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On Representation and Weighted Utilitarian Representation of Preference Orders on Finite Streams

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Abstract

In this paper we re-examine the axiomatic basis of the key result on weighted utilitarian representation of preference orders on finite utility streams. We show that a preference order satisfying the axioms of Minimal Individual Symmetry, Invariance and Strong Pareto need not have a representation, and thus in particular a weighted utilitarian representation. The example establishing this result might also be of interest for the literature on the representation of preference orders. We then establish that whenever a preference order satisfying the axioms of Minimal Individual Symmetry, Invariance and Weak Pareto has a representation, it also has a weighted utilitarian representation. Our approach helps us to view the available results on the weighted utilitarian representation theorem from a different perspective.

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1 Introduction

In developing a theory of social preferences (on utility streams of the individuals in society), the (Weak) Pareto axiom is standard. The Weak Pareto axiom requires that society be better off if every individual in the society is better off. If one wants the social preferences to respond to at least some non-Paretian comparisons, there is a compelling case for imposing the axiom of Minimal Individual Symmetry. Minimal Individual Symmetry allows a loss in utility for one individual to be compensated by a gain in utility of another individual keeping society indifferent between the two social states.[1]

Weighted Utilitarian social welfare functions are typically derived by postulating a third axiom, known as an invariance axiom. The Invariance axiom means that in expressing social preferences between two alternatives x and y in the set of social states, X, account is taken *only* of how much each individual gains or loses. Thus, the preference between x and y in X is the same as that between two other alternatives x' and y' in X, if the gain (loss) of each individual is the same in both comparisons; that is, if $(x_i - y_i) = (x'_i - y'_i)$ for all $i \in \{1, 2, ..., n\}$, where n is the number of individuals in the society. Other aspects of the alternatives do not enter into social preferences. In particular, the actual levels of the individuals' utilities do not figure in the determination of social preferences.

While these axioms might look compelling as a basis for weighted utilitarianism, the fact is that together they are unable to guarantee a weighted utilitarian social welfare function to represent the underlying preferences.[2] The difficulty is that a preference order satisfying the three axioms need not have any representation at all, and we provide an example to establish this point.

The example is also of interest for the literature on the representability of preference orderings.[3] The lexicographic order, which is the pre-eminent example of non-representability of a preference order (since Debreu (1954)), satisfies the Strong Pareto axiom and the Invariance Axiom (in the terminology of social choice theory), but it violates the Minimal Individual Symmetry

[1]A strong form of the latter axiom is ensured by the Anonymity or Equity axiom wherein *interchanging* the utilities of any two individuals leaves society indifferent.

[2]This difficulty is not resolved by strengthening the Weak Pareto axiom to the Strong Pareto axiom.

[3]For an account of the literature, and for the key references, see Bridges and Mehta (1995) and Mehta (1998).

axiom, since it does not allow for any substitution possibilities. So, it does not provide the example we are looking for, and it is not at all obvious whether the problem of non-representability remains when the preference order has (in addition to the Strong Pareto and Invariance axioms) such substitution possibilities. In fact, the example we provide is a rather subtle one, dependent on a decomposition of the irrationals on the real line, a result which might itself be of independent interest.

Given the example, the statement of our version of the weighted utilitarian representation (WUR) theorem is different from the versions available in the literature in that it directly links WUR to representation.[4] We establish the result that if the preference order satisfying the three axioms has a representation, then it also has a weighted utilitarian representation. We believe this result is new [5], and it helps us to understand the existing characterizations of weighted utilitarianism (see d'Aspremont and Gevers (2002) for an account of the available characterization results and for references to the related literature).

The above result is obtained through the following route. We first show that the Invariance axiom actually implies *Strong Invariance* for *rational* multiples; that is, if $x, y \in X$, and $x \succsim y$ then $\rho x \succsim \rho y$ for all positive rational ρ. Thus, the difference between the Invariance axiom and the Strong Invariance axiom is that the latter ensures the property just stated for all positive *real* r. [6]

Assuming that a complete preference order satisfying the three axioms has a representation, we use the Weak Pareto axiom to ensure that the representation has some point of continuity on the "diagonal" of the state space, X. Then, using the Strong Invariance for rational multiples, we are able to completely characterize one social indifference curve (passing through this point of continuity) as a set on which a weighted sum of utilities remains constant. Invariance then allows us to characterize all social indifference curves in this way, and this yields the result that the preference order has a

[4]In the available versions, either the Invariance axiom is strengthened to a *Strong* Invariance axiom (see Section 2 for a definition), or a Continuity axiom is postulated on the preference order. As an axiom, Strong Invariance is less compelling than Invariance, even though both are clearly implied whenever one has a weighted utilitarian representation.

[5]In general, a complete preference order which has a real-valued representation need not be continuous (see the Appendix for an example). Thus, our result does not follow from the available characterizations of WUR.

[6]This implies at once that invariance and continuity ensure strong invariance.

weighted utility representation.

2 Preliminaries

2.1 Notation and Definitions

We denote by I the set $\{1, 2, ..., n\}$, by J the set $\{2, ..., n\}$, and by S the set $\{s \in \mathbb{R}^n : s_i > 0 \text{ for all } i \in I \text{ and } \sum_{i=1}^n s_i = 1\}$.We denote by $\mathbb{N}$ the natural numbers, by $\mathbb{Q}$ the rationals in $\mathbb{R}$ (including 0, by convention) and by $\mathbb{Q}_{++}$ the subset of positive rationals.

