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On Equitable Social Welfare Functions Satisfying the Weak Pareto Axiom: A Complete Characterization

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On Equitable Social Welfare Functions satisfying the Weak Pareto Axiom: A Complete Characterization*

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Abstract

The paper examines the problem of aggregating infinite utility streams with a social welfare function which respects the Anonymity and Weak Pareto Axioms. It provides a complete characterization of domains (of the one period utilities) on which such an aggregation is possible. A social welfare function satisfying the Anonymity and Weak Pareto Axioms exists on precisely those domains which do not contain any set of the *order type* of the set of positive and negative integers. The criterion is applied to decide on possibility and impossibility results for a variety of domains. It is also used to provide an alternative formulation of the characterization result in terms of the accumulation points of the domain.

Journal of Economic Literature Classification Numbers: D60, D70, D90.

Keywords and Phrases: Anonymity Axiom, Weak Pareto Axiom, Social Welfare Function, Infinite Utility Streams, Domain Restrictions, Order Types, Accumulation Points.

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1 Introduction

The possible conflict between equity and efficiency criteria in the aggregation of infinite utility streams has received considerable attention in the literature on intertemporal welfare economics.[1] In particular, formalizing the notion of equity by the Anonymity axiom and of efficiency by some form of the Pareto axiom, a number of results, indicating the impossibility of carrying out such an aggregation, have been established.

In contrast, Basu and Mitra (2007b) have observed that with domain restrictions, it is possible to have social welfare functions on infinite utility streams which satisfy simultaneously the Anonymity and Weak Pareto axioms. The question arises as to what exactly is the nature of the domain restrictions which allows such possibility results to emerge.

We consider the problem of defining social welfare functions on the set X of infinite utility streams, where this set takes the form of $X = Y^{\mathbb{N}}$, with Y denoting a non-empty subset of the reals and $\mathbb{N}$ the set of natural numbers. In discussing "domain restrictions" we refer to the set Y as the "domain" as a short-hand, even though the domain of the social welfare function is actually X. This is because we would like to study the nature of Y that allows for possibility results, and would like to give easily verifiable conditions on Y which can be checked, instead of conditions on the set X, which might be considerably harder to verify.

We start with a brief review of where the literature stands at the present time on this issue. When $Y = \mathbb{N}$, Basu and Mitra (2007b, Theorem 3, p.77) show that there is a social welfare function on $X = Y^{\mathbb{N}}$ which satisfies the Anonymity and Weak Pareto Axioms. In fact, one can write this function explicitly as the "min" function, noting that the minimum for every infinite utility stream in X exists, when $Y = \mathbb{N}$. On the other hand, if $Y = [0, 1]$, Basu and Mitra (2007b, Theorem 4, p.78) also show that there is no social welfare function on $X = Y^{\mathbb{N}}$ which satisfies the Anonymity and Weak Pareto Axioms.

This leads to the question of whether it is the countability of the set Y

[1]For a sample of the literature, see Diamond (1965), Koopmans (1972), Svensson (1980), Chichilnisky (1996), Lauwers (1998, 2010), Shinotsuka (1998), Basu and Mitra (2003, 2007a, 2007b), Fleurbaey and Michel (2003), Asheim and Tungodden (2004), Banerjee (2006), Sakai (2006), Asheim, Mitra and Tungodden (2007), Bossert, Sprumont and Suzumura (2007), Zame (2007) Hara, Shinotsuka, Suzumura and Xu (2008) and Crespo, Núñez and Rincón-Zapatero (2009).

that is crucial in allowing possibility results to emerge. This turns out to be not the case: we show that when $Y = \mathbb{I}$, where $\mathbb{I}$ is the set of positive *and* negative integers, there is no social welfare function on $X = Y^{\mathbb{N}}$ which satisfies the Anonymity and Weak Pareto Axioms.

In fact, we go further and provide a complete characterization of the domains, Y, for which there exists a social welfare function satisfying the Anonymity and Weak Pareto Axioms. These are precisely those domains which *do not* contain any set of the *order type* of the set of positive and negative integers.[2]

The characterization result provides a new perspective on known results in the literature as well as new results for domains for which the existing literature has little to offer.

