omitted.

(A.2+) $u : \Omega \to \mathbb{R}$ is a continuous function, and u is C^2 on $\Lambda = \{(x, z) \in \Omega : x \in (0, 1], z < a(x)\}$.
(A.3+) u is concave on Ω, and if (x, z) and (x, z') belong to Ω, with $u(x, z) \neq u(x, z')$, then for every $\lambda \in (0, 1)$, we have $(x, \lambda z + (1 - \lambda)z') \in \Omega$, and $u(x, \lambda z + (1 - \lambda)z') > \lambda u(x, z) + (1 - \lambda)u(x, z')$. Further, for all $(x, z) \in \Lambda$, $u_{11}(x, z) < 0$, and $u_{11}(x, z)u_{22}(x, z) - [u_{12}(x, z)]^2 \geq 0$.
(A.4+) If $(x, z) \in \Omega$, and $x \leq x' \leq 1$, $0 \leq z' \leq z$, then $(x', z') \in \Omega$, and $u(x', z') \geq u(x, z)$. Also,

$$M \equiv \max_{(x,z)\in\Omega} u(x, z) > \min_{(x,z)\in\Omega} u(x, z) \equiv m$$

Further, defining $\Pi = \{(x, z) \in \Omega : u(x, z) > m\}$, (i) if $(x, z) \in \Pi$, and $x < x' \leq 1$, then $u(x', z) > u(x, z)$; (ii) if $(x, z) \in \Pi$, and $0 \leq z' < z$, then $u(x, z') > u(x, z)$. And, for all $(x, z) \in \Lambda$, $u_1(x, z) > 0, u_2(x, z) < 0$.

These strengthened assumptions reflect the twice continuous differentiability of the utility function on the subset Λ of Ω, which excludes the "upper boundary" of the set Ω, and the differential forms of concavity [(A.3+)] and monotonicity [(A.4+)] on the set Λ.

3.1 Existence of a Stationary Optimal Path

In addition to the differentiability assumptions on u, we make the following end-point assumption[6]:
(A.6) There is some $x^0 \in (0, 1)$, such that $a(x^0) > x^0$. Further, defining $\pi(x) = [-u_2(x, x)]/u_1(x, x)$ for all $x \in (0, 1)$, we have $\lim_{x\to 0} \pi(x) = 0$ and $\lim_{x\to 1} \pi(x) > 1$.

Using (A.6), given any discount factor, $\delta \in (0, 1)$, there is a solution, $k(\delta) \in (0, 1)$, to the equation: $\pi(x) = \delta$. Then, the stationary sequence $(k(\delta), k(\delta), ...)$ is a path from $k(\delta)$, which satisfies the Ramsey-Euler equation. Since it is stationary, it also satisfies the appropriate transversality condition. Thus, the stationary path is optimal from $k(\delta)$, and $k(\delta)$ is a (non-trivial) stationary optimal stock.

[6] We note that the first part of assumption (A.6) ensures that $a(x) > x$ for all $x \in (0, 1)$, and so $(x, x) \in \Lambda$ for all $x \in (0, 1)$. This enables us to legitimately write the second part of the assumption.

3.2 Boundary Behavior of Optimal Paths

The principal result that we would like to note in this subsection is that an optimal path $(x(t))_0^\infty$ from an initial stock in the interior of the state space cannot converge to the boundary of the state space, X. To this end, we assume additionally:

(A.7) $\lim_{x \to 0} u_1(x, x) = \infty$.

We first note the implication of (A.7) on the nature of the optimal policy function, h.

Lemma 1 *The optimal policy function, h, satisfies (i) $h(1) < 1$; and (ii) if $h(0) = 0$, then $h(x) > 0$ for all $x \in (0, 1]$.*

Proof. To establish that $h(1) < 1$, assume on the contrary that $h(1) = 1$. This means that the stationary sequence $(1, 1, ...)$ is optimal from the initial stock 1. Let $z \in (0, 1)$, and consider the sequence $(1, z, z, ...)$. This is a path from the initial stock 1. Consequently, we must have $u(1, 1) + du(1, 1) \geq u(1, z) + du(z, z)$, where $d = \delta/(1-\delta)$. Rearranging terms and using concavity of u, we get:

$$\begin{aligned} d[u_1(z,z) - (-u_2(z,z))](z-1) &\leq d[u(z,z) - u(1,1)] \\ &\leq u(1,1) - u(1,z) \leq u_2(1,z)(1-z) \end{aligned} \tag{3.1}$$

Dividing through by the positive term $(1 - z)u_1(z, z)$, we obtain:

$$d[\pi(z) - 1] \leq u_2(1, z)/u_1(z, z) \tag{3.2}$$

Letting $z \to 1$, we note that $\pi(z)$ converges to a number greater than one, and so the left hand side of (3.2) converges to a positive number. But, the left hand side of (3.2) is negative for all $z \in (0, 1)$. This contradiction establishes (i).

To establish (ii), assume that $h(0) = 0$, and contrary to the claim in the lemma, assume that there is some $y \in (0, 1]$, such that $h(y) = 0$. Then, $(y, 0, 0, ...)$ is optimal from y. Consider the sequence $(y, z, z, ...)$ where $0 < z < y$. Then denoting $[\delta/(1-\delta)]$ by d, we have $u(y, 0) + du(0, 0) \geq u(y, z) + du(z, z)$. Rearranging terms, and using concavity of u, we obtain:

$$\begin{aligned} d[u_1(z,z) - (-u_2(z,z))]z &\leq d[u(z,z) - u(0,0)] \\ &\leq u(y,0) - u(y,z) \leq [-u_2(y,z)]\, z \end{aligned} \tag{3.3}$$

Then, dividing through by the positive term, $z\, u_1(z,z)$, we obtain:

$$d[1-\pi(z)] \leq [-u_2(y,z)]/u_1(z,z) \tag{3.4}$$

Letting $z \to 0$, we note that since $(y,0)$ is in Λ, $[-u_2(y,z)] \to [-u_2(y,0)] < \infty$, while $u_1(z,z) \to \infty$ by (A.7), so that $[-u_2(y,z)]/u_1(z,z) \to 0$. But, since $\pi(z) \to 0$ as $z \to 0$ by (A.6), $d[1-\pi(z)] \to d$. This contradicts (3.4) and establishes (ii). ■

Proposition 2 *Let $(x(t))_0^\infty$ be an optimal path from $x \in Y$. Then (i) $x(t)$ cannot decrease to* 0*, as $t \to \infty$; (ii) $x(t)$ cannot converge to* 1*, as $t \to \infty$.*

Proof. To establish (i), assume on the contrary that there is an optimal path $(x(t))_0^\infty$ from some $x \in (0,1)$, such that $x(t) \searrow 0$ as $t \to \infty$. Since $x(t+1) = h(x(t))$ for $t \geq 0$, and h is continuous on X, we must have $h(0) = 0$.

We consider two cases:(a) $x(t) = 0$ for some finite t; (b) $x(t) > 0$ for all $t \geq 0$.Clearly, case(a) is ruled out by Lemma 1 (ii). In case (b), we proceed as follows. Pick $\bar{k} > 0$, such that for all $x \in (0,\bar{k})$, we have $[-u_2(x,x)] < \delta\, u_1(x,x)$. By (A.6), this can be done. Since $x(t) \to 0$, we can find some T, such that $x(t) < \bar{k}$ for all $t \geq T$. Denote $x(T)$ by k, $u_1(k,k)$ by p, and $[-u_2(k,k)]$ by q. Then, we have $\delta p > q$. Denote $x(t)$ by $y(t-T)$ for $t \geq T$. Then $(y(t))_0^\infty$ is an optimal path from $y(0) = x(T) = k$. Also, the stationary sequence $(k,k,k,...)$ is a path from k.

For $t \geq 0$, we have:

$$\delta^t[u(y(t),y(t+1)) - u(k,k)] \leq \delta^t p(y(t)-k) - \delta^t q(y(t+1)-k) \tag{3.5}$$

Summing (3.5) from $t=0$ to $t=S$, we obtain:

$$\sum_{t=0}^{S} \delta^t[u(y(t),y(t+1)) - u(k,k)] \;\leq\; \sum_{t=0}^{S-1} \delta^t(\delta p - q)(y(t+1)-k) - \delta^S q(y(S+1)-k) \tag{3.6}$$

Since $y(t+1) \leq y(0) = k$ for all $t \geq 0$, and $\delta p > q$, the sum on the right-hand side of (3.6) is non-positive. The remaining term on the right hand side of (3.6) converges to 0 as $S \to \infty$. This shows that the stationary sequence $(k,k,k,...)$ is optimal from k. Since an optimal policy function exists, this is the unique optimal path from k, which contradicts the optimality of $(y(t))_0^\infty$ from k.

To establish (ii), assume on the contrary that there is an optimal path $(x(t))_0^\infty$ from some $x \in (0,1)$, such that $x(t) \to 1$ as $t \to \infty$. Since $x(t+1) = h(x(t))$ for $t \geq 0$, and h is continuous on X, we must have $h(1) = 1$. But this is ruled out by Lemma 1 (i). ■

4 A Two-Period Optimization Problem

This section contains the key to our approach to the characterization of the turnpike property. Instead of examining the infinite horizon optimization problem (1.1), we now analyze (following Mitra and Nishimura(2001)) an artificial two-period optimization problem, in which the initial and terminal (after two periods) stocks are restricted to be the same.