We refer to $X \equiv \mathbb{R}^n$ as the set of social states. Let $\succsim$ be a social preference order (a complete, transitive binary relation) on $X = \mathbb{R}^n$. We consider the following axioms on this order.

Weak Pareto: For all $x, y \in \mathbb{R}^n$, $x \succ y$ if $x_i > y_i$ for all $i \in \{1, 2, ..., n\}$.

Strong Pareto: For all $x, y \in \mathbb{R}^n$, $x \succ y$ if $x_i \geq y_i$ for all $i \in \{1, 2, ..., n\}$, and $x_i > y_i$ for some $i \in \{1, 2, ..., n\}$.

Minimal Individual Symmetry: For all $i, j \in \{1, 2, ..., n\}$, there exist $x, y \in \mathbb{R}^n$ such that $x_i > y_i$, $x_j < y_j$, $x_k = y_k$ for all $k \in \{1, 2, ..., n\} \backslash \{i, j\}$, and $x \sim y$.

Invariance: For all $x, y \in \mathbb{R}^n$, $x \succsim y$ implies that for all $a \in \mathbb{R}^n$, we have $x + a \succsim y + a$.

Strong Invariance: For all $x, y \in \mathbb{R}^n$, $x \succsim y$ implies that for all $a \in \mathbb{R}^n$, and all $b \in \mathbb{R}_{++}$ we have $(bx + a) \succsim (by + a)$.

Continuity: For all $x \in X$, $LC(x) = \{y \in X : x \succsim y\}$ and $UC(x) = \{y \in X : y \succsim x\}$ are *closed* subsets of X.

2.2 Invariance and Strong Rational Invariance

An elementary implication of the Invariance Axiom is noted below.

Lemma 1 *Suppose $\succsim$ is an order on X satisfying the Invariance axiom. Then,*

(i) If $x, y, a \in X$, and $x \sim y$, then $x + a \sim y + a$.

(ii) If $x, y, a \in X$, and $x \succ y$, then $x + a \succ y + a$.

Proof. (i) Since $x \sim y$, the Invariance axiom implies that $x + a \succsim y + a$. Since $y \sim x$, the Invariance axiom also implies that $y + a \succsim x + a$. Thus, $x + a \sim y + a$ must hold.

(ii) Since $x \succ y$, the Invariance axiom implies that $x + a \succsim y + a$. If $x + a \sim y + a$, then by (i), we have:

$$x = (x + a) + (-a) \sim (y + a) + (-a) = y$$

which contradicts the fact that $x \succ y$. Thus, $x + a \succ y + a$ must hold. ■

A more substantive implication of the invariance axiom is that it implies a version of the Strong Invariance axiom, in which the common multiplicative factor, b, is a positive *rational*. We may define Strong Rational Invariance as follows.

Strong Rational Invariance (SRI):

For all $x, y, a \in X$ and $b \in \mathbb{Q}_{++}$, $x \succsim y$ implies $(bx + a) \succsim (by + a)$.

Lemma 2 *If $\succsim$ is an order on X satisfying the Invariance axiom, then it also satisfies the Strong Rational Invariance axiom.*

Proof. We first show that:

$$\text{If } x, y \in X, \text{and } x \succsim y, \text{then } nx \succsim ny \text{ for all } n \in \mathbb{N} \tag{1}$$

Clearly, (1) is true for $n = 1$. Assume that it is true for $n = m$. Then, we have $mx \succsim my$. By Invariance, we have:

$$mx + x \succsim my + x \tag{2}$$

Also, using Invariance and $x \succsim y$ we have:

$$my + x \succsim my + y \tag{3}$$

Using (2) and (3) and transitivity of $\succsim$, we have $(m + 1)x \succsim (m + 1)y$. This proves (1) by induction.

Next, we show that:

$$\text{If } x, y \in X, \text{and } x \succsim y, \text{then } rx \succsim ry \text{ for all } r \in \mathbb{Q}_{++} \tag{4}$$

Given $r \in \mathbb{Q}_{++}$, we can find positive integers m and n such that $r = (m/n)$. Define $x' = (x/n)$ and $y' = (y/n)$. Note that $x', y' \in X$. We claim that:

$$x' \succsim y' \tag{5}$$

For if (5) does not hold, then $n > 1$, and by completeness of $\succsim$, we must have $y' \succ x'$. Then $(n-1) \in \mathbb{N}$ and by applying (1), we have:

$$(n-1)y' \succsim (n-1)x' \tag{6}$$

Also, using $y' \succ x'$ and Lemma 1(ii), we have:

$$ny' \equiv y' + (n-1)y' \succ x' + (n-1)y' \tag{7}$$

and using (6),

$$x' + (n-1)y' \succsim x' + (n-1)x' \equiv nx' \tag{8}$$

Then transitivity of $\succsim$ and (7),(8) yield:

$$y = ny' \succ nx' = x$$

contradicting the fact that $x \succsim y$. Thus, (5) must hold. Now, applying (1), we have:

$$rx = (m/n)x = mx' \succsim my' = (m/n)y = ry$$

which establishes (4). Combining Invariance with (4), we obtain Strong Rational Invariance. ■

Remark:

Since $\mathbb{Q}_{++}$ is dense in $\mathbb{R}_{++}$, we see from Lemma 2 that if $\succsim$ is continuous, then Invariance implies Strong Invariance.