The possibility part of the result is especially useful since the social welfare function can be written in explicit form by a formula, involving a weighted average of the sup and inf functions on X (using simple monotone transformations of the elements of X, if needed).

The characterization is given in terms of *order types*. This criterion is seen to be applicable to decide on possibility and impossibility results for a variety of domains. This is demonstrated by presenting a number of illustrative examples.

The criterion is also used to provide a reformulation of the characterization result in terms of *right* and *left accumulation points* of the domain. This alternative characterization is shown to be easier to apply to domains to decide on possibility and impossibility results.

2 Formal Setting and Main Result

2.1 Weak Pareto and Anonymity Axioms

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{N}$ the set of positive integers, and $\mathbb{I}$ the set of positive and negative integers. Suppose $Y \subset \mathbb{R}$ is the set of all possible utilities that any generation can achieve. Then $X = Y^{\mathbb{N}}$ is the set of all possible utility streams. If $\langle x_n \rangle \in X$, then $\langle x_n \rangle = (x_1, x_2, ...)$, where, for all $n \in \mathbb{N}$, $x_n \in Y$ represents the amount of utility that the generation of period n earns. For all $y, z \in X$, we write $y \geq z$ if $y_n \geq z_n$, for all $n \in \mathbb{N}$; we write $y > z$ if $y \geq z$ and $y \neq z$; and we write $y >> z$ if $y_n > z_n$ for all $n \in \mathbb{N}$.

[2]The term "order type" is explained in Section 2.

If Y has only one element, then X is a singleton, and the problem of ranking or evaluating infinite utility streams is trivial. Thus, without further mention, the set Y will always be assumed to have at least two distinct elements.

A social welfare function (SWF) is a mapping $W : X \to \mathbb{R}$. Consider now the axioms that we may want the SWF to satisfy. The first axiom is the Weak Pareto condition; this is a version of the Pareto axiom that has been widely used in the literature (see Arrow, 1963; Sen, 1977), and is probably even more compelling than the standard Pareto axiom.[3]

Weak Pareto Axiom: For all $x, y \in X$, if $x >> y$, then $W(x) > W(y)$.

The next axiom is the one that captures the notion of 'inter-generational equity'; we shall call it the 'anonymity axiom'.[4]

Anonymity Axiom: For all $x, y \in X$, if there exist $i, j \in \mathbb{N}$ such that $x_i = y_j$ and $x_j = y_i$, and for every $k \in \mathbb{N} \sim \{i, j\}$, $x_k = y_k$, then $W(x) = W(y)$. [5]

2.2 Domain Types

In this subsection, we recall a few concepts from the mathematical literature dealing with *types* of spaces, which are strictly ordered by a binary relation.

We will say that the set A is *strictly ordered* by a binary relation R if R is *connected* (if $a, a' \in Y$ and $a \neq a'$, then either aRa' or $a'Ra$ holds), *transitive* (if $a, a', a'' \in A$ and aRa' and $a'Ra''$ hold, then aRa'' holds) and *irreflexive* (aRa holds for no $a \in A$). In this case, the strictly ordered set will be denoted by $A(R)$. For example, the set $\mathbb{N}$ is strictly ordered by the binary relation $<$ (where $<$ denotes the usual "less than" relation on the reals).

We will say that a strictly ordered set $A'(R')$ is *similar* to the strictly ordered set $A(R)$ if there is a one-to-one function f mapping A onto A', such that:

$$a_1, a_2 \in A \text{ and } a_1 R a_2 \Longrightarrow f(a_1) R' f(a_2)$$

[3] The standard Pareto axiom is:

Pareto Axiom: For all $x, y \in X$, if $x > y$, then $W(x) > W(y)$.

We caution the reader that in some of the literature, what we are calling "Weak Pareto" is often called "Pareto", with the suffix "strong" added to what we have called the "Pareto axiom".

[4] In informal discussions throughout the paper, the terms "equity" and "anonymity" are used interchangeably.

[5] If A and B are two subsets of S, the *difference* $B \sim A$ is the set $\{z : z \in B \text{ and } z \notin A\}$. This notation follows Royden (1988, p.13).

