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Sustained Positive Consumption in a Model of Stochastic Growth: The Role of Risk Aversion

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Abstract

An intriguing problem in stochastic growth theory is as follows: even when the return on investment is arbitrarily high near zero and discounting is arbitrarily mild, long run capital and consumption may be arbitrarily close to zero with probability one. In a convex one-sector model of optimal stochastic growth with i.i.d. shocks, we relate this phenomenon to risk aversion near zero. For a Cobb-Douglas production function with multiplicative uniformly distributed shock, the phenomenon occurs with high discounting if, and only if, risk aversion diverges to infinity sufficiently fast as consumption goes to zero. We specify utility functions for which the phenomenon occurs even when discounting is arbitrarily mild. For the general version of the model, we outline sufficient conditions under which capital and consumption are bounded away from zero almost surely, as well as conditions under which growth occurs almost surely near zero; the latter ensures a uniform positive lower bound on long run consumption (independent of initial capital). These conditions require the expected marginal productivity at zero to be above the discount rate by a factor that depends on the degree of risk aversion near zero.

JEL Classification: D9, E2, O41.

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1 Introduction.

An important concern in the economic theory of growth and capital accumulation is whether positive consumption levels are sustained in the long run. Even though the technological and resource base of an economy may make it *feasible* to sustain a path where consumption is bounded away from zero, the actual economic incentive to accumulate may be low, so that the decentralized "equilibrium path" of the economy may be one where capital stocks and consumption are reduced to levels arbitrarily close to zero in the long run. This is not necessarily a problem of market failure, for even in convex models of growth with no externalities where the optimal and equilibrium paths coincide, eventual extinction may occur if economic agents are sufficiently impatient relative to productivity (or, the rate of return on investment). The problem is, however, significantly more complex in the presence of technological uncertainty that generates randomness in the return to investment; a run of "bad" shocks can lead to serious depletion of the capital stock from which recovery is costly and time consuming, and as a result, consumption may be driven arbitrarily close to zero with very high probability. Apart from impatience, the attitude towards risk plays an important role in determining the incentive to accumulate near zero in such a stochastic framework. This paper attempts to analyze the sustainability of positive consumption in the presence of technological uncertainty, and to understand how it relates to economic fundamentals including risk aversion, impatience and productivity.

Our analysis is carried out in the framework of the well known one sector model of optimal stochastic growth with strictly concave production and utility functions (Brock and Mirman, 1972). In the *deterministic* version of this model[1], positive consumption and capital are sustained in the long run, and the economy expands near zero if (and only if) the marginal productivity at zero exceeds the discount rate. Indeed, under the latter condition, there is a unique non-zero optimal steady state (the modified golden rule), and from every strictly positive initial stock the economy converges to this steady state. In particular, if the marginal productivity at zero is infinite, then long run consumption and capital are bounded away from zero no matter how heavily the future is discounted.

The situation is qualitatively different when the production technology is affected by random shocks over time. In a striking example, Mirman and Zilcha (1976) show that even if the technology is infinitely productive at zero with probability one, and even if the extent of discounting is arbitrarily mild, optimal capital and consumption may be arbitrarily close to zero in the long run. More specifically, they consider a strictly concave Cobb-Douglas production function with multiplicative random shocks that are independent and uniformly distributed on a positive non-degenerate interval (that can be arbitrarily small). Under this technology, even the "worst" possible realization of the production function exhibits infinite productivity at zero. They show that for each value of the discount factor $\delta \in (0, 1)$, *there exists* a smooth, strictly concave "regular" utility function such that in the dynamic economy with this utility function and the specified stochastic technology, capital and consumption always decline under the worst realization of the shock; as a result, capital and consumption fall below *any* strictly positive threshold infinitely often with probability one,[2] and in particular, there does not exist any invariant distribution (stochastic steady state) whose support is bounded away from zero. This is particularly striking when we consider the fact that if the distribution

[1] In this discussion and the rest of this paper, assume full depreciation of capital every period.
[2] See, Mitra and Roy (2007).

of shocks is degenerate, for this production function, for any strictly concave increasing utility function, and for any $\delta \in (0,1)$, the economy necessarily expands from sufficiently small stocks and every optimal path is bounded away from zero. This marks a fundamental qualitative difference between deterministic and stochastic models of growth.[3]

For the same production technology as in the Mirman and Zilcha (1976) example, if the utility function is logarithmic, then for every $\delta \in (0,1)$, the optimal policy function is one that ensures growth with probability one when the current stock is small enough, and long run consumption is bounded away from zero with probability one.[4] This indicates that in a stochastic model, whether capital and consumption are bounded away from zero in the long run depends, among other factors, on the nature of the utility function, even though in the deterministic version of the model these issues are determined exclusively by the discount factor and the productivity at zero.

There are two important questions that arise at this stage and that we proceed to address in the paper. First, what specific properties or attributes of the utility function lead to the indicated outcome in the Mirman-Zilcha example? This question acquires significance in view of the fact that although Mirman and Zilcha (1976) show the existence of a utility function (for each value of the discount factor) that leads to the indicated outcome, they do not explicitly specify the utility function or characterize it. It is therefore difficult to understand the extent of the problem and the economic significance of the example. Second, given any stochastic technology, under what restrictions on preferences is long run consumption almost surely bounded away from zero and, in particular, under what restrictions does the economy expand near zero even under the worst circumstance so that independent of initial conditions, long run consumption is almost surely uniformly bounded away from zero. The latter is important for understanding the kind of restrictions needed to ensure that the limiting stochastic steady state (if it exists) is bounded away from zero, and that poor economies experience growth. These two questions are related in that deriving a reasonably tight sufficient condition for avoidance of zero requires one to first understand the necessary conditions for the same.

The existing literature provides no answer to the first question. As for the second question relating to sufficient conditions for sustaining positive consumption in the long run, existing models of stochastic growth make strong assumptions to ensure that the limiting distribution of capital is bounded away from zero.[5] Brock and Mirman (1972) and Mirman and Zilcha (1975) impose two conditions that ensure expansion of capital and consumption near zero even under the worst realization of the stochastic technology. The two conditions are as follows: marginal productivity at zero is infinite for all realizations of the random shock and there is a strictly positive probability mass on the "worst" realization of the technology[6]. These

[3] The phenomenon described in Mirman and Zilcha (1976) is qualitatively different from that outlined by Kamihigashi (2006) who shows that if the marginal product at zero is *finite*, then every feasible path (including, therefore, any optimal path) converges *almost surely* to zero, provided the random shocks are "sufficiently volatile". The latter result is not driven by any property of the utility function.

[4] See, Mirman and Zilcha (1975).

[5] A number of papers impose conditions on endogenously determined elements such as the optimal policy function, or the stochastic process of optimal capital stocks. For example, Boylan (1979) imposes conditions that include a requirement that for each realization of the shock, the *optimal* capital stock next period is *concave* in the current stock. Mendelssohn and Sobel (1980) impose conditions on the stochastic kernel of the process of capital stocks generated by the optimal policy. It is not clear how these conditions relate to properties of the fundamentals of the model.

[6] This is the production function corresponding to the lower bound of the support of the random shock if

are clearly strong assumptions. For one thing, they do not allow for random shocks with continuous distributions. Further, the requirement that the worst possible realization of the production technology is infinitely productive at zero rules out economies where productivity may not be high under bad realizations of the technology shock. In a more recent contribution, Chatterjee and Shukayev (2008) weaken the productivity requirement by requiring that the lowest possible marginal productivity at zero exceed the discount rate; they show that if, in addition, the utility function is bounded below, then the economy is bounded away from zero with probability one (though the economy may not necessarily grow near zero under the worst realization of the productivity shock and, in particular, the almost sure lower bound on long run consumption may depend on initial condition). The restriction that the utility function is bounded below rules out a large class of CES utility functions that are widely used in the macroeconomic growth literature[7] (and for some of which, optimal paths have been independently shown to be bounded away from zero for all $\delta \in (0,1)$ and for particular choices of the production technology). Finally, in many situations the economy may not be sufficiently productive relative to the discount rate for bad realizations of the production shock; unless the probability that such states occur is high, one may still expect the economy to be bounded away from zero as long as they are balanced by high productivity in better states of the world. What should matter is the behavior of some kind of suitably modified version of "average" or expected productivity rather than the productivity in the worst case scenario.

The literature on optimal harvesting of renewable resources under uncertainty contains conditions under which resource stocks are bounded away from extinction. In a model that allows for non-concave production functions, Mitra and Roy (2006) outline a *joint restriction* on preferences and the production function that ensures growth near zero.[8] However, the analysis does not shed any light on the restriction such a condition implies on the class of utility functions (for any given production function and discount factor).

In this paper, we begin by trying to understand the kind of conditions that generate the phenomenon outlined in the example in Mirman and Zilcha (1976) where it is optimal for the economy to always reduce its capital and consumption under the worst realization of the shock and, as a result, the optimal capital and consumption are arbitrarily close to zero infinitely often almost surely. We describe this as a situation where the economy is "nowhere bounded away from zero". To understand specifically the conditions on preferences that can give rise to this even when productivity is arbitrarily high near zero and discounting is arbitrarily mild, we consider the stochastic technology used in the Mirman-Zilcha example i.e., a Cobb-Douglas production function with multiplicative and uniformly distributed shock. For this given stochastic technology, we first outline verifiable sufficient conditions on the utility function under which the economy is nowhere bounded away from zero provided the discount factor δ is small enough. In particular, confining attention to utility functions that exhibit decreasing relative risk aversion in a neighborhood of zero, we derive a tight *necessary and sufficient condition* for the economy to be nowhere bounded away from zero (when discounting is sufficiently strong). The condition requires that as consumption goes to zero, the Arrow-Pratt measure of relative risk aversion diverges to infinity at a sufficiently fast rate. Thus,

the production functions are ordered by realizations of the shock.

[7] Examples include $u(c) = \ln c$, $u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \sigma > 1$, where $u(0) = -\infty$.

[8] Olson and Roy (2000) provide a similar condition for avoidance of extinction of a renewable resource in a stochastic non-convex model where the utility depends on resource consumption as well as the resource stock.

the source of the problem identified in the Mirman-Zilcha example lies in risk aversion and, in particular, in the manner in which risk aversion explodes near zero. We illustrate this condition using a family of utility functions. We then consider the situation where discounting is arbitrarily mild and explicitly specify a utility function such that the economy is nowhere bounded away from zero.

Next, we consider the general model of optimal stochastic growth and provide a set of verifiable sufficient conditions under which the economy sustains positive consumption in the long run. In particular, we provide conditions under which the optimal policy function exhibits "growth near zero" under the worst realization of the production function so that independent of initial capital, long run consumption is uniformly bounded below by a positive number. We also provide conditions for a weaker form of avoidance of zero where capital and consumption are bounded away from zero with probability one (though the bound may depend on initial condition). Our general theoretical conditions are restrictions on the (limiting) behavior at zero of the *expected* marginal productivity modified by a factor that involves the ratio of marginal utilities of consumptions (that provides a verifiable bound on the marginal rate of substitution between current consumption and future stochastic consumption). For any given technology and discount factor, the behavior of this ratio of marginal utilities near zero is the key restriction needed on the class of utility functions in order to ensure sustained positive consumption. We show that the behavior of this ratio is closely related to the degree of (Arrow-Pratt) relative risk aversion, and provide a condition that involves explicit restrictions on the degree of risk aversion at zero. Higher the risk aversion at zero, higher the discounted expected marginal productivity at zero needed to ensure sustained positive consumption. If the expected marginal productivity at zero is infinite, sustained positive consumption is always ensured as long as risk aversion is bounded.

For utility functions that are bounded below, we also provide sufficient conditions for consumption to be bounded away from zero using the first elasticity of the utility function near zero. Unlike some of the existing conditions, our conditions allow for the possibility that for bad realizations of the random shock, the marginal productivity at zero may be lower than the discount rate. We show that if utility is bounded below, under some mild restrictions on the production function, infinite *expected* marginal productivity at zero is sufficient for sustained positive consumption no matter how small the discount factor.

The rest of the paper is organized as follows. Section 2 discusses the model and some preliminary results. Section 3 discusses the problem of the economy being nowhere bounded away from zero and provides necessary and sufficient conditions for this phenomenon for a specific Cobb Douglas stochastic technology. Section 4 contains general theoretical results providing sufficient conditions for the economy to exhibit growth near zero which ensures a uniform positive lower bound on long run consumption independent of initial condition. Section 5 discusses sufficient conditions for a slightly weaker property of the optimal policy that also ensures sustained positive consumption in the long run. All proofs are contained in the Appendix.

2 Model

We consider an infinite horizon one-good representative agent economy. Time is discrete and is indexed by $t = 0, 1, 2, ...$ The initial stock of output $y_0 > 0$ is given. At each date $t \geq 0$, the representative agent observes the current stock of output $y_t \in \mathbb{R}_+$ and chooses the level of current investment x_t, and the current consumption level c_t, such that

$$c_t \geq 0, x_t \geq 0, c_t + x_t \leq y_t$$

This generates y_{t+1}, the output next period through the relation

$$y_{t+1} = f(x_t, r_{t+1})$$

where $f(.,.)$ is the "aggregate" production function and r_{t+1} is a random production shock realized at the beginning of period $(t+1)$. The capital stock depreciates fully every period. Given current output $y \geq 0$, the feasible set for consumption and investment is denoted by $\Gamma(y)$ i.e.,

$$\Gamma(y) = \{(c, x) : c \geq 0, x \geq 0, c + x \leq y\}$$

The following assumption is made on the sequence of random shocks:

(A.1) $\{r_t\}_{t=1}^{\infty}$ *is an independent and identically distributed random process defined on a probability space* $(\Omega, \mathcal{F}, P)$, *where the marginal distribution function is denoted by* F. *The support of this distribution is a compact set* $A \subset \mathbb{R}$.

The production function $f : \mathbb{R}_+ \times A \to \mathbb{R}_+$ is assumed to satisfy the following:

(T.1) *For all* $r \in A, f(x, r)$ *is concave in* x *on* $\mathbb{R}_+$.

(T.2) *For all* $r \in A, f(0, r) = 0$.

(T.3) $f(x, r)$ *is continuous in* (x, r) *on* $\mathbb{R}_+ \times A$. *For each* $r \in [a, b], f(x, r)$ *is differentiable in* x *on* $\mathbb{R}_{++}$ *and, further,* $f'(x, r) = \frac{\partial f(x,r)}{\partial x} > 0$ *on* $\mathbb{R}_{++} \times A$.

Assumptions (T.1)-(T.3) are standard monotonicity, concavity and smoothness restrictions on production. For any investment level $x \geq 0$, let the upper and lower bound of the support of output next period be denoted by $\overline{f}(x)$ and $\underline{f}(x)$, respectively. In particular,

$$\overline{f}(x) = \max_{r \in A} f(x, r), \underline{f}(x) = \min_{r \in A} f(x, r). \tag{1}$$

It is easy to check that $\underline{f}(x)$ is continuous, concave and strictly increasing on $\mathbb{R}_+$.[9] Further, $\overline{f}(x)$ is continuous and strictly increasing on $\mathbb{R}_+$.

We assume that:

(T.4) $\limsup_{x \to 0}[\frac{\overline{f}(x)}{\underline{f}(x)}] < \infty$.