2.3 Weights Associated with a Preference Order

Given a preference order $\succsim$ satisfying the Invariance axiom and the axiom of Minimal Individual Symmetry (MIS), we can associate with it certain social welfare weights, which we now describe. These weights will be precisely the weights placed on the utilities of the individuals in the social welfare function, when the preference order has a weighted utility representation (as we will see in Section 4).

Using MIS, we can find for each $j \in J$, a vector $z^{(j)} \in \mathbb{R}^n$ and $\alpha_j, \beta_j \in \mathbb{R}_{++}$, such that:

$$z^{(j)} \sim z^{(j)} + \alpha_j e^{(1)} - \beta_j e^{(j)} \tag{9}$$

where $e^{(j)}$ is the j-th unit vector in $\mathbb{R}^n$. Using (9) and Invariance, for each $j \in J$, we have:

$$0 \sim \alpha_j e^{(1)} - \beta_j e^{(j)} \tag{10}$$

Define $\gamma_1 = 1$, and for each $j \in J$, define:

$$\gamma_j \equiv (\alpha_j / \beta_j) \tag{11}$$

Then, denoting $(\gamma_1 + \cdots + \gamma_n)$ by σ, and (γ_i / σ) by q_i for each $i \in I$, we see that $q \in S$. This $q \in S$ will be the vector of weights associated with the preference order.

2.4 On States Socially Indifferent to the Zero Vector

As a prerequisite to characterizing social indifference curves (in Section 4), we can use the Invariance axiom and MIS axiom to identify a subset of the set of states for which a weighted sum of utilities remains constant (where the weights are precisely those described in the previous subsection), such that each state in this subset is socially indifferent to the zero vector.

Denote for each $j \in J$,

$$a^{(j)} \equiv \alpha_j e^{(1)} - \beta_j e^{(j)} \tag{12}$$

Then, by (10), we have $a^{(j)} \sim 0$ for each $j \in J$. Further, for each $j \in J$, we have:

$$qa^{(j)} = q_1 \alpha_j - q_j \beta_j = (1/\sigma)(\alpha_j - \gamma_j \beta_j) = 0 \tag{13}$$

Thus, defining:

$$L = \{z \in X : qz = 0\} \tag{14}$$

we note that $a^{(j)} \in L$ for each $j \in J$, and so any linear combination of these vectors is also in L.

The set L is our candidate for the social indifference curve passing through the zero vector. We will establish this in Section 4 when the preference order satisfies in addition the Weak Pareto axiom and is representable. That is, we will show that L is identical to the set $\mathcal{I}$ where:

$$\mathcal{I} = \{z \in X : z \sim 0\} \tag{15}$$

For the time being, we can identify a subset L' of L which is dense in L and which consists of states which are all socially indifferent to the zero vector. We proceed as follows.

The set $A = \{a^{(2)}, ..., a^{(n)}\}$ is clearly linearly independent, and so the rank of L is at least $(n-1)$. Any set of n vectors in L is linearly dependent, since

$q \neq 0$. Thus, the rank of L is $(n-1)$, and so A is a basis of L. Consequently, any $z \in L$ can be expressed as a linear combination of the vectors in A.

We can now rewrite L as follows:

$$\begin{aligned} L = \{z \in X : &\text{ there exists some } \lambda_j \in \mathbb{R} \text{ for each } j \in J, \\ &\text{such that } \ z = \sum_{j=2}^{n} \lambda_j a^{(j)}\} \end{aligned} \tag{16}$$

We can define a subset of L as follows:

$$\begin{aligned} L' = \{z \in X : &\text{ there exists some } \lambda_j \in \mathbb{Q} \text{ for each } j \in J, \\ &\text{such that } \ z = \sum_{j=2}^{n} \lambda_j a^{(j)}\} \end{aligned} \tag{17}$$

Since $a^{(j)} \sim 0$ for each $j \in J$, we can use the invariance axiom and Lemma 2 to infer that $z \sim 0$ for all $z \in L'$. Then L' is dense in L, and all points in L' are indifferent to the zero vector.

3 Example of an Order with no Representation

We provide in this section an example of a social preference order which shows that the axioms of Minimal Individual Symmetry, Invariance and Strong Pareto are together insufficient to guarantee existence of a real-valued representation and thus specifically a weighted utilitarian representation for the given order.

3.1 On a Decomposition of the Irrationals

Before proceeding to define the social preference order, we provide a result on decomposition of the irrationals on the real line. The proof of this result is presented in Appendix B.

Let $\mathbb{I}$ denote the set of irrationals in $\mathbb{R}$.

Theorem 1 *There exist subsets $\mathbb{A}$ and $\mathbb{B}$ of the set of irrationals $\mathbb{I}$, satisfying the following four properties:*

(i) $\mathbb{A} \cup \mathbb{B} = \mathbb{I}$, *and* $\mathbb{A} \cap \mathbb{B} = \emptyset$

(ii) $\mathbb{A} = -\mathbb{B}$

(iii) *If* $a, a' \in \mathbb{A}$, *then* $(a + a') \in \mathbb{A}$

(iv) *If* $a \in \mathbb{A}$ *and* $q \in \mathbb{Q}$, *then* $(a + q) \in \mathbb{A}$

We add a few remarks to clarify the nature of the decomposition. By (i) and the uncountability of irrationals, at least one of the two sets must be uncountable. By (ii), both must be uncountable. But, properties (i) and (ii) by themselves are not of particular interest. For example, the set of positive irrationals and the set of negative irrationals will also provide a decomposition of $\mathbb{I}$ satisfying properties (i) and (ii).