We now specialize to strictly ordered subsets of the reals. With Y a non-empty subset of $\mathbb{R}$, let us define[6] some *order types* as follows. We will say that the strictly ordered set $Y(<)$ is:

(i) of order type ω if $Y(<)$ is similar to $\mathbb{N}(<)$;

(ii) of order type σ if $Y(<)$ is similar to $\mathbb{I}(<)$;

(iii) of order type μ if Y contains a non-empty subset Y', such that the strictly ordered set $Y'(<)$ is of order type σ.

The characterization of these types of strictly ordered sets is facilitated by the concepts of a cut, a first element and a last element of a strictly ordered set.

Given a strictly ordered set $Y(<)$, let us define a *cut* $[Y_1, Y_2]$ of $Y(<)$ as a partition of Y into two non-empty sets Y_1 and Y_2 (that is, Y_1 and Y_2 are non-empty, $Y_1 \cup Y_2 = Y$ and $Y_1 \cap Y_2 = \emptyset$), such that for each $y_1 \in Y_1$ and each $y_2 \in Y_2$, we have $y_1 < y_2$.

An element $y_0 \in Y$ is called a *first element* of $Y(<)$ if $y < y_0$ holds for no $y \in Y$. An element $y^0 \in Y$ is called a *last element* of $Y(<)$ if $y^0 < y$ holds for no $y \in Y$.

The following result can be found in Sierpinski (1965, p. 210).

Lemma 1 *A strictly ordered set $Y(<)$ is of order type σ if and only if the following two conditions hold:*

(i) Y has neither a first element nor a last element.

(ii) For every cut $[Y_1, Y_2]$ of Y, the set Y_1 has a last element and the set Y_2 has a first element.

2.3 The Characterization Result

The complete characterization result of the paper can now be stated as follows.

Theorem 1 *Let Y be a non-empty subset of $\mathbb{R}$. There exists a social welfare function $W : X \to \mathbb{R}$ (where $X = Y^{\mathbb{N}}$) satisfying the Weak Pareto and Anonymity axioms if and only if $Y(<)$ is not of order type μ.*

[6] The name "order type ω " appears in Sierpinski (1965). The name "order type σ " is our own, but it is discussed and characterized in Sierpinski (1965). The name "order type μ " is our own, and it appears to be the crucial concept for the problem we are studying.

The result implies that there is a social welfare function $W : X \to \mathbb{R}$ satisfying the Weak Pareto and Anonymity axioms (where $X = Y^{\mathbb{N}}$), when $Y = \mathbb{N}$, but that there is no such function when $Y = \mathbb{I}$. Additional examples show that the criterion given is easy to check to decide on possibility and impossibility results.

3 The Possibility Result

We first present the possibility part of the result in Theorem 1 for domains $Y \subset [0,1]$.This enables one to explicitly define a social welfare function, and verify that, when the domain Y is such that $Y(<)$ is not of type μ, the function satisfies the Weak Pareto and Anonymity axioms.

The explicit form of the social welfare function makes this possibility result potentially useful for policy purposes. In addition, the social welfare function has the desirable property[7] that it satisfies the following "monotonicity condition":

$$\text{For all } x, x' \in X, \text{ if } x > x', \text{ then } W(x) \geq W(x') \tag{M}$$

Proposition 1 *Let Y be a non-empty subset of $[0,1]$, and suppose that $Y(<)$ is not of order type μ. For $x = (x_n)_{n=1}^{\infty} \in X \equiv Y^{\mathbb{N}}$, define:*

$$W(x) = \alpha \inf\{x_n\}_{n\in\mathbb{N}} + (1-\alpha)\sup\{x_n\}_{n\in\mathbb{N}}$$

where $\alpha \in (0,1)$ is a parameter. Then W satisfies the Anonymity and Weak Pareto axioms.

Proof. (Anonymity) For any $x \in X$, $W(x)$ depends only on the set $\{x_n\}_{n\in\mathbb{N}}$. This set does not change with any finite permutation of its elements. So, $W(x)$ also does not change with any such permutation. Thus, W satisfies the Anonymity axiom.