Given any $x \in X$, consider the following optimization problem:

$$\left.\begin{array}{ll} Maximize & u(x,y) + \delta u(y,x) \\ subject \;\; to & (x,y) \in \Omega, (y,x) \in \Omega \\ and & x \in X \end{array}\right\} \tag{4.1}$$

Given our assumptions, for each $x \in X$, problem (4.1) has a solution. We denote the value function for this problem by W. Clearly, W is a concave function from X to $\mathbb{R}$.

It can be shown that for each $x \in X$, problem (4.1) has a unique solution[7]. We denote the unique solution of (4.1), for each $x \in X$, by $g(x)$. Clearly, the function g maps X to X. It can be checked that the value function, W, and the policy function, g, associated with problem (4.1), are both continuous functions on X.

In order to characterize the solution to problem (4.1), for $x \in Y = (0,1)$, by its first order condition, we want to ensure that for each $x \in Y$, the solution $g(x)$ is an interior one. To this end, we proceed to make the following end-point assumption:

(A.8) (i) Given any $x^0 \in Y$, and $0 \leq y < a(x^0)$, $[-u_2(x^0,y)] \to \infty$ as $y \to a(x^0)$. (ii) If $(0,y^0) \in \Omega$ for some $y^0 \in Y$, and $x \in (0,y^0)$, then $u_1(x,y^0) \to \infty$ as $x \to 0$. (iii) If $x^0 \in Y$, and $a(x^0) \in Y$, and $(x,a(x^0)) \in \Lambda$, then $u_1(x,a(x^0)) \to \infty$ as $x \to x^0$.

[7] The proof is fairly straightforward, using (2.2), (2.3) and (A.4), and is therefore omitted.

Using (A.8), it can be checked that:

$$0 < g(x) < a(x) \text{ and } x < a(g(x)) \text{ for all } x \in Y \tag{4.2}$$

Given these properties of the function, g, we can return to the problem (4.1), and note that for each $x \in Y$, the following first-order condition must hold:

$$u_2(x, g(x)) + \delta u_1(g(x), x) = 0 \tag{4.3}$$

Since $[u_{22}(x, g(x)) + \delta u_{11}(g(x), x)] < 0$, we can use the implicit function theorem to conclude that g is continuously differentiable on Y, and obtain:

$$u_{21}(x, g(x)) + u_{22}(x, g(x))g'(x) + \delta u_{11}(g(x), x)g'(x) + \delta u_{12}(g(x), x) = 0 \tag{4.4}$$

This yields a convenient formula for $g'(x)$ in terms of the second-order partials of the utility function:

$$g'(x) = -\frac{[u_{21}(x, g(x)) + \delta u_{12}(g(x), x)]}{[u_{22}(x, g(x)) + \delta u_{11}(g(x), x)]} \tag{4.5}$$

We will use this formula in later sections.

5 A Condition for the Absence of Cycles

In this section, we introduce the condition(Condition U), which is *equivalent* to the simultaneous validity of the following two properties:(i) the existence of a unique stationary optimal stock (Condition USS), and (ii) the absence of period-two optimal cycles(Condition NPT). We show the sufficiency part of this claim in the first subsection below, and the necessity part in the second subsection.

The result stated above is valid in a framework in which the "boundary cycle" $(1, 0, 1, 0, 1, ...)$ is *not* optimal starting from the initial stock, 1. This possibility cannot be ruled out based on the set of assumptions we have made. In fact, if $\Omega = X \times X$, and $u(x, z) = x^\alpha(1-z)^{1-\alpha}$, where $(1/2) < \alpha < 1$, then it can be shown that all our maintained assumptions are satisfied, and the boundary cycle $(1, 0, 1, 0, 1, ...)$ is optimal from 1 if and only if $\delta \leq (1-\alpha)/\alpha$.

We now make an additional end-point assumption to rule out such boundary optimal cycles :

(A.9) If $(0, 1) \in \Omega$, then for $y \in Y$, we have $u_1(y, 1-y) \to \infty$ as $y \to 0$.

To verify that this assumption rules out the boundary optimal cycle, suppose on the contrary that $h(1) = 0$, and $h(0) = 1$. Pick $y \in Y$, and note that $(y, 1-y) \in \Lambda$, and $(1-y, y) \in \Lambda$. Consider the sequence $(y(t))_0^\infty$ from 1, defined by $y(t) = y$ for $t = 1, 3, 5, ...$, and $y(t) = (1-y)$ for $t = 2, 4, 6, ...$ Denoting $[1/\ (1-\delta^2)]$ by D, we have:

$$\begin{aligned} Du(1,0) + \delta Du(0,1) &\geq \sum_{t=0}^{\infty} \delta^t u(y(t), y(t+1)) \\ &\geq Du(1-y, y) + \delta Du(y, 1-y) \end{aligned} \tag{5.1}$$

Rearranging terms and using the concavity of u, we obtain:

$$\begin{aligned} u_1(1-y,y)(-y) + u_2(y,1-y)y &\leq u(1-y,y) - u(1,0) \qquad (5.2) \\ &\leq \delta[u(0,1) - u(y,1-y)] \\ &\leq \delta[u_1(y,1-y)(-y) + u_2(y,1-y)y] \end{aligned}$$

This yields the following inequality for all $y \in Y$:

$$\delta[u_1(y,1-y) + [-u_2(y,1-y)]] \leq u_1(1-y,y) + [-u_2(1-y,y)] \tag{5.3}$$

Since $(1,0) \in \Lambda$, the right hand side expression in (5.3) converges to $u_1(1,0) + [-u_2(1,0)]$ as $y \to 0$. But, the left-hand side expression goes to infinity by (A.9). This contradiction establishes our claim.

We now proceed to formally state the main result of this section. The analysis of Section 4 indicates that for $x \in Y$, we must have $(x, g(x))$ and $(g(x), x)$ in the interior of Ω, and consequently $(x, g(x))$ and $(g(x), x)$ are in Λ. So, we can define a function, $f : Y \to \mathbb{R}$, as follows:

$$f(x) = u_2(g(x), x) + \delta u_1(x, g(x)) \tag{5.4}$$

Note that a stationary optimal stock, $k(\delta) \in Y$, whose existence we established in Section 3.1, satisfies the property that $h(k(\delta)) = k(\delta)$, and consequently it also satisfies the property that $g(k(\delta)) = k(\delta)$. Using this information in (5.4), we see that $f(k(\delta)) = 0$. We can now state Condition U, which is a uniqueness condition on the roots of the equation $f(x) = 0$.

Condition U: The equation $f(x) = 0$ has a unique solution in Y.

We will show that Condition U is equivalent to the simultaneous validity of the following two conditions:

Condition USS: There is a unique stationary optimal stock in Y.

Condition NPT: The dynamical system (X, h) has no period-two cycles.

5.1 Condition U is Sufficient

In this subsection, we establish the following proposition.

Proposition 3 *Suppose Condition U is satisfied. Then Condition USS and Condition NPT must hold.*

Proof. Denote by $\bar{x}$ the unique solution in Y to the equation $f(x) = 0$. If k is any stationary optimal stock in Y, then $h(k) = k$, and so $g(k) = k$. Since (k, k) is in the interior of the set Ω, we must have by (4.3):

$$u_2(k, k) + \delta u_1(k, k) = 0 \tag{5.5}$$

Since $g(k) = k$, (5.4) and (5.5) imply that $f(k) = 0$. Thus, $k = \bar{x}$. Since we know that there exists a stationary optimal stock, $k(\delta) \in Y$, Condition USS must hold.

To establish Condition NPT, suppose on the contrary that there is a period two optimal cycle. That is, suppose there exist x, y in X, with $x \neq y$, such that $y = h(x)$ and $x = h(y)$. We will show that this supposition leads to a contradiction. Note, right at the outset, that we must also have $y = g(x)$ and $x = g(y)$, thus enabling us to exploit the results of Section 4. We will find it convenient to break up our proof into the following three cases: (i) $0 < x < 1$; (ii) $x = 0$; (iii) $x = 1$.

In case(i), we must have $0 < g(x) < a(x) \leq 1$, so that $0 < y < 1$. This in turn implies that $0 < g(y) < a(y) \leq 1$, and so $0 < x < a(y)$. Thus, we have (x, y) and (y, x) in the interior of Ω and the following first-order conditions must hold:

$$u_2(y, g(y)) + \delta u_1(g(y), y) = 0 \tag{5.6}$$

$$u_2(x, g(x)) + \delta u_1(g(x), x) = 0 \tag{5.7}$$

Since $y = g(x)$, and $x = g(y)$, (5.6) implies that $f(x) = 0$, while (5.7) implies that $f(y) = 0$. Thus, we must have $x = y = \bar{x}$, a contradiction.

In case (ii), we distinguish between two sub-cases:(a) $y \in Y$; (b) $y = 1$. Case (a) cannot occur, because in this case we must have $x = g(y) > 0$. Case

(b) would imply that the sequence $(1, 0, 1, 0, 1, ...)$ is optimal from 1, which we have ruled out above.