The properties of interest arise from (iii) and (iv), when taken in conjunction with (i) and (ii). Because of (ii), these properties hold of course for the set $\mathbb{B}$ as well; that is, we also have:

(iii') If $b, b' \in \mathbb{B}$, then $(b + b') \in \mathbb{B}$

(iv') If $b \in \mathbb{B}$ and $q \in \mathbb{Q}$, then $(b + q) \in \mathbb{B}$

To appreciate (iii), note that this property is clearly not satisfied by the set of positive irrationals. For instance, π and $(4 - \pi)$ are positive irrationals, but their sum is not an irrational. Further, even though π and e are positive irrationals, the present state of knowledge about the theory of numbers does not indicate whether $(\pi + e)$ is irrational or not.[7] However, the above decomposition manages to avoid these problems: if a and a' are in the set $\mathbb{A}$, then not only is their sum an irrational, but it is also in the set $\mathbb{A}$.

To appreciate (iv), note that given any $a \in \mathbb{A}$, the set $\{a + q : q \in \mathbb{Q}\}$ is a countable dense subset of the reals, $\mathbb{R}$. Since $\mathbb{A}$ is uncountable, $\mathbb{A}$ must contain the uncountable union of all such sets. This means the elements of the (disjoint) sets $\mathbb{A}$ and $\mathbb{B}$ are very finely interlaced along the entire real line.

3.2 Definition of the Order

We now define the binary relation on $X = \mathbb{R}^n$ as follows:

For all $x, y \in \mathbb{R}^n$,

[7] See Morandi (1996, p.174) for a discussion of this observation.

$$x \succsim y \text{ iff } [\sum_{i=1}^{n}(x_i - y_i), \xi(x_1 - y_1)] \succsim_L 0 \tag{18}$$

where $\succsim_L$ is the Lexicographic preference order defined on $\mathbb{R}^2$ and $\xi(.)$ is an indicator function on $\mathbb{R}$ defined as:

$$\xi(r) = \begin{cases} 1 \text{ if } r \in \mathbb{A} \\ 0 \text{ if } r \in \mathbb{Q} \\ -1 \text{ if } r \in \mathbb{B} \end{cases} \tag{19}$$

3.3 Verifying the Axioms

Clearly, $\succsim$ is complete. We check transitivity of $\succsim$ as follows. For any $x, y, z \in X$ with $x \succsim y$ and $y \succsim z$, we have $\sum_{i \in I} x_i \geq \sum_{i \in I} y_i$ and $\sum_{i \in I} y_i \geq \sum_{i \in I} z_i$. Thus, we must have $\sum_{i \in I} x_i \geq \sum_{i \in I} z_i$. If $\sum_{i \in I} x_i > \sum_{i \in I} z_i$, then by (18) we have $x \succsim z$. If $\sum_{i \in I} x_i = \sum_{i \in I} z_i$, then $(x_1 - y_1)$ and $(y_1 - z_1)$ must both belong to $\mathbb{A} \cup \mathbb{Q}$ by (18) and thus by Theorem 1, $(x_1 - z_1) \in \mathbb{A} \cup \mathbb{Q}$, showing that $x \succsim z$. Hence, $\succsim$ is a social preference order.

For any $i, j \in I$, $e_i^{(i)} = e_j^{(j)} = 1 > 0 = e_j^{(i)} = e_i^{(j)}$ and $e_k^{(i)} = 0 = e_k^{(j)}$ for every $k \in I \backslash \{i, j\}$. Also we have $\sum_{k \in I} e_k^{(i)} = \sum_{k \in I} e_k^{(j)}$ and $e_1^{(i)} - e_1^{(j)} \in \{-1, 0, 1\}$. Thus by (18), $e^{(i)} \sim e^{(j)}$,establishing that $\succsim$ satisfies the MIS axiom.

Let $x, y \in X$ such that $x_i \geq y_i$ for all $i \in I$ and $x_i > y_i$ for some $i \in I$. Then $\sum_{i \in I} x_i > \sum_{i \in I} y_i$ and hence by (18) $x \succ y$. Thus, the Strong Pareto axiom is satisfied.

Note that for any $x, y, a \in X$, the inequalities $\sum_{i \in I} x_i \geq \sum_{i \in I} y_i$ and $\sum_{i \in I}(x_i + a_i) \geq \sum_{i \in I}(y_i + a_i)$ are equivalent. We have also $\xi(x_1 - y_1) = \xi((x_1 + a_1) - (y_1 + a_1))$. Therefore by (18), $x \succsim y$ if and only if $x + a \succsim y + a$, showing that the Invariance axiom is satisfied.

3.4 Non-existence of a Real-Valued Representation

We claim that there does not exist any real-valued function representing the social preference order $\succsim$ defined above in (18). To show this, using the fact that the order is defined through the Lexicographic order on $\mathbb{R}^2$, we employ a proof similar to Debreu (1954).