Assumption (T.4) imposes a bound on the extent of fluctuation in output that can be caused by the random shock. Note that (T.4) is always satisfied when the production shock is multiplicative i.e., $f(x, r) = rh(x)$ as long as A is a compact subset of $\mathbb{R}_{++}$. For the production function:

$$f(x, r) = x^r$$

with the random shock r having a non-degenerate distribution F with support $A \subset (0, 1)$, it is easy to check that as $x \to 0$, $\frac{f(x,r)}{\underline{f}(x)} \to +\infty$ for each $r < \sup A$ and that, in particular, (T.4) is violated.

[9] Continuity follows from the maximum theorem. To see concavity observe that for any $x_1, x_2 \in \mathbb{R}_+, \lambda \in [0, 1]$, $\underline{f}(\lambda x_1 + (1-\lambda)x_2) = f(\lambda x_1 + (1-\lambda)x_2, \widehat{r})$ for some $\widehat{r} \in A$ which is $\geq \lambda f(x_1, \widehat{r}) + (1-\lambda)f(x_2, \widehat{r}) \geq \lambda \underline{f}(x_1)$ + $(1-\lambda)\underline{f}(x_2)$.

For each $r \in A$,the let $D_+f(0,r)$ denote the "limiting" marginal product at zero investment where

$$D_+f(0,r) = \lim_{x \downarrow 0} f_x(x,r).$$

Using the concavity of $f(x,r)$ in x and $f(0,r) = 0$

$$D_+f(0,r) = \lim_{x \downarrow 0} \frac{f(x,r)}{x} \tag{2}$$

Let

$$\nu = \lim_{x \downarrow 0} \frac{\underline{f}(x)}{x}. \tag{3}$$

It is easy to check that

$$\nu \geq \inf_{r \in A} D_+f(0,r).$$

$\nu = +\infty$ if the production function satisfies the well-known Uzawa-Inada condition at zero.

We assume that

(T.5) $\nu > 1$ and $\limsup_{x \to \infty} \frac{\overline{f}(x)}{x} < 1$.

The first part of assumption (T.5) ensures that it is feasible for capital and output to grow with probability one in a neighborhood of zero i.e., even under the most adverse realization of the random shock. The second part of the assumption implies that the technology exhibits bounded growth.

Let $\delta \in (0,1)$ denote the time discount factor. Given the initial stock $y_0 > 0$, the representative agent's objective is to maximize the discounted sum of expected utility from consumption:

$$E\left[\sum_{t=0}^{\infty} \delta^t u(c_t)\right]$$

where u is the one period utility function from consumption.

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. The utility function $u : \mathbb{R}_+ \to \overline{\mathbb{R}}$ satisfies the following restrictions:

(U.1) *u is strictly increasing, continuous and strictly concave on* $\mathbb{R}_+$ (on $\mathbb{R}_{++}$ if $u(0) = -\infty$); $u(c) \to u(0)$ as $c \to 0$.

(U.2) *u is twice continuously differentiable on* $\mathbb{R}_{++}$; $u'(c) > 0, u''(c) < 0, \forall c > 0$.

(U.3) $\lim_{c \to 0} u'(c) = +\infty$.

Assumptions (U.1) and (U.2) are standard. Note that we allow the utility of zero consumption to be $-\infty$. (U.3) requires that the utility function satisfy the Uzawa-Inada condition at zero and ensures that optimal consumption and investment lie in the interior of the feasible set.

The *partial history* at date t is given by $h_t = (y_0, x_0, c_0, \ldots, y_{t-1}, x_{t-1}, c_{t-1}, y_t)$. A *policy* π is a sequence $\{\pi_0, \pi_1, \ldots\}$ where π_t is a conditional probability measure such that $\pi_t(\Gamma(y_t)|h_t) = 1$. A policy is *Markovian* if for each t, π_t depends only on y_t. A Markovian policy is *stationary* if π_t is independent of t. Associated with a policy π and an initial state y is an expected discounted sum of social welfare:

$$V_\pi(y) = E \sum_{t=0}^{\infty} \delta^t u(c_t),$$

where $\{c_t\}$ is generated by π, f in the obvious manner and the expectation is taken with respect to P.

The *value function* $V(y)$ is defined on $\mathbb{R}_{++}$ by:

$$V(y) = sup\{V_\pi(y) : \pi \textit{ is a policy}\}.$$

Under assumption **(T.5)**, it is easy to check that

$$-\infty < V(y) < +\infty, \forall y > 0.$$

A policy, π^*, is *optimal* if $V_{\pi^*}(y) = V(y)$ for all y. Standard dynamic programming arguments imply that there exists a unique optimal policy, that this policy is stationary and that the value function satisfies the functional equation:

$$V(y) = \sup_{x \in \Gamma(y)} [u(y-x) + \delta E[V(f(x,r)]]. \tag{4}$$

It can be shown that $V(y)$ is continuous, strictly increasing and strictly concave on $\mathbb{R}_{++}$. Further, the maximization problem on the right hand side of (4) has a unique solution, denoted by $x(y)$. The stationary policy generated by the function $x(y)$ is the optimal policy and we refer to $x(y)$ as the optimal investment function. $c(y) = y - x(y)$ is the optimal consumption function. Using standard arguments in the literature, (U.3) can be used to show that:

Lemma 1 *For all* $y > 0, x(y) > 0$ *and* $c(y) > 0$.

Lemma 2 $x(y)$ *and* $c(y)$ *are continuous and strictly increasing in* y *on* $\mathbb{R}_+$.

Lemma 1 implies that consumption is bounded away from zero along any realized path of the economy if, and only if, capital and output are bounded away from zero. Further, Lemma 2 implies that between any two periods, consumption expands if, and only if, capital and output expand.

Given initial stock $y > 0$, the stochastic process of optimal output $\{y_t(y,\omega)\}$ evolves over time according to the transition rule:

$$y_t(y,\omega) = f(x(y_{t-1}(y,\omega)), \omega_t) \textit{ for } t \geq 1 \tag{5}$$

and $y_0(y,\omega) = y$.

Next, we note that the stochastic Ramsey-Euler equation holds:

Lemma 3 *For all* $y > 0$,

$$u'(c(y)) = \delta E[u'(c(f(x(y),r)))f'(x(y),r)]. \tag{6}$$

Finally, let $R(c)$ denote the Arrow-Pratt measure of relative risk aversion defined by:

$$R(c) = -\frac{cu''(c)}{u'(c)} \text{ for all } c > 0. \tag{7}$$

We state a useful lemma that provides an estimate of the marginal rate of substitution using the Arrow-Pratt measure of relative risk aversion.

Lemma 4 *For any* $c > 0, \eta > 1$, *let* $\underline{R}(c,\eta) = \inf\{R(z) : z \in [c, \eta c]\}, \overline{R}(c,\eta) = \sup\{R(z) : z \in [c, \eta c]\}$. *Then:*

$$\frac{u'(\eta c)}{u'(c)} \leq (\frac{1}{\eta})^{\underline{R}(c,\eta)} \tag{8}$$

$$\frac{u'(\eta c)}{u'(c)} \geq (\frac{1}{\eta})^{\overline{R}(c,\eta)} \tag{9}$$

3 The economy may not be bounded away from zero.

The central focus of this paper is the possibility that consumption and capital may not be bounded away from zero in the long run even when the marginal return to investment at zero is "sufficiently large". In this section, we examine the extent of this problem and shed some light on the economic factors that lead to this problem.

3.1 The problem.

To begin, consider the *deterministic* version of the stochastic growth model outlined in the previous section; in particular, suppose that the probability distribution of $f(x,r)$ is degenerate and

$$\overline{f}(x) = \underline{f}(x) = h(x), \forall x \geq 0.$$

This is the well known Cass-Koopmans discounted classical optimal growth model. As is well-known, if

$$\lim_{x \to 0} h'(x) > \frac{1}{\delta} \tag{10}$$

i.e., the marginal productivity at zero exceeds the discount rate (the technology is "delta-productive" at zero), the sequence of optimal capital stocks from every strictly positive initial stock, converges monotonically to a unique strictly positive limit, the "modified golden rule" capital stock x^* defined by

$$h'(x^*) = \frac{1}{\delta}. \tag{11}$$

In other words, under (10), optimal capital and consumption are always bounded away from zero and indeed, if $0 < y_0 < h(x^*)$, capital and consumption exhibit growth over time. Further, if

$$\lim_{x \to 0} h'(x) = +\infty,$$

i.e., the technology satisfies the Uzawa-Inada condition, then (10) is satisfied for every $\delta \in (0,1)$ so that positive consumption and capital are sustained in the long run no matter how heavily the future is discounted.

The situation may, however, be qualitatively different in the stochastic model where the probability distribution of the production function $f(x,r)$ is non-degenerate. This was first pointed out by Mirman and Zilcha (1976) in a striking example that we briefly summarize now. Consider the stochastic aggregative growth model outlined in the previous section with the following specific form of the production function:

$$f(x,r) = rx^{\frac{1}{2}} \tag{12}$$

and assume that the distribution F of the random shocks is the uniform distribution on the interval $[\alpha, \beta], 0 < \alpha < \beta$. Note that for the above production function, the (limiting) marginal productivity at zero is infinite for all possible realizations of the random shock:

$$D_+ f(0, r) = +\infty, \forall r \in [\alpha, \beta].$$

Mirman and Zilcha (1976) show that for each $\delta \in (0,1)$, there exists a smooth "well-behaved" utility function u_δ satisfying, for instance, the assumptions on u outlined in the previous section, such that the optimal investment policy function for the economy (u_δ, f, δ, F) denoted by $x_\delta(y)$ satisfies:

$$\underline{f}(x_\delta(y)) < y, \forall y > 0. \tag{13}$$

The property (13) of the policy function implies that *no matter what the current state of the economy*, capital, output and consumption necessarily decline under the worst realization of the current technology. For the production function (12), this implies that capital and output are not bounded away from zero in the long run and there is no invariant distribution whose support is bounded away from zero.

This example illustrates a fundamental discrepancy between the stochastic and the deterministic growth models. For the production function (12), if the distribution of r_t is degenerate, for instance if $\alpha = \beta = 1$, the optimal capital path from every $y_0 > 0$ converges to the modified golden rule capital stock $x^* = \frac{\delta^2}{4} > 0$, independent of the choice of utility function u and discount factor $\delta \in (0,1)$. The example shows that even with a little bit of uncertainty in the production function and no matter how mildly one discounts the future, there is some utility function for which capital and consumption may be driven arbitrarily close to zero in the long run.

Now, in an economy with the same production function (12) as used in this example, if the utility function is given by $u(c) = \ln c$, then it is easy to check (see, for instance, Mirman and Zilcha, 1975) that for every $\delta \in (0,1)$, the optimal policy is one where the economy expands near zero even under the worst realization of the production shock and indeed, optimal capital stocks converge in distribution to a unique invariant distribution whose support is bounded away from zero. Therefore, in the stochastic growth model (and in contrast to the deterministic model), the nature of the utility function plays an important role in determining the long run destiny of the economy and in particular, the possibility of consumption and capital being bounded away from zero.

3.2 Nowhere bounded away from zero.

In what follows, we will refer to an economy where the optimal policy function is of the kind described in the Mirman-Zilcha example as being *nowhere bounded away from zero*. More precisely, define the lowest optimal transition function $\underline{H}(y)$ by:

$$\underline{H}(y) = \underline{f}(x(y)), y \geq 0.$$

Thus, $\underline{H}(y)$ is the lower bound of the support of output *next period* when the current output is y and the optimal investment policy function $x(y)$ is used to determine the amount invested.

Definition 1 *The economy (u, δ, f, F) is* <u>nowhere bounded away from zero</u> *(NBZ) if the optimal policy satisfies:*

$$\underline{H}(y) < y, \forall y > 0. \tag{14}$$

Using Proposition 2 in Mitra and Roy (2007), one can check that (14) implies that for every initial stock $y_0 = y > 0$, the stochastic process of optimal output $\{y_t(y, \omega)\}$ defined by (5) must satisfy the property:

$$\Pr\{\liminf_{t\to\infty} y_t(y, \omega) = 0\} = 1$$

so that output and capital are arbitrarily close to zero infinitely often with probability one; in particular, the Markov process $\{y_t(y, \omega)\}$ has no invariant distribution whose support is bounded away from zero.

The Mirman-Zilcha example described above shows that given a specific form of f and F, for each $\delta \in (0, 1)$, there is some u (that potentially depends on δ) for which the economy exhibits NBZ. This does not, however, shed any light on the qualitative properties of the utility functions that can makes the economy exhibit NBZ (for any given stochastic technology). In the rest of this section, we will attempt to address this issue. In particular, we consider the Cobb-Douglas production function with uniformly distributed multiplicative shock and outline verifiable necessary and sufficient conditions on the utility functions u under which the economy exhibits NBZ if the discount factor δ is small.

Consider the production function:

$$f(x, r) = rx^\gamma \text{ for all } x \in \mathbb{R}_+ \text{ and } r \in A \tag{15}$$

where $A = [\alpha, \beta]$, with $0 < \alpha < \beta < \infty$ and $\gamma \in (0, 1)$. The common distribution F is the uniform distribution function given by:

$$F(r) = \begin{cases} 0 & \text{for } r < \alpha \\ (r - \alpha)/(\beta - \alpha) & \text{for } \alpha \le r \le \beta \\ 1 & \text{for } r > \beta \end{cases} \tag{16}$$

Note that the production function considered in the Mirman-Zilcha example is a special case of this where $\gamma = \frac{1}{2}$. Also, observe that $D_+f(0, r) = +\infty$, for all $r \in [\alpha, \beta]$. We begin with a sufficient condition for NBZ. Let

$$\lambda = \frac{\beta}{\alpha}, \zeta = \frac{\gamma}{1 - \gamma}.$$

Clearly, $\lambda > 1$. Further,

$$\frac{1}{\zeta} = \frac{1}{\gamma} - 1.$$

so that $\frac{1}{\zeta}$ is decreasing in γ, and is therefore directly related to the curvature or degree of concavity of the production function and the rate at which marginal productivity goes to infinity as investment tends to zero.

Proposition 1 *Consider the stochastic technology* (f, F) *described by* (15) *and* (16)*. Suppose that:*

$$\limsup_{c\to 0} \int_1^\lambda \left[c^{-\frac{1}{\zeta}} \{ \frac{u'(\mu^\theta c)}{u'(c)} \} \right] d\mu < \infty \tag{17}$$

for some $\theta \in (0, 1)$*. Then, there exists* $\delta_0 > 0$ *such that for every* $\delta \in (0, \delta_0)$*, the economy* (f, F, u, δ) *is nowhere bounded away from zero.*

For the specific technology described above, Proposition 1 provides a verifiable sufficient condition (17) on the utility function for the economy to be nowhere bounded away from zero when the future is discounted sufficiently. The integrand on the left hand side of the inequality in (17) consists of two terms. As $\gamma \in (0,1)$, $\frac{1}{\zeta} = \frac{1}{\gamma} - 1 > 0$ and therefore, as $c \to 0$, $c^{\frac{1}{\zeta}} \to 0$ i.e., the term $c^{-\frac{1}{\zeta}} \to \infty$ and the rate at which this occurs depends on γ. On the other hand, the term $\frac{u'(\mu^\theta c)}{u'(c)} < 1$ is the marginal rate of substitution between consumption in the current and next time periods when the consumption next period grows by a factor μ^θ, where μ actually reflects the ratio of the realized multiplicative shock r to α, the worst possible shock; this term is actually an upper bound of the marginal rate of substitution between the *optimal* consumption in the current and next periods when the current output y is in fact, the largest fixed point, if any, of $\underline{H}(y)$. The condition (17) requires that this marginal rate of substitution converge to zero as $c \to 0$ at a rate that outweighs the rate at which $c^{\frac{1}{\zeta}} \to 0$.