In case (iii), we similarly distinguish between two sub-cases: (a') $y \in Y$; (b') $y = 0$. Case (a') cannot occur, because in this case we must have $x = g(y) < a(y) \leq 1$. Case (b') would imply that the sequence $(1, 0, 1, 0, 1, ...)$ is optimal from 1, which we have ruled out above. ■

5.2 Condition U is Necessary

In this subsection, we establish the following proposition:

Proposition 4 *Suppose Condition USS and Condition NPT are satisfied. Then Condition U must hold.*

Proof. Denote the unique stationary optimal stock in Y, given by Condition USS, by $\bar{x}$. Then, as noted in the proof of Proposition 3, $\bar{x}$ must satisfy $f(\bar{x}) = 0$. To establish Condition U, we need to show that there is no other solution to the equation $f(x) = 0$ in Y. Suppose, on the contrary, that there was another such solution. Call this solution $\hat{x}$. Then $\hat{x} \in Y$, $f(\hat{x}) = 0$, and $\hat{x} \neq \bar{x}$. Then, we have $0 < g(\hat{x}) < a(\hat{x})$, and $\hat{x} < a(g(\hat{x}))$. Thus, the following first-order condition must hold:

$$u_2(\hat{x}, g(\hat{x})) + \delta u_1(g(\hat{x}), \hat{x}) = 0 \tag{5.8}$$

And, since $f(\hat{x}) = 0$, we must also have:

$$u_2(g(\hat{x}), \hat{x}) + \delta u_1(\hat{x}, g(\hat{x})) = 0 \tag{5.9}$$

Thus, defining $\hat{y} = g(\hat{x})$, we note that the sequence $(\hat{x}, \hat{y}, \hat{x}, \hat{y}, ...)$ is a path from $\hat{x}$, which satisfies the Ramsey-Euler equations. Since $\hat{x} \in Y$ and $\hat{y} \in Y$ are the only two values attained on this path, and $(\hat{x}, \hat{y})$ and $(\hat{y}, \hat{x})$ are both in the interior of Ω, it is straightforward to check that the path also satisfies the transversality condition. Hence, it is optimal from $\hat{x}$. Since $\hat{x} \neq \bar{x}$, Condition USS implies that $\hat{y} \neq \hat{x}$.But, this means that $(\hat{x}, \hat{y})$ is a period two cycle of the dynamical system (X, h), which violates Condition NPT. This contradiction establishes the result. ■

6 A Characterization of the Turnpike Property

We are now in a position to characterize the "turnpike property" (also known as the global asymptotic stability property). We first formally state the property as follows.

Condition TP: There is some $\tilde{x} \in Y$, such that if $(x(t))_0^\infty$ is an optimal path from any $x \in (0,1]$, then $x(t) \to \tilde{x}$ as $t \to \infty$.

We will show that Condition TP is equivalent to the simultaneous validity of Condition USS and Condition NPT. Given Propositions 3 and 4 of section 5, Condition TP is then characterized precisely by Condition U.

Proposition 5 *Suppose Condition TP is satisfied. Then, Condition USS and Condition NPT must hold.*

Proof. We know that there exists a stationary optimal stock, $k(\delta) \in Y$. Given Condition TP, we must therefore have $\tilde{x} = k(\delta)$. If $\hat{x}$ is any stationary optimal stock in Y, then by Condition TP, $\hat{x} = \tilde{x}$. Thus, Condition USS must hold. To establish Condition NPT, suppose on the contrary that there is a period two cycle of the dynamical system (X, h). Denoting the period two cycle by (x, y), we note that at least one of the values, say x, must be in $(0,1]$. But, then, by Condition TP, both x and y must be equal to $\tilde{x}$. This contradiction establishes the result. ■

Proposition 6 *Suppose Condition USS and Condition NPT are both satisfied. Then Condition TP must hold.*

Proof. Let us denote the unique stationary optimal stock in Y, ensured by Condition USS, by $\bar{x}$. Given Condition NPT, we can use Proposition 1 to infer that the trajectory $(h^t(x))_0^\infty$ from any $x \in (0,1]$ must converge to a fixed point of h. By Condition USS, $\bar{x}$ is the only fixed point of h in Y. By Lemma 1, we know that 1 is not a fixed point of h. Thus, denoting $h^t(x)$ by $x(t)$ for $t \geq 0$, we note that there are only two possibilities: (i) $x(t) \to \bar{x}$ as $t \to \infty$; (ii) $x(t) \to 0$ as $t \to \infty$. We will rule out the second possibility, thereby establishing Condition TP.

In case (ii), by continuity of h, we have $h(0) = 0$. By Lemma 1, we must have $h(x) > 0$ for all $x \in (0,1]$. Thus, $x(t) > 0$ for $t \geq 0$, and $x(t) \to 0$

as $t \to \infty$. Since $\bar{x}$ is the only fixed point of h in Y, we must have either (A) $h(x) < x$ for all $x \in (0, \bar{x})$, or (B) $h(x) > x$ for all $x \in (0, \bar{x})$. Since $x(t) \to 0$ as $t \to \infty$, we can find $T \geq 1$, such that $x(t) \in (0, \bar{x})$ for all $t \geq T$. Denoting $x(t)$ by $y(t-T)$ for $t \geq T$, we obtain a sequence $(y(s))_0^\infty$, which is an optimal path from $y(0) \in (0, \bar{x})$. In sub-case (A), we must have $y(t) \searrow 0$ as $t \to \infty$, which contradicts Proposition 2. Thus, sub-case (B) must hold. But, in this sub-case, we must have $x(t) \geq x(T)$ for all $t \geq T$, so that $x(t)$ cannot converge to zero. Thus, sub-case (B) also leads to a contradiction. We have, therefore, established that case (ii) cannot occur at all. Thus, case (i) holds, and Condition TP is verified. ■

We can now collect together the results of this section and the previous one to provide the following characterization of the turnpike property.

Theorem 1 *Condition TP holds if and only if Condition U holds.*

7 Sufficient Conditions for the Turnpike Property

The turnpike property is completely characterized by Condition U, which is a uniqueness condition on the solutions of the equation $f(x) = 0$. Given (Ω, u, δ), one can, in principle, solve problem (4.1) to obtain the function, g, and then examine the solutions of the equation $f(x) = 0$. It would be interesting, though, to see whether one can state some sufficient conditions on (Ω, u, δ), which one can check directly, and which will ensure that Condition U will be satisfied. Then, these conditions would be sufficient to ensure the turnpike property.

To this end, we first state a sufficient condition ensuring that there is at most one solution to an equation in a single variable. This is a very special case of a more general result in differential topology, known as the Poincare-Hopf index theorem[8].

7.1 Unique Solution to a Single-Variable Equation

Let a, b be real numbers with $a < b$. Let us denote (a, b) by I, and let F be a function from I to $\mathbb{R}$, which is continuously differentiable on I. Define

[8] For our one-dimensional case, it can be verified quite easily, using elementary analytical methods.

$E = \{x \in I : F(x) = 0\}$. Consider the following condition on F :

Condition D: If $x \in E$, then $F'(x) < 0$.

The implication of this condition is summarized in the following result.

Lemma 2 *If f satisfies Condition D, then E has at most one element.*

7.2 On Verification of Condition U

Using the result of the previous subsection, we can proceed to verify Condition U in our model as follows. Let $x \in Y$ be an arbitrary point satisfying $f(x) = 0$. This gives us quite a bit of information about the point x. Denote $g(x)$ by z, and note that $x = g(z)$, and that (x, z) and (z, x) are both in the interior of Ω. Further, $z = h(x)$ and $x = h(z)$. We want to verify that $f'(x) < 0$ at any such point.

Differentiating (5.4) we obtain:

$$\begin{aligned} f'(x) &= u_{21}(g(x), x)g'(x) + u_{22}(g(x), x) + \\ &\quad \delta u_{11}(x, g(x)) + \delta u_{12}(x, g(x))g'(x) \end{aligned} \tag{7.1}$$

Using the expression for $g'(x)$ given in (4.5) we obtain:

$$\begin{aligned} f'(x) &= [u_{22}(g(x), x) + \delta u_{11}(x, g(x))] + \\ &\quad \left\{ -\frac{[u_{21}(x, g(x)) + \delta u_{12}(g(x), x)]}{[u_{22}(x, g(x)) + \delta u_{11}(g(x), x)]} \right\} [u_{21}(g(x), x) + \delta u_{12}(x, g(x))] \end{aligned} \tag{7.2}$$

Since $[u_{22}(x, g(x)) + \delta u_{11}(g(x), x)] < 0$, (7.2) implies that $f'(x) < 0$ if and only if:

$$\begin{aligned} & [u_{22}(g(x), x) + \delta u_{11}(x, g(x))][u_{22}(x, g(x)) + \delta u_{11}(g(x), x)] \\ > & [u_{21}(x, g(x)) + \delta u_{12}(g(x), x)][u_{21}(g(x), x) + \delta u_{12}(x, g(x))] \end{aligned} \tag{7.3}$$

Our analysis yields the following result, which we state for ready reference.