Suppose that there exists a real-valued representation $f(.)$. Associate with each pair of numbers $(c, d) \in \mathbb{R}^2$, a non-empty subset of X, $D(c, d) = \{x \in X : \sum_{i \in I} x_i = c \text{ and } x_1 = d\}$. For any $x, x' \in D(c, d)$, we have $x \sim x'$ by (18) and thus:

$$f(x) = f(x') \tag{20}$$

Consider two fixed numbers d_1, d_2 with $(d_2 - d_1) \in \mathbb{A}$. Given any $c \in \mathbb{R}$, we can pick a unique element $g(c)$ from $D(c, d_1)$ and a unique element $h(c)$ from $D(c, d_2)$, by using the Axiom of Choice. Define $\alpha(c) = f(g(c))$ and $\beta(c) = f(h(c))$. It follows from (18) that $h(c) \succ g(c)$ for every $c \in \mathbb{R}$, and $g(c') \succ h(c)$ whenever $c, c' \in \mathbb{R}$ and $c' > c$. Thus, we have (i) $\alpha(c) < \beta(c)$ for every $c \in \mathbb{R}$, and (ii) $\beta(c) < \alpha(c')$ for all c, $c' \in \mathbb{R}$ satisfying $c < c'$. Define for all $c \in \mathbb{R}$, the interval $E(c) = [\alpha(c), \beta(c)]$.

Then, whenever $c, c' \in \mathbb{R}$ with $c \neq c'$, we must have $E(c)$ disjoint from $E(c')$. Thus, there is a one to one correspondence between the set of real numbers (which is uncountable) and a set of non-degenerate pairwise disjoint intervals (which is countable), a contradiction, establishing our claim.

4 Weighted Utilitarian Representation

In this section we present a characterization of weighted utilitarian representation. We show that a complete preference order which satisfies Weak Pareto, Invariance and Minimal Individual Symmetry, and has a representation, will always have a weighted utilitarian representation, the weights being the ones associated with the preference order, already introduced in Section 2. The converse, of course, is trivially true.

After presenting our result, we provide remarks to indicate how the existing characterizations in the literature may be viewed as particular cases.

Given invariance, the characterization result rests on the demonstration that the set L defined in (14) is identical to the set $\mathcal{I}$ defined in (15); that is, the set L is the indifference curve passing through the zero vector. So, we state and prove this result first.

Proposition 1 *Let $\succsim$ be a social preference order on X satisfying Weak*

Pareto, Invariance and Minimal Individual Symmetry. If $\succsim$ *can be represented by a real valued function, then* $L \subset \mathcal{I}$.

Proof. Let z belong to L. We will show that $z \sim 0$.

Let $W : X \to \mathbb{R}$ be a representation of the preference order. Define:

$$w(t) = W(te) \ \text{ for all } t \in \mathbb{R} \tag{21}$$

where $e = (1, 1, ..., 1)$ in X. Then, $w : \mathbb{R} \to \mathbb{R}$ is an increasing function by Weak Pareto and therefore has at most a countable number of discontinuities. Let t' be a point of continuity of w. Define $V : X \to \mathbb{R}$ by:

$$V(x) = W(x + t'e) \ \text{ for all } x \in X \tag{22}$$

Then, by Invariance, V also represents the preference order. Further, defining:

$$v(t) = V(te) = W(te + t'e) = w(t + t') \ \text{ for all } t \in \mathbb{R} \tag{23}$$

we see that v is continuous at $t = 0$. We now claim that given any $\varepsilon > 0$,

$$v(-\varepsilon) < V(z) < v(\varepsilon) \tag{24}$$

Since $z \in L$, there exist real numbers $\lambda_2, ..., \lambda_n$ such that:

$$z = \lambda_2 a^{(2)} + \cdots + \lambda_n a^{(n)} \tag{25}$$

Recalling the definition of α_j, β_j for all $j \in J$, let us denote:

$$M = \max\{1, \alpha_2, ..., \alpha_n, \beta_2, ..., \beta_n\} \tag{26}$$

Now, given any $\varepsilon > 0$, we can find $\rho_2, ..., \rho_n$ in $\mathbb{Q}$ such that:

$$|\rho_j - \lambda_j| \ < (\varepsilon/nM) \ \text{ for all } j \in J \tag{27}$$

Define:

$$\left.\begin{array}{l} y = \rho_2 a^{(2)} + \cdots + \rho_n a^{(n)} \\ z' = y + \varepsilon e \\ z'' = y - \varepsilon e \end{array}\right\} \tag{28}$$

Then, we have:

$$\begin{aligned} z' - z &= [\rho_2 - \lambda_2]a^{(2)} + \cdots + [\rho_n - \lambda_n]a^{(n)} + \varepsilon e \\ &\geq -(n-1)(\varepsilon/nM)Me + \varepsilon e \\ &= (\varepsilon/n)e \end{aligned} \tag{29}$$

Since $y \in L'$, we have $y \sim 0$, so by Invariance we must have $z' \sim \varepsilon e$, and:

$$V(z') = V(\varepsilon e) = v(\varepsilon) \tag{30}$$

By (29) we also have $z' >> z$, so by Weak Pareto we must have $z' \succ z$, and:

$$V(z') > V(z) \tag{31}$$

Thus, combining (30) and (31), we obtain:

$$V(z) < v(\varepsilon) \tag{32}$$

Similarly, we obtain:

$$\begin{aligned} z'' - z &= [\rho_2 - \lambda_2]a^{(2)} + \cdots + [\rho_n - \lambda_n]a^{(n)} - \varepsilon e \\ &\leq (n-1)(\varepsilon/nM)Me - \varepsilon e \\ &= -(\varepsilon/n)e \end{aligned} \tag{33}$$

Since $y \in L'$, we have $y \sim 0$, so by Invariance we must have $z'' \sim -\varepsilon e$, and:

$$V(z'') = V(-\varepsilon e) = v(-\varepsilon) \tag{34}$$

By (33) we also have $z'' << z$, so by Weak Pareto we must have $z \succ z''$, and:

$$V(z'') < V(z) \tag{35}$$

Thus, combining (34) and (35), we obtain:

$$V(z) > v(-\varepsilon) \tag{36}$$

Combining (32) and (36), the claim in (24) is established. Thus, by continuity of v at $t = 0$, we must have $V(z) = v(0)$; thus, $V(z) = V(0)$, and so $z \sim 0$. ■

In the following result, our statement deliberately invokes the technical condition $L \subset \mathcal{I}$, since this becomes a convenient way of tying together the various available characterizations.