(Weak Pareto) Let $x, x' \in X$ with $x' >> x$. We claim that $W(x') > W(x)$. Clearly, by definition of W, we have $W(x') \geq W(x)$ and so if the claim is false, it must be the case that:

$$W(x') = W(x) \tag{1}$$

[7] A weak version of Pareto, which requires that the "monotonicity condition" (M), together with what we have called Weak Pareto axiom, be satisfied, is quite appealing, and has been proposed and examined by Diamond (1965).

Since:

$$\inf\{x'_n\}_{n\in\mathbb{N}} \geq \inf\{x_n\}_{n\in\mathbb{N}} \quad \text{and} \quad \sup\{x'_n\}_{n\in\mathbb{N}} \geq \sup\{x_n\}_{n\in\mathbb{N}} \tag{2}$$

and $\alpha \in (0,1)$, it follows that:

$$a \equiv \inf\{x'_n\}_{n\in\mathbb{N}} = \inf\{x_n\}_{n\in\mathbb{N}} \quad \text{and} \quad b \equiv \sup\{x'_n\}_{n\in\mathbb{N}} = \sup\{x_n\}_{n\in\mathbb{N}} \tag{3}$$

Clearly $a, b \in [0,1]$ and $b \geq a$. In fact, we must have $b > a$, since:

$$b \equiv \sup\{x'_n\}_{n\in\mathbb{N}} \geq x'_1 > x_1 \geq \inf\{x'_n\}_{n\in\mathbb{N}} \equiv a \tag{4}$$

We now break up our analysis into the following cases:
(i) $\{x'_n\}_{n\in\mathbb{N}}$ has a minimum.
(ii) $\{x'_n\}_{n\in\mathbb{N}}$ does not have a minimum.
Case (ii) is further subdivided as follows:
(ii) (a) $\{x'_n\}_{n\in\mathbb{N}}$ does not have a minimum, and $\{x_n\}_{n\in\mathbb{N}}$ has a maximum.
(ii) (b) $\{x'_n\}_{n\in\mathbb{N}}$ does not have a minimum, and $\{x_n\}_{n\in\mathbb{N}}$ does not have a maximum.

In case (i), let $k \in \mathbb{N}$ be such that $x'_k = \min\{x'_n\}_{n\in\mathbb{N}}$. Then, we have:

$$a \equiv \inf\{x'_n\}_{n\in\mathbb{N}} = \min\{x'_n\}_{n\in\mathbb{N}} = x'_k > x_k \geq \inf\{x_n\}_{n\in\mathbb{N}} \equiv a$$

a contradiction.

In case (ii) (a), let $s \in \mathbb{N}$ be such that $x_s = \max\{x_n\}_{n\in\mathbb{N}}$. Then, we have:

$$b \equiv \sup\{x_n\}_{n\in\mathbb{N}} = \max\{x_n\}_{n\in\mathbb{N}} = x_s < x'_s \leq \sup\{x'_n\}_{n\in\mathbb{N}} = b$$

a contradiction.

Finally, we turn to case (ii) (b). Choose $c \in (a,b)$. Then, we can find $c < x_{n_1} < x_{n_2} < x_{n_3} < \cdots$ with $x_{n_k} \in (c,b)$ for $k = 1,2,3,...$, and $x_{n_k} \uparrow b$ as $k \uparrow \infty$. Similarly, we can find $x'_{m_1} > x'_{m_2} > x'_{m_3} > \cdots$ with $x'_{m_r} \in (a,c)$ for $r = 1,2,3,...$, and $x'_{m_r} \downarrow a$ as $r \uparrow \infty$. That is, we have:

$$a < \cdots x'_{m_3} < x'_{m_2} < x'_{m_1} < c < x_{n_1} < x_{n_2} < x_{n_3} < \cdots < b \tag{5}$$

Consider the set $Y' = \{x_{n_1}, x_{n_2}, x_{n_3}, ...\} \cup \{x'_{m_1}, x'_{m_2}, x'_{m_3}, ...\}$. Clearly, Y' is a subset of Y and because of (5), we note that (A) Y' has neither a maximum nor a minimum, and (B) for every cut $[Y'_1, Y'_2]$ of Y', the set Y'_1 has a last element and the set Y'_2 has a first element. Thus, by Lemma 1, $Y'(<)$ is of order type σ. This means $Y(<)$ is of order type μ, a contradiction.