Lemma 3 *Let $\hat{x} \in Y$ satisfy $f(\hat{x}) = 0$. Then, $f'(\hat{x}) < 0$ if and only if:*

$$\begin{aligned} & [u_{22}(g(\hat{x}), \hat{x}) + \delta u_{11}(\hat{x}, g(\hat{x}))][u_{22}(\hat{x}, g(\hat{x})) + \delta u_{11}(g(\hat{x}), \hat{x})] \\ > & [u_{21}(\hat{x}, g(\hat{x})) + \delta u_{12}(g(\hat{x}), \hat{x})][u_{21}(g(\hat{x}), \hat{x}) + \delta u_{12}(\hat{x}, g(\hat{x}))] \end{aligned} \tag{S}$$

7.3 On Verification of Condition S

In the context of our aggregative reduced-form model, we can view the sufficient conditions for global asymptotic stability that have been proposed in the literature as conditions which ensure that whenever $\hat{x} \in Y$ satisfies $f(\hat{x}) = 0$, it also satisfies Condition S. We examine below the conditions proposed by Brock and Scheinkman (1978), Cass and Shell (1976), Rockafellar (1976), and Araujo and Scheinkman (1977) from this perspective.

7.3.1 The Q-Condition of Brock and Scheinkman

We state the Q-condition of Brock and Scheinkman in the following way:

Q-Condition: For all $x \in Y$, we have $(x, h(x)) \in int\Omega$, and the matrix:

$$Q = \begin{bmatrix} \delta u_{11}(x, h(x)) & \delta u_{12}(x, h(x)) \\ u_{21}(x, h(x)) & u_{22}(x, h(x)) \end{bmatrix}$$

is quasi negative definite.

If the Q-Condition is satisfied, then we can proceed to verify Condition S as follows. Let $x \in Y$ be an arbitrary point satisfying $f(x) = 0$. Denote $g(x)$ by z, and note that $x = g(z)$, and that (x, z) and (z, x) are both in the interior of Ω. Further, $z = h(x)$ and $x = h(z)$. Using the Q-Condition, the matrices:

$$Q = \begin{bmatrix} \delta u_{11}(x, h(x)) & \delta u_{12}(x, h(x)) \\ u_{21}(x, h(x)) & u_{22}(x, h(x)) \end{bmatrix}; R = \begin{bmatrix} \delta u_{11}(h(x), x) & \delta u_{12}(h(x), x) \\ u_{21}(h(x), x) & u_{22}(h(x), x) \end{bmatrix}$$

are both quasi negative definite. Defining the matrix:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

we note that, since P is non-singular, the matrix S, defined by:

$$S = P'RP = \begin{bmatrix} u_{22}(h(x), x) & u_{21}(h(x), x) \\ \delta u_{12}(h(x), x) & \delta u_{11}(h(x), x) \end{bmatrix}$$

is also quasi negative definite. Consequently, the matrix, $Q + S$, must be quasi negative definite. That is, the 2×2 matrix $A = (a_{ij})$, defined by:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \equiv \begin{bmatrix} \delta u_{11}(x, h(x) + u_{22}(h(x), x) & \delta u_{12}(x, h(x)) + u_{21}(h(x), x) \\ u_{21}(x, h(x)) + \delta u_{12}(h(x), x) & u_{22}(x, h(x)) + \delta u_{11}(h(x), x) \end{bmatrix}$$

is quasi negative definite. Since $a_{11} < 0$ by our maintained assumptions, A is quasi negative definite if and only if $a_{11}a_{22} > [(a_{12}+a_{21})/2]^2$. Since $[(a_{12}+a_{21})/2]^2 \geq a_{12}a_{21}$,we have verified that $a_{11}a_{22} > a_{12}a_{21}$. This means that:

$$\begin{aligned} & [u_{22}(h(x),x)+\delta u_{11}(x,h(x))][u_{22}(x,h(x))+\delta u_{11}(h(x),x)] \\ > \; & [u_{21}(x,h(x))+\delta u_{12}(h(x),x)][u_{21}(h(x),x)+\delta u_{12}(x,h(x))] \end{aligned}$$

which verifies Condition S, since $h(x) = g(x)$.

7.3.2 The Hamiltonian Curvature Condition of Cass and Shell

The Hamiltonian approach to global asymptotic stability is followed by Cass and Shell (1976) and Rockafellar (1976). However, Rockafellar treats only the continuous-time case of the optimal growth model, and, as noted by Cass and Shell (1976), the discrete and continuous treatments of time do create a difference in the conditions to be verified, on the curvature of the Hamiltonian, for ensuring the turnpike property. Consequently, we will examine here only the discrete-time treatment of Cass and Shell.

Denote by Γ the set $\mathbb{R}_+ \times X$, and by Θ the set $\mathbb{R}_{++} \times Y$. For $(q,k) \in \Gamma$, consider the optimization problem, given by:

$$\left.\begin{array}{ll} Maximize & u(k,k+\zeta)+q\zeta \\ subject\ to & (k,k+\zeta) \in \Omega \end{array}\right\} \tag{7.4}$$

Denote the value of this problem by $H(q,k)$, and the policy by $J(q,k)$. It is usual to refer to the function H as the Hamiltonian. It is sraightforward to show that H is concave in k and convex in q. The Hamiltonian approach consists in identifying a condition, involving the discount rate, the degree of concavity of H in k, and the degree of convexity of H in q, which will ensure global asymptotic stability.

We confine our attention to the problem (7.4) when $(q,k) \in \Theta$. If, for $(q,k) \in \Theta$, we have $(k, J(q,k)) \in int\Omega$, then the first-order condition for problem (7.4) will yield the following identity in (q,k) on Θ :

$$u_2(k, k+J(q,k)) + q = 0 \tag{7.5}$$

In fact, given the concavity of u in ζ, (7.5) characterizes a solution to (7.4).

Since u is twice continuously differentiable in $int\Omega$, and $u_{22}(x,z) < 0$ for all $(x,z) \in int\Omega$, we can use the implicit function theorem to infer that

$J(q,k)$ is continuously differentiable in Θ.Differentiating the identity (7.5) with respect to q, we obtain:

$$\frac{\partial J(q,k)}{\partial q} = \frac{1}{[-u_{22}(k,k+J(q,k))]} \tag{7.6}$$

By the envelope theorem, one obtains:

$$\frac{\partial H(q,k)}{\partial q} = J(q,k) \tag{7.7}$$

Differentiating this identity with respect to q, and using (7.6), we get an expression for the degree of convexity of the Hamiltonian with respect to q :

$$\frac{\partial^2 H(q,k)}{\partial q^2} = \frac{\partial J(q,k)}{\partial q} = \frac{1}{[-u_{22}(k,k+J(q,k))]} \tag{7.8}$$

Differentiating the identity (7.5) with respect to k, we obtain:

$$u_{21}(k,k+J(q,k)) + u_{22}(k,k+J(q,k))[1+\frac{\partial J(q,k)}{\partial k}] = 0 \tag{7.9}$$

By the envelope theorem, one obtains:

$$\frac{\partial H(q,k)}{\partial k} = u_1(k,k+J(q,k)) + u_2(k,k+J(q,k)) \tag{7.10}$$

Differentiating this identity with respect to k, one obtains an expression for the degree of concavity of the Hamiltonian with respect to k :

$$\begin{aligned} \frac{\partial^2 H(q,k)}{\partial k^2} &= u_{11}(k,k+J(q,k)) + u_{12}(k,k+J(q,k))[1+\frac{\partial J(q,k)}{\partial k}] \quad (7.11) \\ &= u_{11}(k,k+J(q,k)) - \frac{u_{12}(k,k+J(q,k))u_{21}(k,k+J(q,k))}{u_{22}(k,k+J(q,k))} \end{aligned}$$

We can now introduce the Hamiltonian curvature condition of Cass and Shell as follows:

Condition HC: For all $(q,k) \in \Theta$ for which $(k,k+J(q,k)) \in int\Omega$, denote:

$$\alpha(q,k) = \frac{\partial^2 H(q,k)}{\partial k^2}; \beta(q,k) = \frac{\partial^2 H(q,k)}{\partial q^2}; \gamma(q,k) = \frac{\partial J(q,k)}{\partial k}$$

and $(1/(1+\delta))$ by ρ. Then, the following inequality is satisfied:

$$4\alpha(q,k)\beta(q,k) > \rho^2\ [(1+\gamma(q,k))^2/(1+\rho)] \tag{7.12}$$

If the Condition HC is satisfied, then we can proceed to verify Condition S as follows. Let $x \in Y$ be an arbitrary point satisfying $f(x) = 0$. Denote $g(x)$ by z, and note that $x = g(z)$, and that (x,z) and (z,x) are both in the interior of Ω. Further, $z = h(x)$ and $x = h(z)$.Using the fact that $(x,h(x)) \in int\Omega$, we can define $q = -u_2(x,h(x))$, and verify that $[h(x)-x]$ solves the optimization problem (7.4) for $(x,q) \in \Theta$. That is, $[h(x)-x] = J(q,x)$, so that $(x, x+J(q,x)) \in int\Omega$. Then, by Condition HC,

$$4\alpha(q,x)\beta(q,x) > \rho^2\ [(1+\gamma(q,x))^2/(1+\rho)] \tag{7.13}$$

Similarly, using the fact that $(z,h(z)) \in int\Omega$, we can define $p = -u_2(z,h(z))$, and verify that $[h(z)-z]$ solves the optimization problem (7.4) for $(z,p) \in \Theta$. That is, $[h(z)-z] = J(p,z)$, so that $(z, z+J(p,z)) \in int\Omega$. Then, by Condition HC,

$$4\alpha(p,z)\beta(p,z) > \rho^2\ [(1+\gamma(p,z))^2/(1+\rho)] \tag{7.14}$$

We can re-write (7.13) as follows:

$$4\frac{[u_{11}(x,z)u_{22}(x,z) - u_{12}(x,z)^2]}{u_{22}(x,z)^2} > \frac{(1-\delta)^2 u_{12}(x,z)^2}{\delta u_{22}(x,z)^2} \tag{7.15}$$