Proposition 2 *Let $\succsim$ be a social preference order on X satisfying Weak Pareto, Invariance and Minimal Individual Symmetry. If $L \subset \mathcal{I}$, then for all $x, y \in X$,*

$$x \succsim y \quad \textit{iff} \quad qx \geq qy \tag{37}$$

and $L = \mathcal{I}$.

Proof. In view of invariance, in order to establish (37), it is enough to show that for all $z \in X$,

$$\left.\begin{array}{l}(i)\ z \sim 0 \ \text{ implies } \ qz = 0 \\ (ii)\ z \succ 0 \ \text{ implies } \ qz > 0\end{array}\right\} \tag{38}$$

Let $z \in X$ be given. Denote qz by r. Then, we have:

$$q(z - re) = qz - r = 0$$

and so $z' \equiv z - re \in L$. Since $L \subset \mathcal{I}$, we have $z' \sim 0$, and by Invariance:

$$z \sim re \tag{39}$$

If $z \sim 0$, then by (39) and Weak Pareto, we must have $r = 0$, and so $qz = 0$, establishing (i) of (38). If $z \succ 0$, then by (39) and Weak Pareto, we must have $r > 0$, and so $qz > 0$, establishing (ii) of (38).

The result in (38)(i) establishes that $\mathcal{I} \subset L$, and so $\mathcal{I} = L$. ■

The result, characterizing weighted utilitarianism, can now be stated as follows.

Theorem 2 *Let $\succsim$ be a social preference order on X satisfying Weak Pareto, Invariance and Minimal Individual Symmetry. If $\succsim$ can be represented by a real valued function, then $L = \mathcal{I}$, and for all $x, y \in X$,*

$$x \succsim y \quad \textit{iff} \quad qx \geq qy \tag{40}$$

Conversely, if $\succsim$ is a social preference order on X satisfying (40), then it satisfies Weak Pareto, Invariance and Minimal Individual Symmetry, $L = \mathcal{I}$, and $\succsim$ can be represented by a real valued function.

Proof. The first statement follows from Propositions 1 and 2. The converse statement is trivial. ■

Remarks:

(i) If $\succsim$ is a social preference order on X satisfying Weak Pareto, Invariance, Minimal Individual Symmetry and Continuity, then $\succsim$ has a real valued representation, and so it has a weighted utility representation by Theorem 2.

(ii) If $\succsim$ is a social preference order on X satisfying Weak Pareto, Strong Invariance and Minimal Individual Symmetry, then $L \subset \mathcal{I}$ holds by Strong

Invariance, using the fact that L is characterized by (16). So, $\succsim$ has a weighted utility representation by Proposition 2.

(iii) If $\succsim$ is a social preference order on X satisfying Weak Pareto, Invariance and Anonymity, then Minimal Individual Symmetry is satisfied, and furthermore one can take $\alpha_j = \beta_j = 1$ for all $j \in J$, so that $q_i = (1/n)$ for all $i \in I$. Anonymity, in fact, ensures that for each $j \in J$, $\lambda a^{(j)} \sim 0$ for all $\lambda \in \mathbb{R}$, and so $L \subset \mathcal{I}$ holds using Invariance and the fact that L is characterized by (16). So, $\succsim$ has a weighted utility representation by Proposition 2. Further, since $q_i = (1/n)$ for all $i \in I$, the preference order has a utilitarian representation.

(iv) The connection noted in Theorem 2 between representation and weighted utility representation (when the preference order satisfies Weak Pareto, Invariance and Minimal Individual Symmetry) does not hold for infinite utility streams.

5 Appendix

5.1 A: Representation and Continuity

A preference order might be representable by a numerical function, but it need not be continuous. [When it is not continuous, it cannot be represented by a continuous numerical function]. We provide a concrete example to illustrate this for a preference order in $\mathbb{R}^2$,which also satisfies the Strong Pareto and MIS axioms.[8]

Let $X = \mathbb{R}^2$, and let us define the function $H : X^2 \to \mathbb{R}$ as follows:

$$H(x_1, x_2) = f(x_1) + g(x_2) \tag{41}$$

where:

$$f(x_1) = \begin{cases} (2/3)x_1 & \text{for } x_1 < (1/2) \\ (1/3) + (2/3)x_1 & \text{for } x_1 \geq (1/2) \end{cases} \; ; \; g(x_2) = (1/3) + (1/3)x_2 \tag{42}$$

Define $\succsim$ by:

$$\text{For all } x, y \in X, \quad x \succsim y \text{ iff } H(x_1, x_2) \geq H(y_1, y_2) \tag{43}$$

[8] A simple example in $\mathbb{R}$, illustrating the point, which however does not satisfy the Weak Pareto axiom, appears in Fishburn (1970).