Observe that the ratio $\frac{u'(\mu^\theta c)}{u'(c)}$ depends on the curvature of the utility function and therefore, the degree of risk aversion. The behavior of risk aversion near zero plays an important role here; higher the degree of risk aversion, the smaller the ratio $\frac{u'(\mu^\theta c)}{u'(c)}$. In lemma 4, we provide explicit bounds on this ratio in terms of the Arrow-Pratt measure of relative risk aversion; if this measure diverges to ∞ as $c \to 0$, then the ratio$\frac{u'(\mu^\theta c)}{u'(c)} \to 0$ and the rate at which this occurs depends on the rate at which risk aversion becomes infinitely large at zero.

In what follows, we derive an explicit necessary and sufficient condition on the utility function for the economy to be nowhere bounded away from zero for sufficiently low discount factor. In order to obtain a simple condition that is easy to understand and verify, we impose a further restriction: the utility function is assumed to satisfy decreasing relative risk aversion in a neighborhood of zero

$$\exists\ s > 0 \text{ such that } R(c) \text{ is non-increasing in } c \text{ on } (0, s). \tag{18}$$

As we will see in Proposition 5 of the next section, if $R(c)$ is bounded as $c \to 0$, then infinite expected marginal productivity at zero (as exhibited by the production function (15)) implies that the economy is always bounded away from zero. Therefore, for the economy to exhibit NBZ, it is necessary that $R(c) \to \infty$ as $c \to 0$. The assumption (18) is therefore not a very strong restriction.

Proposition 2 *Consider the stochastic technology* (f, F) *described by* (15) *and* (16)*. Suppose that the utility function* u *satisfies* (18)*. If*

$$\lim_{c \to 0} \inf\ [R(c) c^{\frac{1}{\zeta}}] > 0, \tag{19}$$

then there exists $\delta_0 > 0$ *such that for every* $\delta \in (0, \delta_0)$*, the economy* (u, δ, f, F) *is nowhere bounded away from zero. Conversely, if there is some* $\delta \in (0,1)$ *such that the economy* (u, δ, f, F) *is nowhere bounded away from zero, then (19) holds.*

Condition (19) provides a tight characterization of the kind of utility function that can lead to the phenomenon illustrated in the Mirman-Zilcha example. This condition brings out explicitly the tension between marginal productivity becoming infinitely large at zero (at a rate depending on γ) and the degree of risk aversion exploding as consumption goes to zero.

It requires that as $c \to 0$, risk aversion $R(c) \to \infty$ at a rate faster than the rate at which $c^{\frac{1}{\zeta}} = c^{(\frac{1}{\gamma}-1)} \to 0$.

Example 1 *Consider the "expo-power" utility function:*

$$u(c) = -\exp(pc^{-q}), c > 0$$

where $p > 0, q > 0$. It is easy to check that this utility function satisfies $(U.1), (U.2)$, $(U.3)$ *and* (18)*. The Arrow-Pratt measure of relative risk-aversion for this utility function is given by:*

$$R(c) = 1 + q + pqc^{-q}$$

so that

$$R(c)c^{\frac{1}{\zeta}} = (1+q)c^{\frac{1}{\zeta}} + pqc^{-(q-\frac{1}{\zeta})}$$

and therefore, (19)*is satisfied if, and only if, $q \geq \frac{1}{\zeta}$ i.e.,*

$$\gamma \geq \frac{1}{1+q}.$$

The results outlined above provide explicit and verifiable conditions on the utility function under which the economy (with the Cobb-Douglas production function and multiplicative shock) is nowhere bounded away from zero provided the future is discounted sufficiently i.e., δ is small enough. In their example, Mirman and Zilcha (1976) showed that for every $\delta \in (0,1)$, there exists some utility function for which the economy is nowhere bounded away from zero. This leads us to the question about whether we can explicitly outline a utility function for which the economy is nowhere bounded away from zero even when discounting is sufficiently mild. The next proposition outlines such a condition.

Choose $\delta' \in (0,1)$. Consider the (uniform) distribution function F with support $[\alpha, \beta]$ as described in (16). We impose the following restriction on the parameters of the distribution:

$$\alpha = 1 < \beta < \frac{2}{(\delta')^{\frac{1}{4}}} - 1. \tag{20}$$

Clearly, if $\beta > 1$ is chosen sufficiently close to 1, then (20) can be satisfied. The production function f is given by:

$$f(x,r) = r\sqrt{x}, r \in [\alpha, \beta], x \geq 0. \tag{21}$$

Finally, let the utility function $u : \mathbb{R}_+ \to \bar{\mathbb{R}}$ be given by

$$u(c) = \begin{cases} -e^{(1/c^{\nu})} & \text{if } c > 0 \\ -\infty & \text{if } c = 0 \end{cases}, \nu > 1. \tag{22}$$

We will now impose a restriction on the parameter ν, given β. For $t \in I \equiv (0, (1/\beta))$, define:

$$\phi(t) = 1 - \frac{\beta t}{(2-t)} \tag{23}$$

Note that ϕ maps I to $\mathbb{R}_{++}$, and $\phi(t) \to 1$ as $t \to 0$. Choose $\theta \in I$ such that:

$$\left. \begin{array}{l} (i)\ \phi(\theta) > \sqrt{\delta'} \\ (ii)\ 2^{(1/\theta)} > \dfrac{\beta^2}{(\beta-1)} \end{array} \right\} \tag{24}$$

Clearly, if θ is chosen sufficiently close to 0, then both conditions in (24) can be satisfied.[10] Fix any such θ and set

$$\nu = (1/\theta). \tag{25}$$

Proposition 3 *Choose any $\delta' \in (0,1)$. Given δ', consider the utility function u defined in* (22), *the production function f defined in (21), and the distribution F for the random shocks defined in* (16) *subject to parametric restrictions* (20), (23), (24) *and* (25). *Then, for every $\delta \in (0, \delta')$, the economy (f, F, u, δ) is nowhere bounded away from zero.*

Note that the restrictions on the parameters of the utility function u and the distribution F in Proposition 3 depend on the given δ', but not on the $\delta \in (0, \delta')$. For example, for $\delta' = 0.9801$, by choosing $\beta = 1.01$ and $\theta = 0.01$, we have a utility function u, a production function f and a distribution F for the random shocks such that for every $\delta \in (0, 0.9801)$, the economy (f, F, u, δ) is nowhere bounded away from zero.

4 Growth with Certainty Near Zero (GNZ).

4.1 The Concept.

In this section, we focus on a strong concept of sustaining positive consumption and avoidance of zero. The concept requires that when current output is close enough to zero, the economy expands even under the worst realization of the production technology. We shall refer to this as *growth with certainty near zero.* Recall that $\underline{H}(y) = \underline{f}(x(y))$ is the lower bound of the support of output *next period* when the current output is y and the optimal investment policy function $x(y)$ is used to determine the amount invested.

Definition 2 *The economy (u, δ, f, F) exhibits <u>growth with certainty near zero</u> (GNZ) if there exists $\alpha > 0$ such that*

$$\underline{H}(y) > y, \forall y \in (0, \alpha). \tag{26}$$

Consider $\{y_t(y, \omega)\}$, the Markov process of optimal output from initial stock $y > 0$, defined by (5). Let $\{c_t(y, \omega)\}$, be the Markov process of optimal consumption from initial stock $y > 0$ defined by

$$c_t(y, \omega) = c(y_t(y, \omega)).$$

(26) implies that $f(x(y), r) > y$ i.e., $y_1(y, \omega) > y_0 = y$ almost surely. Indeed, for each $y \in (0, \alpha)$,there exists $T(y) \geq 0$

$$\Pr\{y_{t+1}(y, \omega) > y_t(y, \omega), \forall t = 0, ..., T(y)\} = 1$$

i.e., the economy grows with probability one for at least $T(y)$ periods if the current stock is small enough. Thus, GNZ ensures sufficiently poor economies experience growth almost surely on their transition path. Indeed, GNZ implies that for all $y > 0$,

$$\Pr\{\lim_{t\to\infty} \inf \ y_t(y, \omega) > \alpha\} = 1$$

[10] Suppose we chooose $\delta' = 0.9801$.Then, (20) can be satisfied by choosing $\beta = 1.01$. Further, by choosing $\theta = 0.01$, both conditions in (24) are satisfied.

i.e., independent of initial stock y, optimal output eventually exceeds α with probability one. Using Lemma 2, we have then:

$$\Pr\{\lim_{t\to\infty}\inf\ c_t(y,\omega) > c(\alpha)\} = 1$$

i.e., independent of initial stock y, optimal consumption eventually exceeds $c(\alpha) > 0$ with probability one. Thus, GNZ ensures a uniform positive lower bound on long run consumption that is independent of initial condition.

In their pioneering analysis of the optimal stochastic growth model, Brock and Mirman (1972) impose strong conditions to ensure growth with certainty near zero (GNZ) and use this to show the existence of a unique invariant distribution whose support is bounded away from zero. The conditions they impose are as follows: the marginal productivity at zero is infinite for all realizations of the random shock and there is a strictly positive probability mass on the "worst" realization of the technology. These conditions rule out economies where the production shock is continuously distributed and economies where productivity at zero may be finite, at least for bad realizations of the technology shock. The subsequent literature on stochastic growth theory has not developed any alternative set of conditions that ensures GNZ. In this section, we develop sufficient conditions for GNZ that can be satisfied even when the distribution of the random production shock has no mass point and when marginal productivity is bounded.

4.2 General Sufficient Conditions for GNZ.

Recall the definition of $\overline{f}(x)$ and $\underline{f}(x)$ in (1). We begin by outlining a general sufficient conditions for growth with certainty near zero.

Proposition 4 *Suppose that*

$$\delta\{\lim_{x\to 0}\inf E(\frac{u'(f(x,r))}{u'(\underline{f}(x)-x)}f'(x,r))\} > 1. \tag{27}$$

Then, the economy (u,δ,f,F) *exhibits growth with certainty near zero.*

In the deterministic version of the model, growth near zero is ensured as long as the discounted marginal productivity at zero exceeds 1. This is often referred to as a "delta-productivity condition". One can view condition (27) as an *expected* "welfare-modified" delta-productivity condition at zero that reflects the stochastic nature of our model. Fluctuation in productivity always causes fluctuation in consumption next period and therefore, even if we know the current stock and whether tomorrow's stock lies above or below the current stock in the worst state of nature, we still need to estimate the marginal utility of consumption tomorrow in each state of nature in order to determine the value of marginal product. In particular, the factor $(\frac{u'(f(x,r))}{u'(\underline{f}(x)-x)})$ in (27) reflects a bound on the marginal rate of substitution between consumption in current and next period for each realization r of the random shock. Note that the condition allows the marginal productivity at zero to be below the discount rate for "bad" realizations of the random shock.

Next, define the function $\mu(r)$ on A by

$$\mu(r) = \lim_{x\to 0}\sup\frac{f(x,r)}{\underline{f}(x)} \tag{28}$$

Let

$$\lambda = \lim_{x \to 0} \sup \frac{\overline{f}(x)}{\underline{f}(x)}. \tag{29}$$

Note that under assumption (T.4), $\lambda < \infty$ and further,

$$1 \le \mu(r) \le \lambda, \forall r \in A.$$

For $r \in A$, define $n(r)$ by

$$\begin{aligned} n(r) &= \mu(r)\frac{\nu}{\nu - 1}, \text{ if } \nu < +\infty \\ &= \mu(r), \text{ if } \nu = +\infty. \end{aligned} \tag{30}$$

Let $\overline{n}, \underline{n}$ be defined by:

$$\begin{aligned} \overline{n} &= \lambda\frac{\nu}{\nu - 1}, \text{ if } \nu < +\infty \\ &= \lambda, \text{ if } \nu = +\infty. \end{aligned} \tag{31}$$

$$\begin{aligned} \underline{n} &= \frac{\nu}{\nu - 1}, \text{if } \nu < +\infty \\ &= 1, \text{if } \nu = +\infty. \end{aligned} \tag{32}$$

Then,

$$n(r) \ge \mu(r) \ge 1, \forall r \in A.$$

Further, $\overline{n} > \underline{n}$, and

$$\underline{n} \le n(r) \le \overline{n}, \forall r \in A. \tag{33}$$

Note that if the production function has the form:

$$f(x, r) = rh(x) \tag{34}$$

where the random shock is multiplicative and

$$\alpha = \inf A > 0, \beta = \sup A > \alpha,$$

then

$$\mu(r) = \frac{r}{\alpha}, \lambda = \frac{\beta}{\alpha}.$$

If $h'(0) = \lim_{x \to 0} h'(x) < \infty$, then

$$n(r) = \frac{r}{\alpha}\frac{\alpha h'(0)}{\alpha h'(0) - 1}, \overline{n} = \frac{\beta}{\alpha}\frac{\alpha h'(0)}{\alpha h'(0) - 1}, \underline{n} = \frac{\alpha h'(0)}{\alpha h'(0) - 1}.$$

If $h'(0) = \infty$,

$$n(r) = \frac{r}{\alpha}, \overline{n} = \frac{\beta}{\alpha}, \underline{n} = 1.$$

The next result refines the sufficient condition in Proposition 4 and provides a more transparent condition for GNZ.

Corollary 1 *Suppose that*

$$\delta E[\liminf_{x\to 0} \frac{u'(\sigma n(r)(\underline{f}(x)-x))}{u'(\underline{f}(x)-x)} f'(x,r)] > 1. \tag{35}$$

for some $\sigma > 1$. Then, the economy (u, δ, f, F) exhibits growth with certainty near zero.

Condition (35) is also an expected welfare-modified delta productivity condition at zero that has a somewhat clearer economic interpretation. $(\underline{f}(x) - x)$ is the level of consumption that sustains current output $y = \underline{f}(x)$ under the worst realization of the random shock. Following investment x, it can be shown that $n(r)(\underline{f}(x) - x)$ is the maximum consumption next period for realization r of the random shock. The factor $[\frac{u'(\sigma n(r)(\underline{f}(x)-x))}{u'(\underline{f}(x)-x)}]$ in (35) is a lower bound on the marginal rate of substitution between consumption in the current and next periods for realization r of the random shock. Note that this ratio is essentially of the form $\frac{u'(pc)}{u'(c)}$ where $p > 1$, and the behavior of this ratio as $c \to 0$ plays an important role in (35). This, in turn, can be easily related to the curvature of the utility function near zero and hence, to risk aversion near zero.

4.3 Risk Aversion and GNZ.

In this subsection, we outline sufficient conditions for growth with certainty near zero that explicitly impose restrictions on the degree of risk aversion near zero. Our discussion in Section 3 highlighted the important role played by risk aversion near zero in determining the whether the economy is nowhere bounded away from zero in the long run. When the production technology is stochastic, high risk aversion creates a disincentive to invest in an intrinsically uncertain future prospect. This may overwhelm the incentive to invest resulting from high productivity near zero.