This inequality simplifies to:

$$\delta u_{11}(x,z)u_{22}(x,z) > [(1+\delta)^2/4]u_{12}(x,z)^2 \tag{7.16}$$

Since the right-hand side expression in (7.16) is equal to $\{[u_{21}(x,z) + \delta\ u_{12}(x,z)]/2\}^2$, and $u_{11}(x,z) < 0$, we can infer from (7.16) that the matrix:

$$Q = \begin{bmatrix} \delta u_{11}(x,h(x)) & \delta u_{12}(x,h(x)) \\ u_{21}(x,h(x)) & u_{22}(x,h(x)) \end{bmatrix}$$

is quasi negative definite. Similarly, starting from the inequality (7.14), we can show that the matrix:

$$R = \begin{bmatrix} \delta u_{11}(h(x),x) & \delta u_{12}(h(x),x) \\ u_{21}(h(x),x) & u_{22}(h(x),x) \end{bmatrix}$$

is quasi negative definite. Now, simply note that in our analysis of the Q-Condition, these two pieces of information were all that were used to verify that Condition S is satisfied.

7.3.3 The Dominant Diagonal Condition of Araujo and Scheinkman

Araujo and Scheinkman (1977) use a dominant diagonal condition on the matrix of second-order partials of the utility function to obtain the turnpike property. The discount factor does not explicitly appear in the statement of this condition, but it should be remembered that if the dominant diagonal condition is required to hold along the optimal path, then since the point of evaluation of the second-order partials depends on the discount factor, so does the validity of the dominant diagonal condition.

Let us state this condition in the following way:

Condition DD: For all $x \in Y$, we have $(x, h(x)) \in int\Omega$, and:

$$|u_{11}(x, h(x))| > |u_{12}(x, h(x))| \; and \; |u_{22}(x, h(x))| > |u_{21}(x, h(x))| \tag{7.17}$$

If the Condition DD is satisfied, then we can proceed to verify Condition S as follows. Let $x \in Y$ be an arbitrary point satisfying $f(x) = 0$. Denote $g(x)$ by z, and note that $x = g(z)$, and that (x, z) and (z, x) are both in the interior of Ω. Further, $z = h(x)$ and $x = h(z)$. Then, by Condition DD, we have:

$$|u_{11}(x, h(x))| > |u_{12}(x, h(x))| \; and \; |u_{22}(x, h(x))| > |u_{21}(x, h(x))|$$

and:

$$|u_{11}(h(x), x)| > |u_{12}(h(x), x)| \; and \; |u_{22}(h(x), x)| > |u_{21}(h(x), x)|$$

Since $u_{11}(x, z)$, $u_{11}(z, x)$, $u_{22}(x, z)$, $u_{22}(z, x)$ are all negative, we have:

$$\begin{aligned} & [u_{22}(h(x), x) + \delta u_{11}(x, h(x))][u_{22}(x, h(x)) + \delta u_{11}(h(x), x)] \\ > \; & [|u_{21}(x, h(x))| + \delta |u_{12}(h(x), x)|][|u_{21}(h(x), x)| + \delta |u_{12}(x, h(x))|] \\ \geq \; & [u_{21}(x, h(x)) + \delta u_{12}(h(x), x)][u_{21}(h(x), x) + \delta u_{12}(x, h(x))] \end{aligned}$$

which verifies Condition S.

8 The Weitzman-Samuelson Example

In this section, we analyze the example due to Weitzman, discussed in Samuelson (1973). This is a convenient example in which we can evaluate the sharpness of the sufficient conditions for global asymptotic stability, discussed in the previous section.

The Weitzman-Samuelson example can be described as follows. A representative agent has one unit of labor. She can allocate it between two sectors: one of these sectors produces bread, the other produces grape juice. It takes one unit of labor per unit of bread production, and one unit of labor per unit of grape juice production. A third sector produces wine, and its production process requires one unit of grape juice per unit of wine, and one period of time (for the juice to ferment). As a consumer, the agent derives satisfaction from the consumption of bread and wine, but not from grape juice.

Let $x(t)$ denote the allocation of labor in period t to grape-juice production, with $x(0)$ (initial allocation) as given, and equal to $x > 0$. Then, $(1 - x(t))$ of labor is allocated to production of bread in period t. Thus, in period $(t + 1)$, where $t \geq 0$, we have bread production of $(1 - x(t + 1))$, grape-juice production of $x(t + 1)$, and wine production of $x(t)$.

Assume that the consumer's felicity function is:

$$\phi(b, w) = b^{\beta} w^{\alpha} \quad with \quad (\alpha, \beta) \gg 0, (\alpha + \beta) \leq 1$$

where b and w are the amounts of bread and wine consumed.

The agent's optimization problem can be written as:

$$\left.\begin{array}{lll} Maximize & \sum_{t=0}^{\infty} \delta^t \, \phi(b(t+1), w(t+1)) & \\ subject\ to & b(t+1) = 1 - x(t+1) & for\ t \geq 0 \\ & w(t+1) = x(t) & for\ t \geq 0 \\ & 0 \leq x(t) \leq 1 & for\ t \geq 0 \\ & x(0) = x > 0 & \end{array}\right\} \tag{8.1}$$

To convert problem (8.1) to its reduced form, we can define the state space $X = [0, 1]$, the transition possibility set, $\Omega = \{(x, z) \in X^2\}$, and the utility function, u, to be:

$$u(x, z) = \phi(1 - z, x) = x^{\alpha}(1 - z)^{\beta} \tag{8.2}$$

Then an optimal solution to (8.1) corresponds exactly to an optimal solution to (1.1). In what follows, we restrict attention to utility functions for which $(\alpha + \beta) < 1$, and $\alpha > (1/2) > \beta$.

8.1 Verifying Turnpike Behavior

The complete bifurcation analysis of this model, examining the nature of the long-run behavior of optimal paths as the parameters are varied, was undertaken in the paper by Mitra and Nishimura(2001). We can summarize the results obtained there as follows. There is a critical discount factor:

$$\bar{\delta} = [\beta(2\alpha - 1)/\alpha(1 - 2\beta)] \tag{8.3}$$

such that (i) if $\bar{\delta} < \delta < 1$, there is a unique stationary optimal stock, $k(\delta) \in Y$, given by:

$$k(\delta) = \delta\alpha/(\beta + \delta\alpha) \tag{8.4}$$

and every optimal path $(x(t))_0^\infty$ from $x \in Y$ converges to this stationary optimal stock as $t \to \infty$; (ii) if $0 < \delta < \bar{\delta}$, then there is a unique stationary optimal stock, $k(\delta) \in Y$, given by (8.4), and two stocks, $k^1(\delta)$ and $k^2(\delta)$, with $0 < k^1(\delta) < k(\delta) < k^2(\delta) < 1$, such that every optimal path $(x(t))_0^\infty$ from $x \in Y$ converges to the period-two cycle, $(k^1(\delta), k^2(\delta))$, as $t \to \infty$. Thus, $\bar{\delta}$, given by (8.3), is a bifurcation value of the discount factor, separating the regions of turnpike and cyclical behavior in the long-run.

We can contrast this bifurcation value of the discount factor from the lower bound on the discount factor for global asymptotic stability indicated by the Q-condition of Brock and Scheinkman, or equivalently the Hamiltonian curvature condition of Cass and Shell (and Rockafellar). The Q-condition is equivalent to (7.16) holding, which yields the inequality:

$$\frac{(1-\alpha)(1-\beta)}{\alpha\beta} > \frac{(1+\delta)^2}{4\delta}$$

This, in turn, can be rewritten as:

$$\frac{4\delta}{(1-\delta)^2} > \frac{\alpha\beta}{1-\alpha-\beta} \tag{8.5}$$

The function $\theta(\delta) = [4\delta/(1-\delta)^2]$ is a continuous, monotone increasing function of δ on Y, with $\theta(\delta) \to \infty$ as $\delta \to 1$, and $\theta(\delta) \to 0$ as $\delta \to 0$. Thus, there is a unique solution, $\hat{\delta} \in Y$, to the equation:

$$\theta(\delta) = \frac{\alpha\beta}{1-\alpha-\beta} \tag{8.6}$$

The result of Brock and Scheinkman and of Cass and Shell (discussed in Section 7) then says that if $1 > \delta > \hat{\delta}$, the turnpike property holds.