Since we are led to a contradiction in cases (i), (ii)(a) and (ii)(b), and these exhaust all logical possibilities, (1) cannot hold, and our claim that $W(x') > W(x)$ is established. ■

While the possibility result in Proposition 1 is stated for domains $Y \subset [0,1]$, we will see that the result actually holds for all non-empty domains $Y \subset \mathbb{R}$ (as claimed in Theorem 1) because of an *invariance* result. This states that any possibility result is invariant with respect to monotone transformations of the domain.

Proposition 2 *Let Y be a non-empty subset of $\mathbb{R}$, $X \equiv Y^{\mathbb{N}}$, and $W : X \to \mathbb{R}$ be a function satisfying the Weak Pareto and Anonymity axioms. Suppose f is a monotone (increasing or decreasing) function from I to Y, where I is a non-empty subset of $\mathbb{R}$. Then, there is a function $V : J \to \mathbb{R}$ satisfying the Weak Pareto and Anonymity axioms, where $J = I^{\mathbb{N}}$.*

Proof. We treat two cases: (i) f is increasing, and (ii) f is decreasing.

(i) Let f be an increasing function. Define $V : J \to \mathbb{R}$ by:

$$V(z_1, z_2, ...) = W(f(z_1), f(z_2), ...) \tag{6}$$

Then, V is well-defined, since f maps I into Y.

To check that V satisfies the Anonymity axiom, let $z, z' \in J$, with $z'_r = z_s, z'_s = z_r$, and $z'_i = z_i$ for all $i \neq r, s$. Without loss of generality, assume $r < s$. Then,

$$\begin{aligned} V(z'_1, z'_2, ...) &= W(f(z'_1), f(z'_2), ..., f(z'_r), ..., f(z'_s), ...) \\ &= W(f(z'_1), f(z'_2), ..., f(z'_s), ..., f(z'_r), ...) \\ &= W(f(z_1), f(z_2), ..., f(z_r), ..., f(z_s), ...) \\ &= V(z_1, z_2, ...) \end{aligned} \tag{7}$$

the second line of (7) following from the fact that W satisfies the Anonymity axiom on X. Note that the fact that f is increasing is nowhere used in this demonstration.

To check that V satisfies the Weak Pareto axiom, let $z, z' \in J$ with $z' >> z$. We have:

$$f(z'_i) = f(z_i) + [f(z'_i) - f(z_i)] = f(z_i) + \varepsilon_i \text{ for each } i \in \mathbb{N} \tag{8}$$

where $\varepsilon_i \equiv [f(z_i') - f(z_i)] > 0$ for each $i \in \mathbb{N}$, since f is increasing. Consequently,

$$\begin{aligned} V(z_1', z_2', ...) &= W(f(z_1'), f(z_2'), ...) \\ &= W(f(z_1) + \varepsilon_1, f(z_2) + \varepsilon_2, ...) \\ &> W(f(z_1), f(z_2), ...) \\ &= V(z_1, z_2, ...) \end{aligned} \tag{9}$$

where the third line of (9) follows from the facts that W satisfies the Weak Pareto axiom on X, $f(z_i) \in Y$, $f(z_i) + \varepsilon_i \equiv f(z_i') \in Y$ for all $i \in \mathbb{N}$, and $\varepsilon_i > 0$ for all $i \in \mathbb{N}$.

(ii) Let f be a decreasing function. Define $V : J \to \mathbb{R}$ by:

$$V(z_1, z_2, ...) = -W(f(z_1), f(z_2), ...) \tag{10}$$

Then, V is well-defined, since f maps I into Y.