Recall, that $R(c)$ denotes the Arrow-Pratt measure of relative risk aversion defined by (7).

Proposition 5 *Suppose that*

$$\bar{R} \equiv [\limsup_{c\to 0} R(c)] < \infty. \tag{36}$$

Further, suppose that

$$\delta E[(\frac{1}{n(r)})^{\overline{R}} D_+ f(0,r)] > 1. \tag{37}$$

Then, the economy (u, δ, f, F) exhibits growth with certainty near zero.

Proposition 5 provides a sufficient condition for GNZ for the class of utility function that exhibit bounded relative risk aversion. Note that this includes utility functions that are unbounded below. The sufficient condition (37) is an expected delta-productivity condition modified by the factor $(\frac{1}{n(r)})^{\overline{R}}$ that reflects behavior towards risk. Note that $\frac{1}{n(r)} < 1$ so that lower the risk aversion at zero, easier it is for this condition to be satisfied. Further, $n(r)$ itself reflects the extent of variability in the technology, so that the condition is more easily satisfied if the extent of variability is small.

Proposition 5 immediately yields the following useful corollary:

Corollary 2 *Suppose that*

$$\bar{R} \equiv [\lim_{c \to 0} \sup R(c)] < \infty.$$

Then, the economy (u, δ, f, F) *exhibits growth with certainty near zero if*

$$E[D_+ f(0, r)] > \frac{(\overline{n})^{\overline{R}}}{\delta}$$

If, in particular, $E[D_+ f(0, r)] = +\infty$, *then for every* $\delta \in (0, 1)$, *the economy* (u, δ, f, F) *exhibits growth with certainty near zero.*

Corollary 2 provides a more easily verifiable sufficient condition for GNZ for bounded relative risk aversion utility functions. It requires the expected marginal productivity at zero be larger than the discount rate by a factor that depends on the degree of risk aversion and the extent of variation in production created by the random shock. If the technology satisfies the Uzawa-Inada condition in the sense that the expected marginal productivity at zero is infinite (this is weaker than requiring the marginal productivity at zero to be infinite for every realization of the random shock), then GNZ holds for all utility functions with bounded relative risk aversion.

A widely used category of bounded relative risk aversion utility functions is the class of constant relative risk aversion (CRRA) utility functions u where $u : \mathbb{R}_{++} \to \mathbb{R}$ given by:

$$u(c) = \begin{cases} \frac{c^{1-\rho}}{1-\rho} & \text{if } \rho \neq 1 \\ \ln c & \text{otherwise} \end{cases}$$

with $u(0) = \lim_{c \to 0} u(c)$ when $\rho \in (0, 1)$, and $u(0) = -\infty$ otherwise. For this family of functions, relative risk aversion is given by the parameter ρ. From Corollary 2 it follows that:

Corollary 3 *Suppose that the utility function* u *exhibits CRRA with* $\rho > 0$ *being the relative risk aversion parameter. Further, suppose that*

$$\delta E[D_+ f(0, r)] > 1$$

i.e., expected marginal productivity at zero is greater than the discount rate. Let $\widehat{\rho}$ *be defined by*

$$\widehat{\rho} = \frac{\ln \delta E[D_+ f(0, r)]}{\ln \overline{n}}$$

Then the economy (u, δ, f, F) *exhibits growth with certainty near zero if*

$$\rho < \widehat{\rho}. \tag{38}$$

Further, if $E[D_+ f(0, r)] = +\infty$, *then* (38) *is always satisfied and the economy exhibits growth with certainty near zero no matter how high the level of relative risk aversion.*

Corollary 2 indicates that if relative risk aversion is bounded and the production function exhibits infinite expected productivity at zero, then the stochastic growth model generates growth near zero independent of the level of risk aversion or indeed, of any other property of intertemporal preference; this is qualitatively similar to the behavior of the economy in the deterministic version of the model (where, independent of the utility function, infinite productivity at zero always ensures growth near zero). However, if marginal productivity is bounded (in addition to risk aversion being bounded), then the possibility of growth near zero depends on the level of risk aversion, and the stochastic growth model may generate qualitatively different behavior near zero than that in the deterministic version of the model. The sufficient condition (38) for growth near zero when utility function exhibits CRRA reflects this role of risk aversion; in addition to $\delta E[D_+ f(0,r)] > 1$, the condition requires that relative risk aversion be below $\widehat{\rho}$; smaller the discounted expected marginal productivity at zero, smaller this upper bound $\widehat{\rho}$ on risk aversion.

Corollary 3 does not provide any indication of how the economy behaves near zero (for instance, whether or not growth near zero occurs) when risk aversion ρ exceeds this upper bound $\widehat{\rho}$. While we are not able to provide any general characterization of the outcome for high values ρ, the following example analyzed in Mitra and Roy (2010) provides some indication that with bounded productivity, the economy gets arbitrarily close to zero with probability one when risk aversion is large enough.

Example 2 (*Mitra and Roy,* 2010). *Suppose the utility function satisfies CRRA with relative risk aversion parameter* ρ. *Let*

$$f(x,r) = rh(x) \text{ for all } x \in \mathbb{R}_+ \text{ and } r \in A$$

where $A = [1, \beta]$, *with* $1 < \beta < \infty$, *and:*

$$h(x) = Ax/(1+x) \text{ for all } x \in \mathbb{R}_+$$

with $A > 1$. *Let* F *be the uniform distribution on* $[1, \beta]$. *Assume*

$$\delta(Er)h'(0) = \delta(Er)A > 1$$

It is assumed that $\delta(Er)h'(0)$ *is close enough to* 1. *Then one can explicitly specify* $\rho' > \widehat{\rho}$ (*where* $\widehat{\rho}$ *is as defined in Corollary 3*) *such that for* $\rho > \rho'$, $h(x(y)) < y$ *for all* $y > 0$ *and the economy is nowhere bounded away from zero.*

4.4 GNZ when utility is bounded below.

In this sub-section, we apply the general conditions derived in Section 4.2 to environments where the utility function is bounded below and derive specific conditions under which growth with certainty occurs near zero.

As we focus on utility functions that are bounded below, we may assume (without loss of generality) that

U.4. $u(0) = 0$.

We first state a useful result due to Arrow:

Lemma 5 (*Arrow, 1971*)[11] *Assume* **U.4**. *Then,*

$$\liminf_{c \to 0} R(c) \leq 1.$$

[11] See, Appendix [1] to Essay 3 ("Theory of Risk Aversion) in Arrow (1971).

Using this lemma, it follows that if $R(c)$ is monotonic in a neighborhood of zero so that $\lim_{c\to 0} \sup R(c) = \lim_{c\to 0} \inf R(c)$, then under **U.4**, $\bar{R} \equiv [\lim_{c\to 0} \sup R(c)] \leq 1$ and therefore, using Proposition 5 and Corollary 2, we have the following result:

Proposition 6 *Assume U.4. Further, suppose that there exists $s > 0$ such that $R(c)$ is monotonic (non-increasing or non-decreasing) on $(0, s)$. The economy (u, δ, f, F) exhibits growth with certainty near zero if*

$$\delta E[(\frac{1}{n(r)})D_+ f(0, r)] > 1. \tag{39}$$

A sufficient condition for (39) is given by:

$$E[D_+ f(0, r)] > \frac{\overline{n}}{\delta}.$$

If, in particular, $E[D_+ f(0, r)] = +\infty$, then for every $\delta \in (0, 1)$ the economy (u, δ, f, F) exhibits growth with certainty near zero.

Proposition 6 provides transparent sufficient conditions for GNZ for utility functions that are bounded below; these are modified expected delta-productivity conditions that do not require knowledge of the degree of relative risk aversion near zero. Unfortunately, the proposition also requires that risk aversion be monotonic in a neighborhood of zero. In the rest of this section, we outline alternative conditions for GNZ that do not have such a requirement. These conditions are in terms of the first elasticity of the utility function at zero.

Let

$$\kappa = \lim_{c\to 0} \inf \frac{u'(c)c}{u(c)} \tag{40}$$

$$K = \lim_{c\to 0} \sup \frac{u'(c)c}{u(c)}. \tag{41}$$

Then, $\kappa, K \in [0, 1]$.We begin by establishing a set of weak inequalities.

Lemma 6 *Assume U.4. Fix $\eta > 1$. Then,*

$$\lim_{c\to 0} \sup \frac{u'(\eta c)}{u'(c)} \geq \eta^{\kappa - 1}. \tag{42}$$

$$\lim_{c\to 0} \inf \frac{u'(\eta c)}{u'(c)} \geq \frac{\kappa}{K}\eta^{\kappa-1}, \text{ if } K > 0. \tag{43}$$

One implication of (43) is that if the limit of $\frac{u'(c)c}{u(c)}$ as $c \to 0$ is well defined and strictly positive, then $\lim\inf_{c\to 0} \frac{u'(\eta c)}{u'(c)} \geq \eta^{\kappa-1}$.The next proposition outlines a sufficient condition for GNZ for utility functions that are bounded below.

Proposition 7 *Assume U.4 and that $K > 0$. Suppose that*

$$\delta \frac{\kappa}{K} E[(n(r))^{\kappa-1} D_+ f(0, r)] > 1, \tag{44}$$

then the economy (u, δ, f, F) exhibits growth with certainty near zero.

The sufficient condition (44) for GNZ in the above proposition is, once again, a modified expected delta-productivity condition at zero. As $n(r) > 1$, higher the first elasticity of the utility function at zero, the more likely that this condition is satisfied. As $\kappa \leq 1$, condition (44) in the above proposition always holds as long as:

$$\delta \frac{\kappa}{K} (\overline{n})^{\kappa-1} E[D_+ f(0,r)] > 1.$$

This immediately yields the following useful corollary:

Corollary 4 *Assume U.4 and that $\kappa > 0$. Then, the economy (u, δ, f, F) exhibits growth with certainty near zero if*

$$E[D_+ f(0,r)] > [\frac{K}{\kappa} (\overline{n})^{1-\kappa}] \frac{1}{\delta} \tag{45}$$

Further, if $E[D_+ f(0,r)] = +\infty$, then for every $\delta \in (0,1)$ the economy (u, δ, f, F) exhibits growth with certainty near zero.

Corollary 4 provides an easily verifiable sufficient condition for GNZ for utility functions that are bounded below. It requires the *expected* marginal productivity at zero be larger than the discount rate by a factor that depends on the first elasticity of the utility function and the extent of variation in production created by the random shock.

More generally, the results in this section indicate that if the *expected* marginal productivity at zero is infinite, then GNZ occurs for all utility functions that are bounded below as long as their first elasticity is bounded away from zero or alternatively, relative risk aversion is monotonic near zero.

5 Bounded away from zero (BAZ).

In the previous section, we focused on the concept of growth with certainty near zero under which the economy expands even under the worst realization of the technology when current output is sufficiently close to zero. As mentioned earlier, this implies almost sure uniform positive lower bounds for long run capital and consumption independent of initial stock. While this is certainly sufficient to ensure that from every initial stock, capital and consumption are almost surely bounded away from zero, it is by no means necessary. In this section, we discuss a weaker concept under which capital and consumption are bounded away from zero from every positive initial stock, though the lower bounds may depend on the initial condition.

For $y > 0$, recall that $\{y_t(y, \omega)\}$ be the Markov process of optimal output defined by (5).

Definition 3 *We say that the economy is <u>bounded away from zero</u> (BAZ) if for every $y > 0$, there exists $\alpha(y) > 0$ such that*

$$\Pr\{\lim_{t\to\infty} \inf y_t(y, \omega) \geq \alpha(y)\} = 1. \tag{46}$$

While the above definition of an economy being bounded away from zero (BAZ) is in terms of the asymptotic behavior of the stochastic process of stocks generated by the optimal policy, it is easier to visualize the nature of the optimal policy *function* that generates a stochastic process $\{y_t(y, \omega)\}$ that satisfies the above definition. To this end, consider the lowest optimal transition function $\underline{H}(y) = \underline{f}(x(y)), y \geq 0$.

Lemma 7 *Suppose there exists a sequence* $\{y^n\}_{n=1}^{\infty} \to 0, y^n \in \mathbb{R}_{++}$ *such that*

$$\underline{H}(y^n) \geq y^n, \forall n.$$

Then, the economy is bounded away from zero.

Lemma 7 indicates that BAZ allows for the possibility that the lowest optimal transition function $\underline{H}(y)$ may have infinite number of fixed points that converge to zero. Further, from Lemma 7, it follows immediately, that growth with certainty near zero (GNZ) implies BAZ. Therefore, the sufficient conditions for GNZ provided in the previous section are also sufficient conditions for BAZ. However, BAZ may also be ensured under slightly weaker conditions. Note that BAZ implies that if $\{c_t(y, \omega)\}$ is the Markov process of optimal consumption from initial stock $y > 0$, then

$$\Pr\{\lim_{t\to\infty} \inf c_t(y, \omega) \geq c(\alpha(y))\} = 1.$$

where $c(\alpha(y))$, the optimal consumption from stock $\alpha(y)$, is strictly positive but may depend on the initial stock y.

Chatterjee and Shukayev (2008) provide sufficient conditions for BAZ (that do not necessarily ensure GNZ). Their sufficient condition requires that (i) the utility function is bounded below and (ii) $D_+f'(0, r) > \frac{1}{\delta}, \forall r \in A$. (i) is obviously a strong restriction as it does not allow for some of the widely used utility functions in the macro growth literature (including those of the CES family) where $u(0) = -\infty$. Further, as we have shown in the previous section, if the technology is sufficiently productive at zero, it is certainly possible to ensure GNZ (which is stronger than BAZ) even when $u(0) = -\infty$ if, for instance, risk aversion is bounded. (ii) is also a strong condition in that it requires the technology to be delta-productive at zero even under the worst realization of the production shock. As shown in the previous section, it is possible to ensure GNZ (and therefore, BAZ) under conditions that require the *expected* marginal productivity at zero to be large enough even if the productivity at zero is small under the worst realization of the shock.

We begin by providing a general sufficient condition for the economy to be bounded away from zero that allows for utility functions that are unbounded below and is in terms of expected welfare-modified marginal productivity.

Proposition 8 *Suppose that*

$$\delta\{\lim_{x\to 0} \sup E(\frac{u'(f(x, r))}{u'(\underline{f}(x) - x)} f'(x, r))\} > 1. \tag{47}$$

Then the economy (u, δ, f, F) *is bounded away from zero.*

Observe that the sufficient condition (47) for BAZ in the above proposition is comparable to and, in fact, a weaker version of the sufficient condition (27) for GNZ in Proposition 4. This proposition can be used to derive a more transparent sufficient condition.

Recall the definition of $\mu(r)$ in (28).

Corollary 5 *Assume that as* $x \to 0$

$$\frac{f(x, r)}{\underline{f}(x)} \to \mu(r) \text{ uniformly in } r \text{ on } A \tag{48}$$

If
$$\delta \lim_{x\to 0} \sup E[\frac{u'(\sigma n(r)(\underline{f}(x)-x))}{u'(\underline{f}(x)-x)} f'(x,r))] > 1. \tag{49}$$
for some $\sigma > 1$, then the economy (u,δ,f,F) is bounded away from zero.

Note that (48) is always satisfied for the multiplicative shock production function described in (34). The sufficient condition (49) for BAZ in the above result is a weaker version of the sufficient condition (35) for GNZ in Corollary 5.