To see most clearly how strong this discount factor restriction $\hat{\delta}$ is, compared with what is really needed for the turnpike property $(1 > \delta > \bar{\delta})$, consider the following construct. Let n be any positive number, greater than 1. Define $\alpha(n) = (1/2) + \varepsilon$, $\beta(n) = (1/2) - n\varepsilon$, where $0 < \varepsilon < (1/2n)$. Then, using (8.3), the bifurcation value of the discount factor is:

$$\bar{\delta}(n) = (1/n)\frac{[0.5 - n\varepsilon]}{[0.5 + \varepsilon]} \rightarrow (1/n) \; as \; \varepsilon \rightarrow 0 \tag{8.7}$$

On the other hand, using (8.6), the discount factor, $\hat{\delta}(n)$, satisfies the equation:

$$\theta(\hat{\delta}(n)) = \frac{(0.5 + \varepsilon)(0.5 - n\varepsilon)}{(n-1)\varepsilon} \rightarrow \infty \; as \; \varepsilon \rightarrow 0 \tag{8.8}$$

Thus, if we start with the number n large enough, (8.7) says that for all discount factors except for really low ones, the turnpike property holds. But, (8.8) indicates that the results of Brock and Scheinkman and of Cass and Shell are able to assert the turnpike property only for discount factors very close to 1 [since $\hat{\delta}(n)$ has to approach 1, to make $\theta(\hat{\delta}(n))$ approach infinity, by definition of the function, θ]. As a numerical example, choosing $n = 11$, and $\varepsilon = 0.0000164$, one can check that $\bar{\delta}(n) \approx 0.09$, while $\hat{\delta}(n) \approx 0.95$.

9 Sufficient Conditions for Cycles

In this section, we use our characterization of the turnpike property to obtain conditions for the existence of period-two cycles. We indicate how the sufficient condition for cycles used by Benhabib and Nishimura (1985) arises quite naturally from this approach. An example is also provided to show that their condition is not necessary to generate locally stable period-two cycles.

9.1 Violating Condition U

In order to produce optimal (equivalently, competitive) cycles in our framework, it is necessary to violate Condition U. However, violating Condition U in an arbitrary way will not be sufficient to produce optimal cycles. If, for example, the utility function, u, is supermodular, then the optimal policy function, h, will be monotone increasing, so there will be no optimal cycles. However, this scenario is quite consistent with multiple stationary optimal stocks in Y, so that Condition U will be violated. If we assume that there is a *unique* stationary optimal stock, $\bar{x} \in Y$, and we *violate* Condition U, then we must have a period-two optimal cycle. However, uniqueness of a stationary optimal stock in Y might be too strong a restriction in this context (although it clearly is not in establishing the turnpike property).We, therefore, develop our analysis along somewhat more general lines, keeping in mind that we seek easily verifiable conditions which would violate Condition U, and produce optimal cycles.

To this end, for this section, we add to our set of maintained assumptions:

(A.10) The function, a, satisfies the condition: $\lim_{x \to 0} a(x) = 0$.

This is a natural assumption to make in many models of economic growth, and in particular is satisfied in the standard versions of the one-sector and two-sector models[9].

Under our assumptions, we know that there is a stationary optimal stock, $k(\delta) \in Y$. Let $S = \{y \in Y : y \text{ is a stationary optimal stock}\}$. Using (A.6), we can find $\varepsilon > 0$, such that if $y \in S$, then $y \geq \varepsilon$. We define $\bar{x}$ to be the infimum of the set S. Then, $\bar{x}$ is a stationary optimal stock in Y, $\bar{x} \geq \varepsilon$, and for every $y \in S$, we have $y \geq \bar{x}$. That is, $\bar{x}$ is the smallest stationary optimal stock in Y.

A natural suggestion for *violating* Condition U is to assume that the following condition holds.

Condition V: At the minimal stationary optimal stock, $\bar{x} \in Y$, we have $f'(\bar{x}) > 0$.

We show now that the minimality of the stationary optimal stock, $\bar{x} \in Y$, and Condition V, lead to a violation of Condition U.

First, we claim that for all $x \in (0, \bar{x})$, we must have $g(x) > x$. If this was not the case, then because $\bar{x}$ is a minimal stationary optimal stock in Y, we must have $g(x) < x$ for all $x \in (0, \bar{x})$ [since any $x \in Y$, satisfying $g(x) = x$ is a stationary optimal stock in Y, and g is continuous on Y]. This implies

[9] It is not satisfied in the Weitzman-Samuelson example, discussed in the previous section. It is perhaps possible to proceed without making this assumption; it is certainly convenient for our analysis to proceed with it.

that, since the function $(-u_2)$ is increasing in the second argument, we have:

$$[-u_2(x, g(x))] \leq [-u_2(x, x)] \tag{9.1}$$

and, since the function u_1 is decreasing in the first argument, we also have:

$$\delta u_1(g(x), x) \geq \delta u_1(x, x) \tag{9.2}$$

Since $[-u_2(x, g(x))] = \delta u_1(g(x), x)$, we obtain from (9.1) and (9.2):

$$[-u_2(x, x)] \geq \delta u_1(x, x) \tag{9.3}$$

But, for x close to 0, (9.3) clearly violates assumption (A.6). This establishes our claim.

Next, we claim that there is some $\hat{x} \in (0, \bar{x})$ for which $f(\hat{x}) = 0$. Notice that $f(\bar{x}) = 0$, and by Condition V, we must have $f(x) < 0$ for those $x \in (0, \bar{x})$ which are sufficiently close to $\bar{x}$. Thus, if our claim is not true, we must have (by continuity of f on Y) $f(x) < 0$ for all $x \in (0, \bar{x})$. That is, we must have for all $x \in (0, \bar{x})$:

$$[-u_2(g(x), x)] > \delta u_1(x, g(x)) \tag{9.4}$$

Now, since the function $(-u_2)$ is increasing in the second argument, and $g(x) > x$ for $x \in (0, \bar{x})$, we have:

$$[-u_2(g(x), g(x))] \geq [-u_2(g(x), x)] \tag{9.5}$$

And, since the function u_1 is decreasing in the first argument, and $g(x) > x$ for $x \in (0, \bar{x})$, we also have:

$$\delta u_1(g(x), g(x)) \leq \delta u_1(x, g(x)) \tag{9.6}$$

Combining (9.4), (9.5) and (9.6), we obtain for $x \in (0, \bar{x})$,

$$[-u_2(g(x), g(x))] > \delta u_1(g(x), g(x)) \tag{9.7}$$

Letting $x \to 0$, we note that since $g(x) \leq a(x)$, we must have $g(x) \to 0$, by using assumption (A.10). But, then, (9.7) violates assumption (A.6), and establishes our claim. The existence of $\hat{x} \in (0, \bar{x})$, with $f(\hat{x}) = 0$ clearly violates Condition U. But, more important, since $\bar{x}$ is the minimal stationary optimal stock, we can infer that $\hat{x}$ is *not* a stationary optimal stock. Thus, $\hat{y} \equiv g(\hat{x}) = h(\hat{x})$ is distinct from $\hat{x}$, and $(\hat{x}, \hat{y})$ is a period-two cycle.We summarize our findings in the following proposition.

Proposition 7 *If $\bar{x} \in Y$ is the minimal stationary optimal stock in Y, then, $g(x) > x$ for all $x \in (0, \bar{x})$. If, further, Condition V is satisfied, then Condition U is violated, and there is a period two cycle.*

We can now proceed to relate Condition V to a suitable condition on the utility function and the discount factor. Note first that, if $\bar{x} \in Y$ is the minimal stationary optimal stock in Y, then using (4.5) and the fact that $g(\bar{x}) = \bar{x}$, we have:

$$g'(\bar{x}) = (1+\delta)u_{12}(\bar{x}, \bar{x})/\{[-u_{22}(\bar{x}, \bar{x})] + \delta[-u_{11}(\bar{x}, \bar{x})]\} \tag{9.8}$$

Since $g(x) > x$ for all $x \in (0, \bar{x})$, and $g(\bar{x}) = \bar{x}$, we must have $g'(\bar{x}) \leq 1$; that is, using (9.8), we have:

$$(1+\delta)u_{12}(\bar{x}, \bar{x}) \leq [-u_{22}(\bar{x}, \bar{x})] + \delta[-u_{11}(\bar{x}, \bar{x})] \tag{9.9}$$

Note, next, that from the definition of f, we have for $x \in Y$:

$$f'(x) = u_{21}(g(x), x)g'(x) + u_{22}(g(x), x) + \delta u_{11}(x, g(x)) + \delta u_{12}(x, g(x))g'(x) \tag{9.10}$$

Thus, using $g(\bar{x}) = \bar{x}$, (9.8) and (9.10), the condition that $f'(\bar{x}) > 0$ is equivalent to:

$$(1+\delta)^2[u_{12}(\bar{x}, \bar{x})]^2 > \{[-u_{22}(\bar{x}, \bar{x})] + \delta[-u_{11}(\bar{x}, \bar{x})]\}^2 \tag{9.11}$$

Comparing (9.9) and (9.11), and using (9.8), we see that we must have simultaneously $g'(\bar{x}) \leq 1$, and $[g'(\bar{x})]^2 > 1$. This means, of course, that $g'(\bar{x})$ is negative, and in fact, $g'(\bar{x}) < -1$. In particular, this forces the cross-partial of u at $(\bar{x}, \bar{x})$ to be negative, and:

$$(1+\delta)[-u_{12}(\bar{x}, \bar{x})] > [-u_{22}(\bar{x}, \bar{x})] + \delta[-u_{11}(\bar{x}, \bar{x})]$$

which is the condition used by Benhabib and Nishimura (1985) to generate optimal period-two cycles. In fact, because they assume that $u_{12}(x, z) < 0$ for all $(x, z) \in int\Omega$, the policy function, h, is downward sloping for all $x \in Y$, for which $(x, h(x)) \in int\Omega$. Consequently, their assumptions ensure that a stationary optimal stock in Y is necessarily unique. Our analysis above, together with Proposition 7 yields the following result, which can be compared to Benhabib and Nishimura (1985, Theorem 1, p.288).