Then, $\succsim$ is a complete preference order on X, and H is a numerical representation of $\succsim$. Further, since f and g are strictly increasing functions on $\mathbb{R}$, the preference order satisfies the Strong Pareto axiom. Also, the MIS axiom is satisfied as can be checked by comparing the points $x' = ((1/4), (1/2))$ and $x'' = (0, 1)$.

Define $\bar{x} = ((1/2), 0)$ and $\tilde{x} = ((1/4), 1)$. Now, consider the sequence $\{x(n)\}_{n=1}^{\infty}$, where:

$$x(n) = (n/2(n+1), 1/(n+1)) \text{ for all } n \in \mathbb{N} \tag{44}$$

Then, $H(x(n)) = (2/3)$ for all $n \in \mathbb{N}$. Thus, for all $n \in \mathbb{N}$, we have $H(\tilde{x}) = (5/6) > (2/3) = H(x(n))$ and so $\tilde{x} \succ x(n)$; Since $x(n) \to \bar{x}$ as $n \to \infty$, continuity of $\succsim$ would require that $\tilde{x} \succsim \bar{x}$. But, we clearly have:

$$H(\bar{x}) = 1 > (5/6) = H(\tilde{x}) \tag{45}$$

And, since H represents $\succsim$, we must have $\bar{x} \succ \tilde{x}$;thus, $\succsim$ is not continuous.

5.2 B: Decomposition of the Set of Irrationals

A *groupoid* is an ordered pair $(G, *)$ where G is a non-empty set and $*$ is a binary operation on G; that is, given any two elements $a, b \in G$, there is a unique element $a * b \in G$. A *semigroup* is an ordered pair $(G, *)$ where G is a non-empty set and $*$ is an *associative* binary operation on G; that is, a semigroup is a groupoid $(G, *)$ such that if a, b, c are any three elements of G, then $(a * b) * c = a * (b * c)$. [9]

We will be considering groupoids $(G, *)$, where G will be a non-empty subset of the reals, and $*$ will be the binary operation of addition of reals (denoted as usual by $+$). Since addition is associative on the reals, any such groupoid will necessarily be a semigroup, and so the two concepts coincide in our case. In what follows, we shall refer to the set itself as the semigroup, it being understood that $+$ is the associative binary operation on the set.

Consider the collection of sets:

$$F = \{M \subset \mathbb{I} : M \text{ is a semigroup}\} \tag{46}$$

Note that F is a non-empty collection of sets. To see this, let a be any irrational and define:

$$A(a) = \{z \in \mathbb{R} : z = ma + q \text{ for some } m \in \mathbb{Q}_{++} \text{ and } q \in \mathbb{Q}\} \tag{47}$$

[9] For basic concepts of semigroup theory, see Howie (1995).

It is straightforward to verify that $A(a) \subset \mathbb{I}$ and $(A(a), +)$ is a semigroup.

Set inclusion $\subset$ is a partial ordering on F. By the Hausdorff Maximal Principle (see Royden (1988), p.25)[10], there is a maximal linearly ordered subcollection $\mathcal{F}$ of F. Define:

$$\mathbb{A} = \bigcup_{M \in \mathcal{F}} M \tag{48}$$

and:

$$\mathbb{B} = -\mathbb{A} \tag{49}$$

We will verify that the sets $\mathbb{A}$ and $\mathbb{B}$ defined by (48) and (49) satisfy the four properties stated in Theorem 1.

Property (ii) follows directly from (49). We proceed with the verification of Property (iii).

Let $a, a' \in \mathbb{A}$. Then by (48), there exist $M, M' \in \mathcal{F}$ such that $a \in M$ and $a' \in M'$. Since $\mathcal{F}$ is linearly ordered, we can assume without loss of generality that both a and a' belong to M. But then, since M is a semigroup, $(a + a') \in M$ and thus by (48) $(a + a') \in \mathbb{A}$ which establishes Property (iii).

Property (iii) shows that $\mathbb{A}$ is a groupoid and therefore a semigroup. Since $\mathbb{A}$ is the union of sets which are subsets of $\mathbb{I}$, it is a subset of $\mathbb{I}$ and thus by (46), $\mathbb{A}$ belongs to F. But then, since $M \subset \mathbb{A}$ for all $M \in \mathcal{F}$ by (48) and $\mathcal{F}$ is a maximal linearly ordered subcollection of F, $\mathbb{A}$ is in $\mathcal{F}$. Further, $\mathbb{A}$ is a *maximal* element of F. That is:

$$S \in F \text{ with } \mathbb{A} \subset S \text{ implies that } S = \mathbb{A} \tag{50}$$

We now verify Property (iv). Consider the set $\mathbb{A} + \mathbb{Q} = \{z \in \mathbb{R} : z = a + q$ for some $a \in \mathbb{A}$ and $q \in \mathbb{Q}\}$. Since addition of an irrational with a rational gives an irrational, the set $\mathbb{A} + \mathbb{Q}$ is a subset of $\mathbb{I}$. Furthermore, since sets $\mathbb{A}$ and $\mathbb{Q}$ are semigroups, the set $\mathbb{A} + \mathbb{Q}$ is a semigroup. Thus $\mathbb{A} + \mathbb{Q}$ belongs to F. Note that since $0 \in \mathbb{Q}$ (by convention), $\mathbb{A} \subset \mathbb{A} + \mathbb{Q}$ and therefore by (50), $\mathbb{A} + \mathbb{Q} = \mathbb{A}$,establishing property (iv).