The next proposition outlines a more easily verifiable condition for the economy to be bounded away from zero *when the utility function is bounded below.*

Proposition 9 *Assume U.4 and that as $x \to 0$*
$$\frac{f(x,r)}{\underline{f}(x)} \to \mu(r) \text{ uniformly in } r \text{ on } A.$$
Suppose that
$$\delta E[D_+ f(0,r)] > (\overline{n})^{1-\kappa}. \tag{50}$$
Then, the economy (u,δ,f,F) is bounded away from zero. In particular, if $E[D_+f(0,r)] = +\infty$, then for every $\delta \in (0,1)$, the economy (u,δ,f,F) is bounded away from zero.

The sufficient conditions for BAZ in Proposition 9 are weaker versions of and comparable to the conditions for GNZ outlined in Proposition 7 and Corollary 4. However, unlike the latter, Proposition 9 also requires that $\frac{f(x,r)}{\underline{f}(x)}$ converges uniformly in r as $x \to 0$.

APPENDIX.

Proofs of Lemmas 1, 2 and 3 are standard in the literature and hence omitted.

Proof of Lemma 4

Proof. For all $z \in [c, \eta c]$, we have:
$$\frac{-u''(z)z}{u'(z)} \geq \underline{R}(c,\eta)$$
This can be written as:
$$-\frac{d}{dz}(\ln u'(z)) \geq \frac{\underline{R}(c,\eta)}{z}$$
Integrating from c to ηc, we obtain:
$$-\ln u'(\eta c) - (-\ln u'(c)) \geq \underline{R}(c,\eta)\ln(\eta c) - \underline{R}(c,\eta)\ln c$$
Thus,
$$\ln \frac{u'(c)}{u'(\eta c)} \geq \ln \frac{(\eta c)^{\underline{R}(c,\eta)}}{c^{\underline{R}(c,\eta)}}$$

and:

$$\frac{u'(c)}{u'(\eta c)} \geq \eta^{\underline{R}(c,\eta)}$$

which establishes (8). The proof of (9) follows very similarly using the reverse inequality and upper bound on risk aversion. ■

Proof of Proposition 1.

Proof. Let

$$\xi = [1/(1-\gamma)], \omega = \beta^{\xi}.$$

Denote the left-hand side of (17) by B''. Pick $B' \in (B'', \infty)$. Then, there is $c' \in (0, \omega)$, such that for all $c \in (0, c')$:

$$M(c) = \int_1^{\lambda} \left[\frac{u'(\mu^{\theta} c)}{c^{(1/\zeta)} u'(c)} \right] d\mu < B'$$

The function $M(c)$ is continuous in c on $[c', \omega]$, and consequently, there is $M \in (0, \infty)$, such that $M(c) < M$ for all $c \in [c', \omega]$. Then, denoting $\max\{B', M\}$ by B, we have that for all $c \in (0, \omega]$:

$$\int_1^{\lambda} \left[\frac{u'(\mu^{\theta} c)}{c^{(1/\zeta)} u'(c)} \right] d\mu < B, \tag{51}$$

Define $\delta_0 \in (0,1)$ be defined by:

$$\delta_0 = \min\{\frac{1}{\lambda^{(1/\gamma)}}, \frac{1}{\gamma\lambda} \left[\frac{(\lambda^{1-\theta} - 1)}{(\lambda^{(1/\gamma)-\theta} - 1)} \right], \frac{(\beta - \alpha)}{\gamma\beta\alpha^{1/\gamma}} \frac{1}{B}\} \tag{52}$$

Consider any $\delta \in (0, \delta_0)$ and the economy (u, δ, f, F) as described in the proposition. Fix the economy in what follows. Suppose that contrary to the proposition, there is some $\bar{y} > 0$, such that,

$$\underline{H}(\overline{y}) = \alpha(x(\overline{y}))^{\gamma} \geq \overline{y}$$

Observe that for $y > \alpha^{\frac{1}{1-\gamma}}$,

$$\alpha(x(y))^{\gamma} \leq \alpha y^{\gamma} < y. \tag{53}$$

Thus, the function $\underline{H}(y) = \alpha(x(y))^{\gamma}$ has a strictly positive positive fixed point. Denote the largest fixed point of $\underline{H}(y)$ by z; note that by (53) and continuity of $\underline{H}$ in y, there is a largest fixed point of $\underline{H}(y)$. We now proceed to obtain a sequence of results that follow from this fact. It is useful to separate the results into steps.

Step 1: $[z < \alpha^{\xi}]$.

Since z is a fixed point of H, we have:

$$\alpha(z - c(z))^{\gamma} = \alpha(x(z))^{\gamma} = z \tag{54}$$

so that:

$$c(z) = z - (z/\alpha)^{(1/\gamma)} \tag{55}$$

And, since $c(z) > 0$, we obtain:

$$z > (z/\alpha)^{(1/\gamma)} \tag{56}$$

and this yields $z < \alpha^{[1/(1-\gamma)]} = \alpha^{\xi}$.

Step 2: $[z < (\delta\lambda\gamma)^{\zeta}\alpha^{\xi}$, using the Ramsey-Euler equation with initial stock $z]$.

With initial stock z, the Ramsey-Euler equation (6) yields:

$$u'(c(z)) = \frac{\delta}{\beta - \alpha} \int_{\alpha}^{\beta} r\gamma(x(z))^{\gamma-1} u'(c(r(x(z))^{\gamma}))dr \tag{57}$$

This can be rewritten as:

$$\frac{(x(z))^{1-\gamma}}{\delta\gamma} = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} r \frac{u'(c(r(x(z))^{\gamma}))}{u'(c(z))} dr \tag{58}$$

Substituting the information from (54) into (58), we obtain:

$$\frac{(z/\alpha)^{(1-\gamma)/\gamma}}{\delta\gamma} = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} r \frac{u'(c((r/\alpha)z)}{u'(c(z))} dr \tag{59}$$

Since c is strictly increasing, and $r > \alpha$ for all $r \in (\alpha, \beta]$, we have $c((r/\alpha)z) > c(z)$ for all $r \in (\alpha, \beta]$, and so:

$$\frac{u'(c((r/\alpha)z)}{u'(c(z))} < 1 \ \text{ for all } r \in (\alpha, \beta] \tag{60}$$

and consequently,

$$\frac{(z/\alpha)^{(1-\gamma)/\gamma}}{\delta\gamma} < \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} r dr = \frac{\alpha + \beta}{2} \tag{61}$$

so that:

$$\frac{(z/\alpha)^{(1-\gamma)/\gamma}}{\delta\gamma} < \beta \tag{62}$$

This implies that $(z/\alpha)^{(1/\zeta)} < (\delta\beta\gamma) = (\delta\lambda\gamma)\alpha$ and so:

$$z < (\delta\lambda\gamma)^{\zeta}\alpha^{(\zeta+1)} = (\delta\lambda\gamma)^{\zeta}\alpha^{\xi} \tag{63}$$

This becomes a particularly useful bound in the next step.

Step 3: [A lower bound on the consumption function.]

The principal difficulty in deriving additional properties of z is that we have very little information about the optimal consumption function for output levels beyond z (other than that $0 < c(y) < y$ and c is increasing in y). What one would like is to have the optimal consumption function bounded *below* by a function, whose behavior is known for output levels beyond z. Define $m : \mathbb{R}_+ \to \mathbb{R}$ by:

$$m(y) = y - (y/\alpha)^{(1/\gamma)} \tag{64}$$

Note that, by (55) and (64), $m(z) = c(z) > 0$, so that $m(z)$ is the optimal consumption when the stock is z. We would like to show that $c(y) \geq m(y) > 0$ for $y \in [z, \lambda z]$, so that $c(y)$ is bounded below by the *positive* function $m(y)$ for $y \in [z, \lambda z]$.

We first note that:

$$\begin{aligned} m'(y) &= 1 - (1/\alpha^{(1/\gamma)}\gamma)y^{(1-\gamma)/\gamma} \\ &= 1 - [y^{(1/\zeta)}/\alpha^{(1/\gamma)}\gamma] \end{aligned} \tag{65}$$

Thus, we have, using (63) in (65),

$$\begin{aligned} m'(z) &= 1-[z^{(1/\zeta)}/\alpha^{(1/\gamma)}\gamma] \\ &> 1-\delta\lambda \end{aligned} \tag{66}$$

Since $\delta\lambda < 1$ (using (52) and $\delta < \delta_0$), we must have $m'(z) > 0$.

As z is the *largest* fixed point of $\underline{H}(y)$, we have:

$$\underline{H}(y) = \alpha(x(y))^\gamma < y \text{ for all } y > z \tag{67}$$

so that:

$$c(y) > y - (y/\alpha)^{1/\gamma} = m(y) \text{ for all } y > z \tag{68}$$

The information in (68) is, of course, not useful unless we know that $m(y) > 0$. We now proceed to show that $m'(y) > 0$, and $m(y) > m(z) > 0$ for all $y \in (z, \lambda z]$.

For $y \in [z, \lambda z]$, we have, using (63),

$$y \leq \lambda z \leq \lambda(\delta\lambda\gamma)^\zeta\alpha^\xi = (\delta\lambda^{(1/\gamma)}\gamma)^\zeta\alpha^\xi \tag{69}$$

and:

$$[y^{(1/\zeta)} \leq (\delta\lambda^{(1/\gamma)}\gamma)\alpha^{(1/\gamma)}$$

so that $[y^{(1/\zeta)}/\alpha^{(1/\gamma)}\gamma] \leq \delta\lambda^{(1/\gamma)}$. Using this in (65), we get:

$$m'(y) = 1 - [y^{(1/\zeta)}/\alpha^{(1/\gamma)}\gamma] \geq 1 - \delta\lambda^{(1/\gamma)} > 0 \text{ for all } y \in [z, \lambda z] \tag{70}$$

since $\delta\lambda^{(1/\gamma)} < 1$ (follows from (52) and $\delta < \delta_0$). This implies, of course that:

$$m(y) > m(z) = c(z) > 0 \text{ for all } y \in (z, \lambda z]. \tag{71}$$

Step 4: [The Basic Ramsey-Euler Inequality]

Since we have for all $y \in (z, \lambda z]$, $c(y) > m(y) > 0$, we have for all $r \in (\alpha, \beta]$,

$$c((r/\alpha)z) > m((r/\alpha)z) = (r/\alpha)z - [(r/\alpha)(z/\alpha)]^{(1/\gamma)} > 0 \tag{72}$$

and so:

$$u'(c((r/\alpha)z)) < u'((r/\alpha)z - [(r/\alpha)(z/\alpha)]^{(1/\gamma)}) \tag{73}$$

Using (73) in the Ramsey-Euler equation (59), one obtains:

$$\frac{(z/\alpha)^{(1-\gamma)/\gamma}}{\delta\gamma} < \frac{1}{\beta-\alpha}\int_\alpha^\beta r\frac{u'((r/\alpha)z - [(r/\alpha)(z/\alpha)]^{(1/\gamma)})}{u'(z-(z/\alpha)^{(1/\gamma)})}dr$$

and this yields the basic Ramsey-Euler inequality:

$$\frac{(z/\alpha)^{(1-\gamma)/\gamma}}{\delta\gamma} < \frac{1}{\beta-\alpha}\int_\alpha^\beta r\frac{u'(\mu z - \mu^{(1/\gamma)}(z/\alpha)^{(1/\gamma)})}{u'(z-(z/\alpha)^{(1/\gamma)})})dr \tag{74}$$

where μ is short-hand for the function $\mu(r) = (r/\alpha)$ for $r \in [\alpha, \beta]$.

Step 5: [Another Lower Bound]

We know from (52) and $\delta < \delta_0$ that:

$$\frac{(\lambda^{1-\theta} - 1)}{(\lambda^{(1/\gamma)-\theta} - 1)} > \delta\gamma\lambda \tag{75}$$

We show that:

$$\mu z - \mu^{(1/\gamma)}(z/\alpha)^{(1/\gamma)} > \mu^{\theta} z - \mu^{\theta}(z/\alpha)^{(1/\gamma)} = \mu^{\theta}[z - (z/\alpha)^{(1/\gamma)}] \text{ for all } \mu \in (1, \lambda] \tag{76}$$

The consumption level at which the marginal utility is evaluated in the numerator of the marginal rate of substitution in (74) is $\mu z - \mu^{(1/\gamma)}(z/\alpha)^{(1/\gamma)}$, and (76) places a convenient lower bound on this consumption.

To establish (76), define:

$$L(\mu) = \frac{(\mu^{1-\theta} - 1)}{(\mu^{(1/\gamma)-\theta} - 1)} \text{ for all } \mu > 1 \tag{77}$$

Then, we have $L'(\mu) < 0$ iff:

$$(\mu^{(1/\gamma)-\theta} - 1)(1-\theta)\mu^{-\theta} < (\mu^{1-\theta} - 1)((1/\gamma) - \theta)\mu^{(1/\gamma)-\theta-1}$$

This inequality can be written as:

$$\begin{aligned} (\mu^{(1/\gamma)-\theta} - 1)(1-\theta) &< (\mu^{1-\theta} - 1)((1/\gamma) - \theta)\mu^{(1/\gamma)-1} \\ &= ((1/\gamma) - \theta)\mu^{(1/\gamma)-\theta} - ((1/\gamma) - \theta)\mu^{(1/\gamma)-1} \\ &= ((1/\gamma) - 1)\mu^{(1/\gamma)-\theta} + \mu^{(1/\gamma)-\theta}(1-\theta) - ((1/\gamma) - \theta)\mu^{(1/\gamma)-1} \end{aligned}$$

Canceling common terms, we then obtain:

$$-(1-\theta) < ((1/\gamma) - 1)\mu^{(1/\gamma)-\theta} - ((1/\gamma) - \theta)\mu^{(1/\gamma)-1} \tag{78}$$

In order to determine whether (78) holds, we define the function:

$$f(\mu) = ((1/\gamma) - 1)\mu^{(1/\gamma)-\theta} - ((1/\gamma) - \theta)\mu^{(1/\gamma)-1} + (1-\theta) \text{ for all } \mu \geq 1$$

Note that:

$$f(1) = ((1/\gamma) - 1) - ((1/\gamma) - \theta) + (1-\theta) = 0 \tag{79}$$

Further, we have:

$$\begin{aligned} f'(\mu) &= ((1/\gamma) - \theta)((1/\gamma) - 1)\mu^{(1/\gamma)-\theta-1} - ((1/\gamma) - \theta)((1/\gamma) - 1)\mu^{(1/\gamma)-2} \\ &= ((1/\gamma) - \theta)((1/\gamma) - 1)\mu^{(1/\gamma)-1}[(1/\mu^{\theta}) - (1/\mu)] \end{aligned}$$

Thus, $f'(1) = 0$, and $f'(\mu) > 0$ for all $\mu > 1$, since $\theta \in (0, 1)$. This implies that $f(\mu) > f(1)$ for all $\mu > 1$, and so $f(\mu) > 0$ for all $\mu > 1$ by (79). Thus, the inequality (78) must hold for all $\mu > 1$, and so $L'(\mu) < 0$ for all $\mu > 1$. That is, L is decreasing on $(1, \infty)$, and so it attains a minimum at λ on the interval $(1, \lambda]$. This means:

$$L(\mu) \geq \frac{(\lambda^{1-\theta} - 1)}{(\lambda^{(1/\gamma)-\theta} - 1)} \text{ for all } \mu \in (1, \lambda]$$

and consequently, given (75), we have:

$$\frac{(\mu^{1-\theta}-1)}{(\mu^{(1/\gamma)-\theta}-1)} = L(\mu) \geq \delta\gamma\lambda \text{ for all } \mu \in (1,\lambda]$$

Using (63), we also have:

$$\frac{z^{(1-\gamma)/\gamma}}{\alpha^{(1/\gamma)}} = \frac{z^{(1/\zeta)}}{\alpha^{(\xi/\zeta)}} < \delta\lambda\gamma$$

Thus, we obtain, for all $\mu \in (1,\lambda]$,

$$\frac{(\mu^{1-\theta}-1)}{(\mu^{(1/\gamma)-\theta}-1)} > \frac{z^{(1-\gamma)/\gamma}}{\alpha^{(1/\gamma)}} \tag{80}$$

We can rewrite (80) as:

$$(\mu^{1-\theta}-1)\alpha^{(1/\gamma)} > (\mu^{(1/\gamma)-\theta}-1)z^{(1-\gamma)/\gamma} \tag{81}$$

Multiplying through in (81) by $\mu^\theta z$, we obtain for all $\mu \in (1,\lambda]$

$$(\mu-\mu^\theta)\alpha^{(1/\gamma)}z > (\mu^{(1/\gamma)}-\mu^\theta)z^{(1/\gamma)}$$

so that:

$$(\mu-\mu^\theta)z > (\mu^{(1/\gamma)}-\mu^\theta)(z/\alpha)^{(1/\gamma)} \tag{82}$$

Transposing terms in (82), we have for all $\mu \in (1,\lambda]$,

$$\mu z - \mu^{(1/\gamma)}(z/\alpha)^{(1/\gamma)} > \mu^\theta z - \mu^\theta (z/\alpha)^{(1/\gamma)} = \mu^\theta[z-(z/\alpha)^{(1/\gamma)}] \tag{83}$$

This establishes our claim.