Corollary 1 *If $\bar{x} \in Y$ is the unique stationary optimal stock in Y, then there is a period-two optimal cycle if:*

$$(1+\delta)[-u_{12}(\bar{x},\bar{x})] > [-u_{22}(\bar{x},\bar{x})] + \delta[-u_{11}(\bar{x},\bar{x})] \qquad \textit{(BN)}$$

Remark 1 *Benhabib and Nishimura (1985) obtain their result by looking at the characteristic roots of the Jacobian associated with a Ramsey-Euler path at the stationary optimal stock. Our approach has been to see this result as arising out of violating Condition U, which we have shown completely characterizes the turnpike property.*

9.2 Cycles with a Locally Stable Stationary Optimal Stock

Condition BN is not *necessary* for the existence of cycles. As indicated in Benhabib and Nishimura (1985), when condition BN holds, then both the characteristic roots of the Jacobian associated with the Ramsey-Euler path at the stationary optimal stock are less than (-1), and so the optimal policy function also has a slope less than (-1) at the stationary optimal stock. Thus, as the discount factor moves away from 1,cycles emerge in this case when the optimal policy function becomes locally unstable at the stationary optimal stock. This is the standard scenario in the case of a flip bifurcation.

However, period-two cycles need not always emerge in this way. We present an example now of a case in which the unique stationary optimal stock in Y is locally stable, but there is also a (locally stable) period-two cycle. In this case, the analysis of Benhabib and Nishimura indicates that Condition BN must be violated since one of the characteristic roots of the Jacobian associated with the Ramsey-Euler path at the stationary optimal stock must now be greater than (-1).

We start to describe the example by first writing down a function with the features that we desire of our optimal policy function, and then using the method of Boldrin and Montrucchio (1986) to rationalize this function as an optimal policy function arising from a suitably constructed dynamic optimization model. The function we describe has been studied in detail by Singer (1978):

$$h(x) = 7.86x - 23.31x^2 + 28.75x^3 - 13.30x^4 \; for \; all \; x \in X \qquad (9.12)$$

The function h satisfies $h(0) = h(1) = 0$. It is increasing on $[0, \hat{x}]$ and decreasing on $[\hat{x}, 1]$, where $\hat{x} \approx 0.3239$. It can be checked that it is concave on

X, and $|h'(x)| < 8$, $|h''(x)| < 210$ for all $x \in X$. The unique fixed point of h in Y is $\bar{x}$, where $\bar{x} \approx 0.7264$, and $h'(\bar{x}) \approx -0.8854$.

We can define an economy (Ω, u, δ) as follows. First, the transition possibility set is described by:

$$\Omega = \{(x, z) \in X^2 : z \leq a(x)\} \tag{9.13}$$

where:

$$a(x) = \begin{cases} 8x & for\ x \in [0, (1/8)] \\ 1 & for\ x \in ((1/8), 1] \end{cases} \tag{9.14}$$

Next, the utility function, $u : \Omega \to \mathbb{R}$ is defined by:

$$u(x, z) = Ax - 0.5Bx^2 + zh(x) - 0.5z^2 - 0.0008[Az - 0.5Bz^2 + \{h(z)\}^2] \tag{9.15}$$

where:

$$A = 270\ and\ B = 128 \tag{9.16}$$

Finally, the discount factor, δ, is specified as $\delta = 0.0008$. It can be checked that u, defined by (9.15) and (9.16) is twice continuously differentiable on Ω, with:

$$\begin{aligned} u_1(x, z) &> 0, u_2(x, z) < 0; \\ u_{11}(x, z) &< 0, u_{22}(x, z) < 0, [u_{11}(x, z)u_{22}(x, z)] > [u_{12}(x, z)]^2 \end{aligned} \tag{9.17}$$

Following Boldrin and Montrucchio (1986), it can also be verified that $h(x)$ is the optimal policy function of the dynamic optimization model (Ω, u, δ).

Thus, the unique stationary optimal stock of the model (Ω, u, δ) in Y is given by $\bar{x}$, and since $h'(\bar{x}) = -0.8854$, it is locally stable. It can be checked, following Singer (1978), that there is a period-two cycle (x', z'), where $x' \approx 0.3218$, $z' \approx 0.9309$, and that $h'(x')h'(z') \approx -0.06236$, so that the period-two cycle is also locally stable.

10 Cycles in the Two-Sector Model

In this section, we apply the sufficient conditions, developed in section 9, to the well-known two-sector model, and show how the capital-intensity condition for the existence of cycles, due to Benhabib and Nishimura (1985), arises naturally from this approach.

The well-known two-sector model of optimal economic growth can be described as follows. Production uses two inputs, capital and labor. Given the total amounts of capital and labor available to the economy, the inputs are allocated to two sectors of production, the consumption good sector, and the investment good sector. Output of the former sector is consumed (and cannot be used for investment purposes), and output of the latter sector is used to augment the capital stock of the economy (and cannot be consumed). Thus, the consumption-investment decision amounts to a decision regarding allocation of capital and labor between the two production sectors.

The discounted sum of welfares obtained from outputs of the consumption good sector is to be maximized to arrive at the appropriate sectoral allocation of inputs in each period. We use an appropriate welfare function on the output of the consumption good sector, instead of the output of the consumption good sector itself, in the objective function, in order to ensure interior optimal solutions.

Formally, the model is specified by (F, G, d, w, δ), where:
(a) the production function in the consumption good sector, $F : \mathbb{R}^2_+ \to \mathbb{R}_+$, satisfies:
(i) F is continuous and homogeneous of degree one on $\mathbb{R}^2_+$, and twice continuously differentiable on $\mathbb{R}^2_{++}$.
(ii) F is non-decreasing and concave on $\mathbb{R}^2_+$, with $F_i > 0$ for $i = 1, 2$, and $F_{ii} < 0$ for $i = 1, 2$ on $\mathbb{R}^2_{++}$.
(iii) $F(1, 0) = F(0, 1) = 0$; $F_1(K, 1) \to \infty$ as $K \to 0, F_1(K, 1) \to 0$ as $K \to \infty$.
(b) the production function in the investment good sector, $G : \mathbb{R}^2_+ \to \mathbb{R}_+$, satisfies:
(i) G is continuous and homogeneous of degree one on $\mathbb{R}^2_+$, and twice continuously differentiable on $\mathbb{R}^2_{++}$.
(ii) G is non-decreasing and concave on $\mathbb{R}^2_+$, with $G_i > 0$ for $i = 1, 2$, and $G_{ii} < 0$ for $i = 1, 2$ on $\mathbb{R}^2_{++}$.
(iii) $G(1, 0) = G(0, 1) = 0$; $G_1(K, 1) \to \infty$ as $K \to 0$, and $G_1(K, 1) \to 0$ as $K \to \infty$.
(c) the depreciation factor, d, satisfies $0 < d \leq 1$.
(d) the welfare function, $w : \mathbb{R}_+ \to \mathbb{R}$ is continuous, strictly increasing and strictly concave on $\mathbb{R}_+$; further, it is twice continuously differentiable on $\mathbb{R}_{++}$, with $w'(c) > 0$ and $w''(c) < 0$ for all $c \in \mathbb{R}_{++}$, and $w'(c) \to \infty$ as $c \to 0$.
(e) the discount factor, δ, satisfies $0 < \delta < 1$.

The optimal growth problem can be written as:

$$\left.\begin{array}{lll} Maximize & \sum_{t=0}^{\infty} \delta^t w(c(t+1)) & \\ subject\ to & c(t+1) = F(K(t), L(t)) & for\ t \geq 0 \\ & x(t+1) = G(x(t) - K(t), 1 - L(t)) + (1-d)x(t) & for\ t \geq 0 \\ & 0 \leq K(t) \leq x(t), 0 \leq L(t) \leq 1 & for\ t \geq 0 \\ & x(0) = x > 0 & \end{array}\right\} \tag{10.1}$$

Here, $x(t)$ is the total capital available at date t, which is allocated between the consumption good sector, $(K(t))$ and the investment good sector $(x(t) - K(t))$. Labor is exogenously available at a constant amount (normalized to unity), which is allocated between the consumption good sector $(L(t))$ and the investment good sector $(1-L(t))$. Note that an exogenously growing labor force (at a constant growth rate) can be accomodated easily by interpreting $K(t)$ and $x(t)$ as per-worker capital stocks, and reinterpreting F, G and d suitably.