Finally, we show that Property (i) holds. The sets $\mathbb{A}$ and $\mathbb{B}$ are disjoint. Otherwise, there would exist some $z \in \mathbb{A} \cap \mathbb{B}$. So, $z \in \mathbb{A}$ and by property (ii), $(-z) \in \mathbb{A}$. But then, since $\mathbb{A}$ is a semigroup, $z + (-z) = 0 \in \mathbb{A}$, a contradiction.

[10] The Hausdorff Maximal Principle is the same as version ZL" of Zorn's Lemma in Kaplansky (1972, p.60).

It remains to show that $\mathbb{A}\cup\mathbb{B} = \mathbb{I}$. Since $\mathbb{A} \subset \mathbb{I}$ (by (48)), we have $\mathbb{B} = -\mathbb{A} \subset \mathbb{I}$, and so $\mathbb{A} \cup \mathbb{B} \subset \mathbb{I}$. Suppose that there exists $x \in \mathbb{I}\backslash(\mathbb{A} \cup \mathbb{B})$. We claim that (a) there is some $\bar{m} \in \mathbb{N}$ such that $\bar{m}x \in \mathbb{B}$, and (b) there is some $\bar{n} \in \mathbb{N}$ such that $\bar{n}x \in \mathbb{A}$.

To establish (a), define the set $\mathbb{H} = \{mx : m \in \mathbb{N}\}$. Clearly, $\mathbb{H} \subset \mathbb{I}$ and $\mathbb{H}$ is a semigroup; thus, $\mathbb{H}$ is in F. Since $\mathbb{A}$ and $\mathbb{H}$ are semigroups, so is $(\mathbb{A} + \mathbb{H})$. Next, define the set $\mathbb{A}' = \mathbb{A} \cup (\mathbb{A} + \mathbb{H})$. Note that by property (iii), $\mathbb{A} + (\mathbb{A} + \mathbb{H}) = (\mathbb{A} + \mathbb{A}) + \mathbb{H} \subset (\mathbb{A} + \mathbb{H}) \subset \mathbb{A}'$, and $\mathbb{A}$ and $(\mathbb{A} + \mathbb{H})$ are semigroups, $\mathbb{A}'$ is also a semigroup.

If $\mathbb{A}'\subset \mathbb{I}$ then $\mathbb{A}' \in F$. And, since $\mathbb{A} \subset \mathbb{A}'$, we must have:

$$\mathbb{A}' = \mathbb{A} \tag{51}$$

by (50). Define $\mathbb{G} = \mathbb{A} \cup \mathbb{H}$. Since $\mathbb{A}$ and $\mathbb{H}$ belong to $\mathbb{I}$, $\mathbb{G}$ also belongs to $\mathbb{I}$. Since $\mathbb{A}$ and $\mathbb{H}$ are semigroups and $(\mathbb{A} + \mathbb{H}) \subset \mathbb{A}' = \mathbb{A} \subset \mathbb{G}$, the equality following from (51), $\mathbb{G}$ is a semigroup. Therefore $\mathbb{G}$ is in F. Since $\mathbb{A} \subset \mathbb{G}$, (50) implies that:

$$\mathbb{G} = \mathbb{A} \tag{52}$$

Since $x \in \mathbb{H} \subset \mathbb{G}$, we must therefore have $x \in \mathbb{A}$, a contradiction.

Thus, $\mathbb{A}'$ cannot be a subset of $\mathbb{I}$. This means there is some $q \in \mathbb{Q}$ which belongs to $\mathbb{A}'$. Thus, there is some $a \in \mathbb{A}$ and $\bar{m} \in \mathbb{N}$, such that $(a+\bar{m}x) = q$. By property (iv), $(-\bar{m}x) = a - q \in \mathbb{A}$, and so $\bar{m}x \in \mathbb{B}$ by property (ii). This establishes claim (a).

We can establish claim (b) by applying a similar argument on $\mathbb{B}$ once we show that $\mathbb{B}$ is also a maximal element of F. It is straightfoward to verify that for any $S \in F$, we have $(-S) \in F$. Since $\mathbb{B} = -\mathbb{A}$ by property (ii) and $\mathbb{A} \in F$, we have $\mathbb{B} \in F$. Now consider any $S \in F$ satisfying $\mathbb{B} \subset S$. Then we have, $\mathbb{A} \subset (-S)$ by property (ii) and thus $(-S) = \mathbb{A}$ by (50). That is, $S = -\mathbb{A} = \mathbb{B}$ by property (ii). This shows that $\mathbb{B}$ is a maximal element of F.

Now, applying the argument leading to claim (a), there exist $q \in \mathbb{Q}$, $b \in \mathbb{B}$ and $\bar{n} \in \mathbb{N}$ such that $b + \bar{n}x = q$. Hence $\bar{n}x = -b + q \in \mathbb{A}$ by properties (ii) and (iv). This establishes claim (b).

To complete the proof of property (i), note that since $\mathbb{A}$ and $\mathbb{B}$ are semigroups, we must have $(\bar{m}\bar{n})x \in \mathbb{A} \cap \mathbb{B}$. This, however, contradicts the fact that $\mathbb{A}$ and $\mathbb{B}$ are disjoint sets. Thus, we must have $\mathbb{A} \cup \mathbb{B} = \mathbb{I}$.

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