Step 6: [The Refined Ramsey-Euler Inequality]

Using (76) in the Ramsey-Euler inequality (74), we obtain:

$$\frac{(z/\alpha)^{(1-\gamma)/\gamma}}{\delta\gamma} < \frac{1}{\beta-\alpha}\int_\alpha^\beta r\frac{u'(\mu^\theta c(z))}{u'(c(z))}dr \tag{84}$$

Note that the marginal rate of substitution inside the integral in (84) is now in the form $u'(\eta c)/u'(c)$, with η a function of r.

Step 7: [The Final Step]

From (84):

$$\frac{1}{\delta} < \gamma\alpha^{(1-\gamma)/\gamma}\frac{1}{\beta-\alpha}\int_\alpha^\beta r\frac{u'(\mu^\theta c(z))}{z^{(1-\gamma)/\gamma}u'(c(z))}dr \tag{85}$$

isolating the role of the discount factor. Further, using the fact that in the integral on the right hand side of (85), $r \leq \beta$, and $z > c(z)$, we obtain:

$$\frac{1}{\delta} < \frac{\gamma\beta\alpha^{(1-\gamma)/\gamma}}{(\beta-\alpha)}\int_\alpha^\beta \frac{u'(\mu^\theta c(z))}{c(z)^{(1-\gamma)/\gamma}u'(c(z))}dr \tag{86}$$

Writing c for $c(z)$, we finally obtain:

$$\frac{1}{\delta} < \frac{\gamma\beta\alpha^{(1-\gamma)/\gamma}}{(\beta-\alpha)}\int_\alpha^\beta \frac{u'(\mu^\theta c)}{c^{(1/\zeta)}u'(c)}dr \tag{87}$$

Making the change of variable $\mu = (r/\alpha)$ in (87), we obtain:

$$\frac{1}{\delta} < \frac{\gamma\beta\alpha^{1/\gamma}}{(\beta-\alpha)} \int_1^\lambda \frac{u'(\mu^\theta c)}{c^{(1/\zeta)}u'(c)} d\mu$$

But given (52) and $\delta < \delta_0$, this contradicts (51) and establishes the Proposition. ■

Proof of Proposition 2

Proof. First, we show that if (19) holds then there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$, the economy (u, δ, f, F) is nowhere bounded away from zero. Let $\theta \in (0, 1)$ be given, and denote $[(1-\theta)/\theta]$ by θ'. Note that since $R(c)$ is decreasing in c on $(0, s)$, condition (19) implies:

$$R(c) \uparrow \infty \text{ as } c \downarrow 0 \text{ on } (0, s). \tag{88}$$

Thus, we can find $c' \in (0, \frac{s}{\lambda^\theta})$, such that:

$$R(\lambda^\theta c) - \theta' - 1 > (\frac{1}{2})R(\lambda^\theta c) \text{ for all } c \in (0, c') \tag{89}$$

Our analysis that follows is for $c \in (0, c')$. In view of Proposition 1, it suffices to show that:

$$\limsup_{c \to 0} \int_1^\lambda \left[\frac{u'(\mu^\theta c)}{c^{(1/\zeta)}u'(c)}\right] d\mu < \infty \tag{90}$$

In order to show (90), we evaluate the integral:

$$I(c) = \int_1^\lambda \left[\frac{u'(\mu^\theta c)}{u'(c)}\right] d\mu \tag{91}$$

for all $c \in (0, c')$. Pick an arbitrary $c \in (0, c')$. Using Lemma 4 and the fact that for $c \in (0, c')$, $R(.)$ is non-increasing on $[c, \lambda^\theta c]$, we have that for all $\mu \in (1, \lambda]$:

$$\left[\frac{u'(\mu^\theta c)}{u'(c)}\right] \leq \frac{1}{\mu^{\theta R(\mu^\theta c)}} \tag{92}$$

while for $\mu = 1$, the inequality in (92) holds trivially. Thus, (92) holds for all $\mu \in [1, \lambda]$, and so:

$$I(c) = \int_1^\lambda \left[\frac{u'(\mu^\theta c)}{u'(c)}\right] d\mu \leq \int_1^\lambda \left[\frac{1}{\mu^{\theta R(\mu^\theta c)}}\right] d\mu = J(c) \tag{93}$$

To evaluate $J(c)$, define $t = \mu^\theta$ for all $\mu \in [1, \lambda]$, and denote λ^θ by λ'. Then, the change of variable rule yields:

$$J(c) = \int_1^{\lambda'} \left[\frac{1}{\theta t^{[R(tc)-\theta']}}\right] dt = \int_1^{\lambda'} \left[\frac{1}{\theta t^{\rho(tc)}}\right] dt \tag{94}$$

where $\rho(tc) \equiv R(tc) - \theta' > 1$ for all $t \in [1, \lambda']$ by choice of $c \in (0, c')$. Since ρ is decreasing in its argument, we have:

$$\theta J(c) = \int_1^{\lambda'} \left[\frac{1}{t^{\rho(tc)}}\right] dt \leq \int_1^{\lambda'} \left[\frac{1}{t^{\rho(\lambda' c)}}\right] dt = K(c) \tag{95}$$

Now, since $a(c) \equiv \rho(\lambda' c)$ is not itself a function of t, and $a(c) > 1$, the integral $K(c)$ can be evaluated as:

$$K(c) = \frac{1}{(a(c)-1)} - \frac{1}{(a(c)-1)(\lambda')^{a(c)-1}} \leq \frac{1}{(a(c)-1)} \tag{96}$$

Thus, we obtain for an arbitrary $c \in (0, c')$,

$$I(c) \leq J(c) \leq \frac{K(c)}{\theta} \leq \frac{1}{\theta(a(c)-1)} \leq \frac{2}{\theta R(\lambda' c)} \tag{97}$$

by using (89). Then, for an arbitrary $c \in (0, c')$,

$$\begin{aligned} \int_1^\lambda \left[\frac{u'(\mu^\theta c)}{c^{(1/\zeta)} u'(c)}\right] d\mu &= \frac{I(c)}{c^{(1/\zeta)}} \leq \frac{(2/\theta)}{c^{(1/\zeta)} R(\lambda' c)} \\ &= \frac{(\lambda')^{(1/\zeta)}(2/\theta)}{(\lambda' c)^{(1/\zeta)} R(\lambda' c)} \end{aligned} \tag{98}$$

Let

$$Q = \liminf_{c \to 0} \, [R(c) c^{\frac{1}{\zeta}}].$$

Then, (19) implies $Q > 0$ and we can find $c'' < c'$, such that for all $c \in (0, c'')$,

$$c^{(1/\zeta)} R(c) \geq (Q/2) \tag{99}$$

Thus, for all $c \in (0, \frac{c''}{\lambda'})$, we have $\lambda' c < c''$, and so by (99),

$$(\lambda' c)^{(1/\zeta)} R(\lambda' c) \geq (Q/2)$$

Also, for all $c \in (0, c''/\lambda')$, we have $c \in (0, c')$ and so (98) holds. Consequently, for all $c \in (0, \frac{c''}{\lambda'})$,

$$\int_1^\lambda \left[\frac{u'(\mu^\theta c)}{c^{(1/\zeta)} u'(c)}\right] d\mu \leq (\lambda')^{(1/\zeta)} (\frac{2}{\theta})(\frac{2}{Q})$$

which establishes (90), given $\theta \in (0,1)$. This establishes the first part of the proposition.

Next, we show that if there is some $\delta \in (0,1)$ such that the economy (u, δ, f, F) is nowhere bounded away from zero, then (19) holds. Suppose to the contrary that there exists $\delta \in (0,1)$ such that the economy (u, δ, f, F) is nowhere bounded away from zero but:

$$\liminf_{c \to 0} \, [R(c) c^{(1/\zeta)}] = 0 \tag{100}$$

Proposition 8 in Section 5 provides a sufficient condition (47) for the economy to be bounded away from zero (BAZ), a property that violates NBZ. Since the economy (u, δ, f, F) exhibits NBZ, (47) cannot hold and therefore:

$$\delta \frac{\gamma}{\beta - \alpha} [\limsup_{x \to 0} \{\int_\alpha^\beta \frac{u'(rx^\gamma)}{u'(\alpha x^\gamma - x) x^{1-\gamma}} r dr\}] \leq 1$$

and setting $\mu = \frac{r}{\alpha}, \lambda = \frac{\beta}{\alpha}$, we have :

$$\limsup_{x \to 0} \int_1^\lambda \left[\frac{u'(\mu \alpha x^\gamma)}{x^{1-\gamma} u'(\alpha x^\gamma - x)}\right] d\mu < \infty \tag{101}$$

Given $x > 0$, we write the integral in (101) as:

$$I(x) = \int_1^\lambda \left[\frac{u'(\mu\alpha x^\gamma)}{x^{1-\gamma}u'(\alpha x^\gamma)}\right]\left[\frac{u'(\alpha x^\gamma)}{u'(\alpha x^\gamma - x)}\right] d\mu \tag{102}$$

The second term in the integral in (102) does not involve μ, and we proceed to find a positive lower bound for it. To this end, we make some preliminary observations. Note that since $R(c)$ is decreasing in c on $(0, s)$ we must have $[-u''(c)]$ also decreasing in c on $(0, s)$. Further, defining $x' = (\alpha/2)^\xi$, we see that for all $x \in (0, x')$,

$$x^{1-\gamma} \leq (\alpha/2) \tag{103}$$

Finally, defining $z = (\alpha/2)x^\gamma$ for all $x > 0$, we note by (100), that we can find $x'' \in (0, x')$ such that for all $x \in (0, x'')$,

$$\beta x^\gamma < s.$$

and

$$[R(z)z^{(1/\zeta)}][\frac{2}{\alpha}]^{(1/\gamma)} < (1/2) \tag{104}$$

For $x \in (0, x'')$, we evaluate:

$$\begin{aligned} J(x) &= \frac{u'(\alpha x^\gamma - x)}{u'(\alpha x^\gamma)} = \frac{[u'(\alpha x^\gamma - x) - u'(\alpha x^\gamma)] + u'(\alpha x^\gamma)}{u'(\alpha x^\gamma)} \\ &= 1 + \frac{[-u''(m)]x}{u'(\alpha x^\gamma)} \end{aligned} \tag{105}$$

where $\alpha x^\gamma - x \leq m \leq \alpha x^\gamma$ is given by the mean value theorem. Since $[-u''(c)]$ is decreasing in c on $(0, \beta x^\gamma]$, (105) yields:

$$\begin{aligned} J(x) &\leq 1 + \frac{[-u''(\alpha x^\gamma - x)]x}{u'(\alpha x^\gamma)} = 1 + \frac{[-u''(\alpha x^\gamma - x)]x}{u'(\alpha x^\gamma - x)}\left[\frac{u'(\alpha x^\gamma - x)}{u'(\alpha x^\gamma)}\right] \\ &= 1 + \frac{[-u''(\alpha x^\gamma - x)]x}{u'(\alpha x^\gamma - x)}J(x) = 1 + \frac{[-u''(\alpha x^\gamma - x)](\alpha x^\gamma - x)}{u'(\alpha x^\gamma - x)}\left[\frac{x}{(\alpha x^\gamma - x)}\right] J(x) \\ &= 1 + R(\alpha x^\gamma - x)J(x)\left[\frac{x}{(\alpha x^\gamma - x)}\right] \end{aligned} \tag{106}$$

Now, note that by (103),

$$\begin{aligned} (\alpha x^\gamma - x) &= (\alpha/2)x^\gamma + [(\alpha/2)x^\gamma - x] \\ &\geq (\alpha/2)x^\gamma \end{aligned} \tag{107}$$

So, using (107) in (106), we get:

$$\begin{aligned} J(x) &\leq 1 + R(\alpha x^\gamma - x)J(x)\left[\frac{x}{(\alpha x^\gamma - x)}\right] \leq 1 + R(\alpha x^\gamma - x)J(x)x^{1-\gamma}\left[\frac{x^\gamma}{(\alpha x^\gamma - x)}\right] \\ &\leq 1 + R(\alpha x^\gamma - x)J(x)x^{1-\gamma}(2/\alpha) \end{aligned} \tag{108}$$

Further, since $R(c)$ is decreasing in c on $(0, \beta x^\gamma]$, we can use (107) in (108) to write:

$$J(x) \leq 1 + R((\alpha/2)x^\gamma)J(x)x^{1-\gamma}(2/\alpha) \tag{109}$$

Since $z = (\alpha/2)x^{\gamma}$, we have $x = (2/\alpha)^{(1/\gamma)} z^{(1/\gamma)}$ and so:

$$x^{1-\gamma} = (2/\alpha)^{(1/\zeta)} z^{(1/\zeta)} \tag{110}$$

Using (110) in (109), we get:

$$\begin{aligned} J(x) &\leq 1 + R(z) z^{(1/\zeta)} (2/\alpha)^{(1/\gamma)} J(x) \\ &\leq 1 + (1/2) J(x) \end{aligned} \tag{111}$$

the last line of (111) following from (104). Thus,

$$J(x) \leq 2 \tag{112}$$

and so:

$$2I(x) \geq \int_1^{\lambda} \left[\frac{u'(\mu\alpha x^{\gamma})}{x^{1-\gamma} u'(\alpha x^{\gamma})} \right] d\mu \tag{113}$$