Clearly, one can find a unique value $B > 0$, such that $G(B,1)+(1-d)B = B$; then B is the maximum sustainable capital stock. One can check that if $x \in [0, B]$, then for every sequence $(x(t), K(t), L(t), c(t+1))_{t=0}^{\infty}$, satisfying the constraints of the optimization problem (10.1), we have $x(t) \in [0, B]$ for $t \geq 0$. Thus, $[0, B]$ is the natural state space. We assume, without loss of generality, that $B = 1$, so we can take $X = [0, 1]$ to be the state space.

To convert the optimization problem to its reduced form, we can define the transition possibility set, Ω, as:

$$\Omega = \{(x, z) \in \mathbb{R}_+^2 : z \leq G(x, 1) + (1-d)x\} \tag{10.2}$$

Further, we can define a function, v, from Ω to $\mathbb{R}$, as:

$$\left.\begin{array}{lll} v(x,z) = & Max & F(K, L) \\ & subject\ to & 0 \leq K \leq x, 0 \leq L \leq 1 \\ & and & z \leq G(x-K, 1-L) + (1-d)x \end{array}\right\} \tag{10.3}$$

Also, we can define a utility function, u, from Ω to $\mathbb{R}$ as: $u(x, z) = w(v(x, z))$.Then a solution to (10.1) corresponds exactly to a solution to (1.1). In what follows, for simplicity, we shall consider only the case of full-depreciation ($d = 1$). Also, we will restrict our analysis to welfare functions of the form $w(c) = c^{1-\eta}$, where $0 < \eta < 1$.

Given $(x, z) \in \Omega$, there exists a solution $K(x, z)$, $L(x, z)$ to problem (10.3). When the solution to (10.3) involves "incomplete specialization", that is, $0 < K(x,z) < x$, and $0 < L < 1$, we can characterize the solution in terms of familiar first-order conditions:

$$\frac{F_1(K(x,z),L(x,z))}{G_1(x-K(x,z),1-L(x,z))} = \frac{F_2(K(x,z),L(x,z))}{G_2(x-K(x,z),1-L(x,z))} \tag{10.4}$$

Consider a stationary optimal stock, $x \in Y$. Then, since $G(x-K(x,x),1-L(x,x)) \geq x > 0$, the end-point condition (b)(iii) implies that $K(x,x) < x$, and $L(x,x) < 1$. Also, since the stationary sequence $(x,x,x,...)$ is optimal from x, we must have $F(K(x,x),L(x,x)) > 0$, so that by end-point condition (a)(iii), we must have $K(x,x) > 0$, and $L(x,x) > 0$. Thus, we have incomplete specialization at any stationary optimal stock in Y.

Given $(x,z) \in \Omega$, such that we have incomplete specialization,we can use the envelope theorem to obtain:

$$v_1(x,z) = F_1(K(x,z),L(x,z)) \tag{10.5}$$

and:

$$v_2(x,z) = -\frac{F_1(K(x,z),L(x,z))}{G_1(x-K(x,z),1-L(x,z))} \tag{10.6}$$

We can use (10.5) and (10.6) to obtain expressions for the second-order partials of v. To write these expressions in the standard form, let us identify the consumption good sector as the first sector, and the investment good sector as the second sector. Similarly, let us label capital as the first input and labor as the second input. Then, identifying the first subscript (i) with the input, and the second subscript (j) with the sector, $a_{ij}(x,z)$ is the amount of input i used in sector j per unit of output of sector j, where the amounts of inputs chosen solve problem (9.3) for the given (x,z). For example, in terms of our above notation, we have $a_{21}(x,z) = L(x,z)/F(K(x,z),L(x,z))$.

Let us denote by $b(x,z)$ the following expression, involving the a_{ij} terms:

$$b(x,z) = a_{12}(x,z)\left[\frac{a_{22}(x,z)}{a_{12}(x,z)} - \frac{a_{21}(x,z)}{a_{11}(x,z)}\right] \tag{10.7}$$

One can then establish the following relationships for the second partials of v at those $(x,z) \in \Omega$, where there is incomplete specialization:

$$v_{12}(x,z) = v_{21}(x,z) = b(x,z)[-v_{11}(x,z)] \tag{10.8}$$

and:

$$v_{22}(x,z) = [b(x,z)]^2 v_{11}(x,z) \tag{10.9}$$

We observe that a stationary optimal stock in Y is determined entirely by (F, G, δ), and is independent of w. The equations governing any stationary optimal stock, $x \in Y$, are:

$$\delta G_1(x - K(x,x), 1 - L(x,x)) = 1 \tag{10.10}$$

and:

$$\frac{F_2(K(x,x), L(x,x))}{G_2(x - K(x,x), 1 - L(x,x))} = \frac{F_1(K(x,x), L(x,x))}{G_1(x - K(x,x), 1 - L(x,x))} \tag{10.11}$$

Note that (10.10) determines the capital intensity (a_{22}/a_{12}) in the investment good sector uniquely, independent of x. Then, (10.11) determines the capital intensity (a_{21}/a_{11}) in the consumption good sector uniquely, independent of x. We assume that the consumption good sector is more capital-intensive than the investment good sector; that is:

$$\left[\frac{a_{22}}{a_{12}} - \frac{a_{21}}{a_{11}}\right] < 0$$

It follows that there is a unique stationary optimal stock in Y. For, if there were two, then the larger one has to allocate more labor to the consumption good sector, since the capital intensities in the two sectors are fixed at either stationary optimal stock, and the consumption good sector is more capital intensive. Thus, corresponding to the larger stationary optimal stock, there will be less labor allocated to the investment good sector. Because the capital intensities are fixed at either stationary optimal stock, there must also be less capital allocated to the investment good sector at the larger stationary optimal stock. And this leads to the contradiction that a lower labor input and a lower capital input produces the larger stock.

Let $\bar{x}$ denote the stationary optimal stock in Y. The uniquely determined capital intensities then determine $K(\bar{x}, \bar{x})$ and $L(\bar{x}, \bar{x})$ uniquely. Let us denote $K(\bar{x}, \bar{x})$ by $\bar{K}$, $L(\bar{x}, \bar{x})$ by $\bar{L}$, and $F(\bar{K}, \bar{L})$ by $\bar{c}$. Then the second partials of u at $(\bar{x}, \bar{x})$ can be expressed as follows:

$$u_{11}(\bar{x}, \bar{x}) = w'(\bar{c})\{v_{11}(\bar{x}, \bar{x}) - [\eta v_1(\bar{x}, \bar{x})^2/\bar{c}]\} \tag{10.12}$$

$$u_{12}(\bar{x}, \bar{x}) = w'(\bar{c})\{v_{12}(\bar{x}, \bar{x}) + [\eta \delta v_1(\bar{x}, \bar{x})^2/\bar{c}]\} \tag{10.13}$$

$$u_{22}(\bar{x}, \bar{x}) = w'(\bar{c})\{v_{22}(\bar{x}, \bar{x}) - [\eta \delta^2 v_1(\bar{x}, \bar{x})^2/\bar{c}]\} \tag{10.14}$$

We can now establish the following sufficient condition for optimal cycles in the two-sector model, which can be compared to Benhabib and Nishimura (1985, Theorem 5, p.302).

Proposition 8 *Suppose the stationary optimal stock* $\bar{x} \in Y$, *satisfies* $b(\bar{x}, \bar{x}) \in (-1, -\delta)$, *then there is* $\bar{\eta} \in (0, 1)$, *such that for all* $\eta \in (0, \bar{\eta})$, *an optimal period-two cycle exists.*

Proof. : Recall from the previous section that Condition V holding is equivalent to the validity of the inequality (9.11).

If the condition $b(\bar{x}, \bar{x}) \in (-1, -\delta)$ holds, then we have $[b(\bar{x}, \bar{x})]^2 \in (\delta^2, 1)$. This implies the validity of the following inequality:

$$(1+\delta)^2[b(\bar{x}, \bar{x})]^2 > \{[b(\bar{x}, \bar{x})]^2 + \delta\}^2 \tag{10.15}$$

Using the expressions in (10.08) and (10.09), this yields:

$$(1+\delta)^2[v_{12}(\bar{x}, \bar{x})]^2 > \{[-v_{22}(\bar{x}, \bar{x})] + \delta[-v_{11}(\bar{x}, \bar{x})]\}^2 \tag{10.16}$$

Now, using (10.12), (10.13) and (10.14), we see that there is $\bar{\eta} \in (0, 1)$, such that for all $\eta \in (0, \bar{\eta})$, we have:

$$(1+\delta)^2[u_{12}(\bar{x}, \bar{x})]^2 > \{[-u_{22}(\bar{x}, \bar{x})] + \delta[-u_{11}(\bar{x}, \bar{x})]\}^2 \tag{10.17}$$

Thus, Condition V holds, and the result follows from Proposition 7. ■

Remark 2 *Benhabib and Nishimura (1985) use a dual approach to obtain their result; our approach is essentially primal. They use a linear welfare function, so strictly speaking their case is not a special case of ours. However, for purposes of comparison, one might view their scenario as arising by letting* $\eta \to 0$ *in our framework.*

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