Using (113) in (101), we get:

$$\limsup_{x \to 0} \int_1^{\lambda} \left[\frac{u'(\mu\alpha x^{\gamma})}{x^{1-\gamma} u'(\alpha x^{\gamma})} \right] d\mu < \infty \tag{114}$$

Denoting αx^{γ} by t, we see that $x = t^{(1/\gamma)} \alpha^{(1/\gamma)}$, and so

$$x^{1-\gamma} = t^{(1/\zeta)} \alpha^{(1/\zeta)} \tag{115}$$

Using (115) in (114), we get:

$$\limsup_{t \to 0} \int_1^{\lambda} \left[\frac{u'(\mu t)}{t^{(1/\zeta)} u'(t)} \right] d\mu < \infty. \tag{116}$$

Using Lemma 4 and the fact that $R(c)$ is decreasing in c on $(0, \beta x^{\gamma}]$ we have:

$$\limsup_{t \to 0} \int_1^{\lambda} \left[\frac{1}{t^{(1/\zeta)} \mu^{R(t)}} \right] d\mu < \infty \tag{117}$$

Using Corollary 2 and the fact that for the chosen production function (15), $E[D_+ f(0, r)] = \infty$, we have that since the economy is nowhere bounded away from zero,

$$\limsup_{c \to 0} R(c) = \infty.$$

So, since $R(c)$ is decreasing in c on $(0, s]$, we must have:

$$R(c) \uparrow \infty \;\text{ as }\; c \downarrow 0 \text{ on } (0, s). \tag{118}$$

It follows from (118) that we can find $\bar{x} \in (0, x'')$, such that for all $x \in (0, \bar{x})$, we have:

$$\begin{array}{c} (i)\; R(\alpha x^{\gamma}) > 1 \\ (ii)\; \lambda^{R(\alpha x^{\gamma}) - 1} > 2 \end{array} \tag{119}$$

Confining our attention then to $x \in (0, \bar{x})$, and continuing to denote αx^γ by t, we see that:

$$\begin{aligned} K(t) &\equiv \int_1^\lambda \left[\frac{1}{t^{(1/\zeta)}\mu^{R(t)}}\right] d\mu = \frac{1}{t^{(1/\zeta)}} \int_1^\lambda \left[\frac{1}{\mu^{R(t)}}\right] d\mu \\ &= \frac{1}{t^{(1/\zeta)}} \left[\frac{1}{[R(t)-1]} - \frac{1}{[R(t)-1]\lambda^{R(t)-1}}\right] \geq \frac{1}{t^{(1/\zeta)}} \left[\frac{1}{2[R(t)-1]}\right] \\ &\geq \frac{1}{2R(t)t^{(1/\zeta)}} \end{aligned} \tag{120}$$

the first inequality in (120) following from (119). Combining (117) and (120), we obtain:

$$\limsup_{t \to 0} \left[\frac{1}{2R(t)t^{(1/\zeta)}}\right] \leq \limsup_{t \to 0} \int_1^\lambda \left[\frac{1}{t^{(1/\zeta)}\mu^{R(t)}}\right] d\mu < \infty$$

which implies that:

$$\liminf_{t \to 0} \, [R(t)t^{(1/\zeta)}] > 0$$

This contradicts (100) and establishes the second part of the Proposition. ■

Proof of Proposition 3

Proof. Fix $\delta \in (0, \delta')$.It is sufficient to show that in the economy (f, F, u, δ),the optimal consumption policy function, c satisfies:

$$f(y - c(y), \alpha) = (y - c(y))^{\frac{1}{2}} < y \text{ for all } y > 0 \tag{121}$$

Suppose, contrary to (121), that there is some $y > 0$, such that:

$$(y - c(y))^{\frac{1}{2}} \geq y$$

Note that for $y > \beta^2$, we have $(y - c(y))^{\frac{1}{2}} \leq y^{\frac{1}{2}} < \beta y^{\frac{1}{2}} < y$. Thus, we have:

$$z \equiv \sup\{y > 0 : (y - c(y))^{\frac{1}{2}} \geq y\} \tag{122}$$

to be well-defined. The definition of z in (122) entails that:

$$(y - c(y))^{\frac{1}{2}} < y \text{ for all } y > z. \tag{123}$$

and continuity of c implies that:

$$(z - c(z))^{\frac{1}{2}}) = z \tag{124}$$

From (123) and (124),

$$\left.\begin{array}{l} (i)\ c(y) > y - y^2 \text{ for all } y > z \\ (ii)\ c(z) = z - z^2 \end{array}\right\} \tag{125}$$

We now break up the proof into several steps.

Step 1: (Ramsey -Euler equation with z as initial stock):

Evaluating the Ramsey-Euler equation (6) at $y = z$, we have:

$$\begin{aligned} u'(c(z)) &= \delta \int_{\alpha}^{\beta} u'(c(f(x(z), r))) f'(x(z), r)] dF(r) \\ &= \frac{\delta}{2(\beta - 1)} \int_{1}^{\beta} u'(c(r(z - c(z))^{\frac{1}{2}}))) \frac{r}{(z - c(z))^{\frac{1}{2}}} d(r) \end{aligned}$$

Using (124), we obtain:

$$\frac{1}{\delta} = \frac{1}{2z(\beta - 1)} \int_{1}^{\beta} r \frac{u'(c(rz))}{u'(c(z))}) dr \tag{126}$$

Step 2: [An upper bound on the largest fixed point of the map $H(y, \alpha) = f(x(y), \alpha) = (y - c(y))^{\frac{1}{2}}$]

Using (126), and noting that $g(rz) \leq g(z)$ for all $r \in [1, \beta]$, we get:

$$\frac{1}{\delta} \leq \frac{1}{2z(\beta - 1)} \int_{1}^{\beta} r dr = \frac{\beta + 1}{4z}. \tag{127}$$

This yields the following upper bound on z :

$$\begin{aligned} z &\leq \frac{\delta(\beta + 1)}{4} < \frac{\delta'(\beta + 1)}{4} \\ &= \sqrt{\delta'} \frac{[(\beta + 1)\sqrt{\delta'}]}{4} < \sqrt{\delta'} \frac{1}{(\beta + 1)} \\ &< \frac{\phi(\theta)}{(\beta + 1)} \end{aligned} \tag{128}$$

the third inequality in (128) following from (20), and the last inequality in (128) following from (24)(i).

Step 3: [A lower bound on the optimal consumption function]

We show that the optimal consumption function c has the following lower bound:

$$c(rz) > r^{\theta}(z - z^2) \text{ for all } r \in J \equiv (1, \beta] \tag{129}$$

Pick any $r \in J$, and define:

$$w(x) = r^x \text{ for all } x \geq 0 \tag{130}$$

Clearly, w is a convex function of x, and so we have:

$$\begin{aligned} r^{1+\theta} - r &= w(1 + \theta) - w(1) \leq w'(r^{1+\theta})\theta \\ &= r^{1+\theta} \theta \ln r \end{aligned} \tag{131}$$

and:

$$\begin{aligned} r^2 - r^{\theta} &= w(2) - w(\theta) \geq w'(\theta)(2 - \theta) \\ &= r^{\theta}(2 - \theta) \ln r \end{aligned} \tag{132}$$

Using (131) and (132), we obtain:

$$\frac{r^{1+\theta}-r}{r^2-r^\theta} \leq \frac{r\theta}{(2-\theta)} \leq \frac{\beta\theta}{(2-\theta)} \tag{133}$$

Now, note that:

$$\begin{aligned}\frac{(r-r^\theta)(r+1)}{r^2-r^\theta} &= \frac{r^2-r^{1+\theta}+r-r^\theta}{r^2-r^\theta} \\ &= \frac{r^2-r^\theta}{r^2-r^\theta} - \frac{r^{1+\theta}-r}{r^2-r^\theta} = 1 - \frac{r^{1+\theta}-r}{r^2-r^\theta} \\ &\geq 1 - \frac{\beta\theta}{(2-\theta)} = \phi(\theta) \end{aligned} \tag{134}$$

where the inequality on the last line of (134) follows from (133). Thus, we get:

$$\frac{(r-r^\theta)}{r^2-r^\theta} \geq \frac{\phi(\theta)}{(r+1)} \geq \frac{\phi(\theta)}{(\beta+1)} \tag{135}$$

Combining (128) and (135), we have:

$$z < \frac{\phi(\theta)}{(\beta+1)} \leq \frac{(r-r^\theta)}{(r^2-r^\theta)} \text{ for all } r \in J \tag{136}$$

Noting that $(r^2-r^\theta) > 0$, this can be rewritten as:

$$(r^2z - r^\theta z) = (r^2-r^\theta)z < (r-r^\theta) \tag{137}$$

Multiplying through by $z > 0$, we get $(r^2z^2 - r^\theta z^2) < (rz - r^\theta z)$ which, after transposing terms, yields:

$$r^\theta(z-z^2) < (rz - r^2z^2) \tag{138}$$

Now, using (125)(i) and (138), we have for all $r \in J$,

$$g(rz) > (rz - r^2z^2) > r^\theta(z-z^2)$$

establishing claim (129).

Step 4: [The Ramsey-Euler Inequality]

Using (126), (125)(ii) and claim (129), we obtain:

$$\begin{aligned}\frac{1}{\delta} &< \frac{1}{2(\beta-1)z}\int_1^\beta r\frac{u'(r^\theta(z-z^2))}{u'(z-z^2)}dr \\ &< \frac{1}{2(\beta-1)}\int_1^\beta r\frac{u'(r^\theta(z-z^2))}{(z-z^2)u'(z-z^2)}dr \\ &= \frac{1}{2(\beta-1)}\int_1^\beta r\frac{u'(r^\theta c)}{cu'(c)}dr \end{aligned} \tag{139}$$

where $c \equiv z - z^2$.

Step 5: [Evaluating an Integral]

The idea now, given Step 4, is to show that the right hand side of (139) is actually less than one, by choice of θ in (24)(ii). This would be a contradiction to (139) since the left-hand side of (139) is clearly greater than one. This contradiction would establish (121) and hence the Proposition.

To show that the right hand side of (139) is less than one, we evaluate the integral appearing in it. Recall that $\nu = (1/\theta) > 1$. Define the function $\psi(r)$ as follows:

$$\psi(r) = -e^{(1/c^{\nu})((1/r)-1)} \text{ for all } r \geq 1 \tag{140}$$

Note that:

$$\psi'(r) = \left[\frac{1}{r^2}\right]\left[\frac{1}{c^{\nu}}\right] e^{(1/c^{\nu})((1/r)-1)} \tag{141}$$

Thus, we can write:

$$\begin{aligned} 1 - e^{(1/c^{\nu})((1/\beta)-1)} &= \psi(\beta) - \psi(1) = \int_1^{\beta} \psi'(r)dr \\ &= \int_1^{\beta} \left[\frac{1}{r^2}\right]\left[\frac{1}{c^{\nu}}\right] e^{(1/c^{\nu})((1/r)-1)} dr \end{aligned} \tag{142}$$

which yields:

$$\int_1^{\beta} \left[\frac{e^{(1/c^{\nu})((1/r)-1)}}{r^2 c^{\nu}}\right] dr < 1 \tag{143}$$

Now returning to the integral appearing in the right-hand side of (139), we have:

$$\frac{ru'(r^{\theta}c)}{u'(c)} = \frac{\nu e^{(1/rc^{\nu})}}{r^{\theta}c^{\nu+1}} \frac{c^{\nu+1}}{\nu e^{(1/c^{\nu})}} = \frac{e^{(1/c^{\nu})((1/r)-1)}}{r^{\theta}} \tag{144}$$

Using (144), we obtain:

$$\begin{aligned} \int_1^{\beta} \frac{ru'(r^{\theta}c)}{cu'(c)} dr &= \frac{1}{c}\int_1^{\beta} \frac{ru'(r^{\theta}c)}{u'(c)} dr = \frac{1}{c}\int_1^{\beta} \frac{e^{(1/c^{\nu})((1/r)-1)}}{r^{\theta}} dr \\ &= \frac{c^{\nu}}{c}\int_1^{\beta} r^{2-\theta} \frac{e^{(1/c^{\nu})((1/r)-1)}}{r^{2-\theta}c^{\nu}r^{\theta}} dr \leq c^{\nu-1}\beta^{2-\theta} \int_1^{\beta} \frac{e^{(1/c^{\nu})((1/r)-1)}}{r^2 c^{\nu}} dr \\ &< c^{\nu-1}\beta^{2-\theta} \end{aligned} \tag{145}$$

the last line of (145) following from (143). Using (145) in (139), we get:

$$\begin{aligned} \frac{1}{\delta} &< \frac{1}{2(\beta-1)} \int_1^{\beta} r\frac{u'(r^{\theta}c)}{cu'(c)} dr \\ &< \frac{c^{\nu-1}\beta^{2-\theta}}{2(\beta-1)} < \frac{c^{\nu-1}\beta^2}{2(\beta-1)} < \frac{\beta^2}{2^{\nu}(\beta-1)} \\ &< 1 \end{aligned} \tag{146}$$

the last but one inequality in (146) following from the fact that (using (23) and (128)),

$$0 < c = z - z^2 < z < 1/(\beta+1) < (1/2)$$

and $\nu > 1$, while the last inequality in (146) follows from (24)(ii). Since $\delta \in (0,1)$, (146) yields a contradiction, establishing (121) and hence the Proposition. ■

Proof of Proposition 4

Proof. Suppose not. Then, there exists a sequence $\{y^n\}_{n=1}^{\infty} \to 0, y^n \in \mathbb{R}_{++}$ such that

$$\underline{f}(x(y^n)) \leq y^n, \forall n. \tag{147}$$

Let $x^n = x(y^n)$. Then, using Lemma 1 and(147)

$$0 < x^n < y^n, \underline{f}(x^n) \leq y^n, \forall n \geq N. \tag{148}$$

Further, $\{x^n\} \to 0$. Using (6), (148) and strict concavity of u, for all $n \geq N$,

$$\begin{aligned} & u'(\underline{f}(x^n) - x^n) \geq u'(y^n - x^n) = u'(c(y^n)) \\ = \ & \delta E[u'(c(f(x^n,r)))f'(x^n,r)] \geq \delta E[u'(f(x^n,r))f'(x^n,r)] \end{aligned}$$

so that

$$\delta E[\frac{u'(f(x^n,r))}{u'(\underline{f}(x^n) - x^n)} f'(x^n,r)] \leq 1, \forall n \geq N.$$

As $\{x^n\} \to 0$ and $x^n > 0, \forall n$, we obtain a contradiction to (27). The proof is complete ■

Proof of Corollary 1

Proof. Observe that

$$\begin{aligned} \limsup_{x\to 0} \frac{f(x,r)}{\underline{f}(x) - x} &= \limsup_{x\to 0} \frac{f(x,r)}{\underline{f}(x)} \frac{\underline{f}(x)}{\underline{f}(x) - x} \\ &= \limsup_{x\to 0} \frac{f(x,r)}{\underline{f}(x)} \frac{\frac{\underline{f}(x)}{x}}{\frac{\underline{f}(x)}{x} - 1} = n(r). \end{aligned}$$

Since $\sigma > 1$, for each $r \in A$, there exists $y(r) > 0$ such that for all $x \in (0, y(r))$

$$\frac{f(x,r)}{\underline{f}(x) - x} < \sigma n(r)$$

so that

$$\frac{u'(f(x,r))}{u'(\underline{f}(x) - x)} f'(x,r)) > \frac{u'(\sigma n(r)(\underline{f}(x) - x))}{u'(\underline{f}(x) - x)} f'(x,r), \forall x \in (0, y(r)).$$

In particular, therefore,

$$\liminf_{x\to 0} \frac{u'(f(x,r))}{u'(\underline{f}(x) - x)} f'(x,r)) \geq \liminf_{x\to 0} \frac{u'(\sigma n(r)(\underline{f}(x) - x))}{u'(\underline{f}(x) - x)} f'(x,r). \tag{149}$$

Choose any sequence $\{x_k\} \to 0$. For each k, let $g_k(r)$ be the function:

$$g_k(r) = \frac{u'(f(x_k,r))}{u'(\underline{f}(x_k) - x_k)} f'(x_k,r)$$

Observe that for each k, $g_k(r) \geq 0$ and is integrable (with respect to the probability measure corresponding to the distribution function F). Using Fatou's lemma[12]:

$$\lim_{k\to\infty} \inf \int g_k(r)dF(r) \geq \int [\lim_{k\to\infty} \inf g_k(r)]dF(r)$$

so that

$$\begin{aligned} &\delta \lim_{k\to\infty} \inf E[\frac{u'(f(x_k,r))}{u'(\underline{f}(x_k)-x_k)} f'(x_k,r)] \geq \delta E[\lim_{k\to\infty} \inf \frac{u'(f(x_k,r))}{u'(\underline{f}(x_k)-x_k)} f'(x_k,r)] \\ \geq\ & \delta E[\lim_{k\to\infty} \inf \frac{u'(f(x,r))}{u'(\underline{f}(x)-x)} f'(x,r)] \end{aligned}$$

and as this holds for every sequence $\{x_k\} \to 0$ we have that

$$\begin{aligned} &\delta \lim_{x\to 0} \inf E[\frac{u'(f(x,r))}{u'(\underline{f}(x)-x)} f'(x,r)] \geq \delta E[\lim_{x\to 0} \inf \frac{u'(f(x,r))}{u'(\underline{f}(x)-x)} f'(x,r)]. \\ \geq\ & \delta E[\lim_{x\to 0} \inf \frac{u'(\sigma n(r)(\underline{f}(x)-x))}{u'(\underline{f}(x)-x)} f'(x,r)], \text{using (149)} \\ >\ & 1, \text{ using (35).} \end{aligned}$$

Thus,(27) holds and the result follows from Proposition 4. QED. ■

Using Corollary 1, we can derive the following lemma which is useful in the proof of other results.

Lemma 8 *Let $\underline{g}(\eta) : (1,\infty) \to [0,1]$ be a continuous and non-increasing function such that:*

$$\lim_{c\to 0} \inf \frac{u'(\eta c)}{u'(c)} \geq \underline{g}(\eta), \forall \eta > 1$$

Suppose

$$\delta E[\underline{g}(n(r))D_+ f(0,r)] > 1. \tag{150}$$

Then, the economy exhibits growth with certainty near zero.

Suppose, further, that $\underline{g}(\overline{n}) > 0$. Then, a sufficient condition for growth with certainty near zero is given by

$$\delta E[D_+ f(0,r)] > \frac{1}{\underline{g}(\overline{n})}. \tag{151}$$

If, in particular, $E[D_+ f(0,r)] = +\infty$, then the economy exhibits growth with certainty near zero for every $\delta \in (0,1)$.

Proof. Under (150) and using continuity of $\underline{g}$, there exists $\sigma > 1$ such that

$$\delta E[\underline{g}(\sigma n(r))D_+ f(0,r)] > 1 \tag{152}$$

[12] See, for example, Theorem 3.3 in Bhattacharya and Waymire (1990).

Since,

$$\lim_{x\to 0}\inf \frac{u'(\sigma n(r)(\underline{f}(x)-x))}{u'(\underline{f}(x)-x)} \geq \underline{g}(\sigma n(r))$$

for any $\epsilon > 0$, there exists $h > 0$ such that for all $x \in (0, h)$,

$$\frac{u'(\sigma n(r)(\underline{f}(x)-x))}{u'(\underline{f}(x)-x)} \geq \underline{g}(\sigma n(r)) - \epsilon,$$

so that

$$\begin{aligned} & \delta E[\lim_{x\to 0}\inf \frac{u'(\sigma n(r)(\underline{f}(x)-x))}{u'(\underline{f}(x)-x)} f'(x,r)] \geq \delta E[\{\underline{g}(\sigma n(r)) - \epsilon\} \lim_{x\to 0}\inf f'(x,r)] \\ = \;& \delta E[\{\underline{g}(\sigma n(r)) - \epsilon\} D_+ f'(0,r)] \end{aligned}$$

and since $\epsilon > 0$ is arbitrary

$$\begin{aligned} & \delta E[\lim_{x\to 0}\inf \frac{u'(\sigma n(r)(\underline{f}(x)-x))}{u'(\underline{f}(x)-x)} f'(x,r)] \geq \delta E[\underline{g}(\sigma n(r)) D_+ f'(0,r)] \\ > \;& 1 \text{ using (152).} \end{aligned}$$

Thus, (35) holds and from Corollary 1, we have that economy exhibits growth with certainty near zero. Using the fact that $n(r) \leq \overline{n}, \forall r \in A$ and that $\underline{g}(.)$ is non-increasing, it follows that if $\underline{g}(\overline{n}) > 0$,(151) implies (150). If, in addition, $E[D_+ f(0,r)] = +\infty$, (151) is satisfied for every $\delta \in (0,1)$. This completes the proof. ■

Proof of Proposition 5

Proof. Using (33), we have from (37) that there exists $k > 1$ such that

$$\delta E[(\frac{1}{n(r)})^{k\overline{R}} D_+ f(0,r)] > 1 \tag{153}$$

Then, we can find $\varepsilon > 0$, such that for all $c \in (0, \varepsilon)$, we have:

$$R(c) \leq k\overline{R}. \tag{154}$$

For any $\eta > 1$, for all $c \in (0, \frac{\varepsilon}{\eta}), \eta c < \varepsilon$ so that, using Lemma 4, we have

$$\frac{u'(\eta c)}{u'(c)} \geq (\frac{1}{\eta})^{k\overline{R}}.$$

Defining

$$\underline{g}(\eta) = (\frac{1}{\eta})^{k\overline{R}},$$

we can check that $\underline{g}(\eta) : (1, \infty) \to [0, 1]$ is a continuous and non-increasing function such that:

$$\lim_{c\to 0}\inf \frac{u'(\eta c)}{u'(c)} \geq \underline{g}(\eta), \forall \eta > 1$$

and, using (153),

$$\delta E[\underline{g}(n(r)) D_+ f(0,r)] > 1.$$

The proposition now follows from Lemma 8.

Proof of Lemma 6 ■

Proof. Choose any $b < 1$. Using (40), there exists $h > 0$ such that for all $z \in (0, h)$,

$$\frac{u'(z)z}{u(z)} \geq b\kappa$$

so that

$$\frac{d}{dz}[\ln u(z)] \geq \frac{b\kappa}{z} = b\kappa \frac{d}{dz}[\ln z]. \tag{155}$$

Choose any $c' \in (0, \frac{h}{\eta})$. Then, $\eta c' < h$,and integrating both sides of (155) from c' to $\eta c'$ we have:

$$\ln u(\eta c') - \ln u(c') \geq b\kappa \ln \eta$$

so that

$$\frac{u(\eta c')}{u(c')} \geq \eta^{b\kappa}, \forall c' \in (0, \frac{h}{\eta}). \tag{156}$$

First, we establish (42). Define $v(c) = u(\eta c)$ for all $c \geq 0$. Then $v(0) = u(0) = 0$. By the generalized law of mean[13] (also known as the Cauchy Mean Value Theorem), we have $\xi \in (0, c')$ such that:

$$\frac{v(c') - v(0)}{u(c') - u(0)} = \frac{v'(\xi)}{u'(\xi)} \tag{157}$$

Given the definition of v, we have $v'(c) = u'(\eta c)\eta$ for all $c > 0$. Thus, (157) can be written as:

$$\frac{u(\eta c')}{u(c')} = \frac{u'(\eta\xi)\eta}{u'(\xi)} \tag{158}$$

since $u(0) = v(0) = 0$. Using (156) in (158), we have:

$$\frac{u'(\eta\xi)}{u'(\xi)} > \eta^{-1}\frac{u(\eta c')}{u(c')} \geq \eta^{b\kappa-1}.$$

Thus, for any $c' \in (0, \frac{h}{\eta})$, there exists $\xi \in (0, c')$ such that

$$\frac{u'(\eta\xi)}{u'(\xi)} \geq \eta^{b\kappa-1}$$

so that

$$\limsup_{c\to 0} \frac{u'(\eta c)}{u'(c)} \geq \eta^{b\kappa-1}. \tag{159}$$

As (159) holds for arbitrary $b < 1$, we have (42).

Next, we establish (43). Choose any $c \in (0, \frac{h}{\eta})$. Then,

$$\begin{aligned}
\frac{u'(\eta c)}{u'(c)} &= [\frac{u'(\eta c)\eta c}{u(\eta c)}][\frac{u'(c)c}{u(c)}]^{-1}[\frac{u(\eta c)}{u(c)}]\frac{1}{\eta} \\
&\geq [\frac{u'(\eta c)\eta c}{u(\eta c)}][\frac{u'(c)c}{u(c)}]^{-1}[\frac{1}{\eta}\eta^{b\kappa}], \text{ using (156)},
\end{aligned}$$

[13] See, for example, Goldberg (1964), Theorem 7.7C, p.182.

so that:

$$\begin{aligned}\liminf_{c\to 0} \frac{u'(\eta c)}{u'(c)} &\geq \liminf_{c\to 0} [\frac{\frac{u'(\eta c)\eta c}{u(\eta c)}}{\frac{u'(c)c}{u(c)}}]\eta^{b\kappa-1} \\ &\geq \frac{\kappa}{K}\eta^{b\kappa-1}, \text{ using (40), (41) and } K>0,\end{aligned}$$

and as this holds for every $b<1$, we have

$$\liminf_{c\to 0} \frac{u'(\eta c)}{u'(c)} \geq \frac{\kappa}{K}\eta^{\kappa-1}$$

yielding (43). ■

Proof of Proposition 7

Proof. Defining

$$\underline{g}(\eta) = \frac{\kappa}{K}\eta^{\kappa-1},$$

we can check that $\underline{g}(\eta) : (1,\infty) \to [0,1]$ is a continuous and non-increasing function where (using Lemma 6):

$$\liminf_{c\to 0} \frac{u'(\eta c)}{u'(c)} \geq \underline{g}(\eta), \forall \eta > 1$$

Under (44), $\kappa > 0$ and

$$\delta E[\underline{g}(n(r))D_+ f(0,r)] > 1.$$

The proposition now follows from Lemma 8. ■

Proof of Lemma 7

Proof. Choose any initial stock $y>0$. There exists $N \geq 1$, such that $y^N < y$. Set $\alpha(y) = y^N$. Let $\{\widetilde{y}_t\}_{t=0}^{\infty}$ be the deterministic sequence defined by: $\widetilde{y}_0 = y, \widetilde{y}_{t+1} = \underline{H}(\widetilde{y}_t), t \geq 0$. Then,

$$\Pr\{y_t(y,\omega) \geq \widetilde{y}_t, \forall t \geq 0\} = 1.$$

Further, as $\underline{f}, x(.)$ are non-decreasing in y, $\underline{H}(y)$ is non-decreasing in y. One can check by induction, that $\widetilde{y}_{t+1} = \underline{H}(\widetilde{y}_t) \geq \underline{f}(x(\alpha(y))) \geq \alpha(y), \forall t \geq 0$. This concludes the proof. ■

Proof of Proposition 8

Proof. Suppose not. Then, there exists $\epsilon_1 > 0$ such that

$$\underline{f}(x(y)) < y, \forall y \in (0,\epsilon_1). \tag{160}$$

Using (6), and strict concavity of u, for every $y \in (0,\epsilon_2)$,

$$\begin{aligned}u'(c(y)) &= \delta E[u'(c(f(x(y),r)))f'(x(y),r)] \\ &\geq \delta E[u'(f(x(y),r))f'(x(y),r)]\end{aligned}$$

and since, using (160), $c(y) = y - x(y) > \underline{f}(x(y)) - x(y) > 0, \forall y \in (0,\epsilon_2)$, we have

$$u'(\underline{f}(x(y)) - x(y)) > \delta E[u'(f(x(y),r))f'(x(y),r)], \forall y \in (0,\epsilon_2),$$

As $x(y) > 0$ and continuous in y, this implies that

$$u'(\underline{f}(x) - x) > \delta E[u'(f(x,r))f'(x,r)], \forall x \in (0, x(\epsilon_2)),$$

that contradicts (47). The proof is complete. ■

Proof of Corollary 5

Proof. Observe that

$$\frac{f(x,r)}{\underline{f}(x) - x} = \frac{f(x,r)}{\underline{f}(x)} \frac{\frac{\underline{f}(x)}{x}}{\frac{\underline{f}(x)}{x} - 1}$$

Since $\sigma > 1$ and $\frac{f(x,r)}{\underline{f}(x)} \to \mu(r)$ uniformly in r on A as $x \to 0$, there exists $\epsilon > 0$, such that $\forall x \in (0, \epsilon), \forall r \in A$,

$$\frac{f(x,r)}{\underline{f}(x)} \frac{\frac{\underline{f}(x)}{x}}{\frac{\underline{f}(x)}{x} - 1} < \sigma n(r),$$

so that

$$\frac{u'(f(x,r))}{u'(\underline{f}(x) - x)} f'(x,r) > \frac{u'(\sigma n(r)(\underline{f}(x) - x))}{u'(\underline{f}(x) - x)} f'(x,r).$$

and therefore,

$$\begin{aligned} \lim_{x\to 0} \sup \delta E[\frac{u'(f(x,r))}{u'(\underline{f}(x) - x)} f'(x,r)] &\geq \lim_{x\to 0} \sup \delta E[\frac{u'(\sigma n(r)(\underline{f}(x) - x))}{u'(\underline{f}(x) - x)} f'(x,r)] \\ &> 1, \text{ using (49)}. \end{aligned}$$

Thus, (47) holds. The result follows from Propositions 8. QED. ■

Proof of Proposition 9

Proof. Under (50), there exists $\sigma > 1$ such that

$$\delta(\sigma\overline{n})^{\kappa-1} E[D_+ f(0,r)] > 1.$$

Then,

$$\begin{aligned} &\delta \lim_{x\to 0} \sup E[\frac{u'(\sigma n(r)(\underline{f}(x) - x))}{u'(\underline{f}(x) - x)} f'(x,r))] \geq \delta \lim_{x\to 0} \sup\{\frac{u'(\sigma\overline{n}(\underline{f}(x) - x))}{u'(\underline{f}(x) - x)}\} E[f'(x,r)] \\ \geq\ & \delta(\sigma\overline{n})^{\kappa-1} \lim_{x\to 0} \sup E[f'(x,r)], \text{ using (42)}, \\ =\ & \delta(\sigma\overline{n})^{\kappa-1} E[D_+ f(0,r)] > 1. \end{aligned}$$

From Corollary 5, we have that the economy is bounded away from zero. ■

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