One can check that V satisfies the Anonymity axiom by following the steps used in (i) above. To check that V satisfies the Weak Pareto axiom, let $z, z' \in J$ with $z' >> z$. We have:

$$f(z_i') = f(z_i) + [f(z_i') - f(z_i)] = f(z_i) - \varepsilon_i \;\text{ for each } i \in \mathbb{N} \tag{11}$$

where $\varepsilon_i \equiv [f(z_i) - f(z_i')] > 0$ for each $i \in \mathbb{N}$, since f is decreasing. Consequently,

$$\begin{aligned} -V(z_1', z_2', ...) &= W(f(z_1'), f(z_2'), ...) \\ &= W(f(z_1) - \varepsilon_1, f(z_2) - \varepsilon_2, ...) \\ &< W(f(z_1), f(z_2), ...) \\ &= -V(z_1, z_2, ...) \end{aligned} \tag{12}$$

where the third line of (12) follows from the facts that W satisfies the Weak Pareto axiom on X, $f(z_i) \in Y, f(z_i) - \varepsilon_i \equiv f(z_i') \in Y$ for all $i \in \mathbb{N}$, and $\varepsilon_i > 0$ for all $i \in \mathbb{N}$. Thus, we have:

$$V(z_1', z_2', ...) > V(z_1, z_2, ...) \tag{13}$$

■

We can now state the possibility result claimed in Theorem 1 as follows.

Proposition 3 *Let Y be a non-empty subset of $\mathbb{R}$. There exists a social welfare function $W : X \to \mathbb{R}$ satisfying the Weak Pareto and Anonymity axioms (where $X \equiv Y^{\mathbb{N}}$) if $Y(<)$ is not of order type μ.*

Proof. Let us define a function $f : \mathbb{R} \to \mathbb{R}$ by:

$$f(y) = (\frac{1}{2})\left[1 + \frac{y}{1+|y|}\right] \quad \text{for all } y \in \mathbb{R} \tag{14}$$

Clearly f is an increasing function from $\mathbb{R}$ to $(0, 1)$.

Denote $f(Y)$ by A; then A is a non-empty subset of $(0, 1)$. We claim that $A(<)$ is not of order type μ. For if $A(<)$ is of order type μ, then there is a non-empty subset A' of A such that $A'(<)$ is of order type σ. Define $C = \{y \in Y : f(y) \in A'\}$. Then C is a non-empty subset of Y and f is an increasing function from C onto A'. Thus, $C(<)$ is similar to $A'(<)$ and so $C(<)$ is of order type σ. Clearly C is a non-empty subset of Y, and so $Y(<)$ must be of order type μ, a contradiction. This establishes our claim.

Since A is a non-empty subset of $(0, 1)$, and $A(<)$ is not of order type μ, we can apply Proposition 1 to obtain a function $U : B \to \mathbb{R}$ satisfying the Weak Pareto and Anonymity axioms, where $B \equiv A^{\mathbb{N}}$.

Since f is an increasing function from Y to A and Y is a non-empty subset of $\mathbb{R}$, we can now apply Proposition 2 to obtain a function $W : X \to \mathbb{R}$ satisfying the Weak Pareto and Anonymity axioms, where $X = Y^{\mathbb{N}}$. ■

We now discuss examples which illustrate the usefulness of Proposition 3.

Example 3.1:

Let $Y = \mathbb{N}$ and $X = Y^{\mathbb{N}}$. We claim that $Y(<)$ is not of order type μ. For if $Y(<)$ is of order type μ, then Y contains a non-empty subset Y' such that $Y'(<)$ is of order type σ. Thus, by Lemma 1, $Y'(<)$ has no first element. But, any non-empty subset of $\mathbb{N}(<)$ has a first element (Munkres (1975, Theorem 4.1, p.32)). This contradiction establishes the claim.

Using Proposition 3, there is a function $W : X \to \mathbb{R}$ satisfying the Weak Pareto and Anonymity axioms, where $X = Y^{\mathbb{N}}$. This provides an alternative approach to the possibility result noted in Basu and Mitra (2007b, Theorem 3, p.77), and in Lauwers (2010, p.37).

Example 3.2:

Let Y be defined by:

$$Y = \{1/n\}_{n \in \mathbb{N}}$$

and let $X = Y^{\mathbb{N}}$. We claim that $Y(<)$ is not of order type μ. For if $Y(<)$ is of order type μ, then Y contains a non-empty subset Y' such that $Y'(<)$ is of order type σ. Then, defining:

$$Z = \{(1/y) : y \in Y'\}$$

we see that Z is a non-empty subset of $\mathbb{N}$. Thus, $Z(<)$ has a first element and so $Y'(<)$ has a last element. But, by Lemma 1, $Y'(<)$ cannot have a last element. This contradiction establishes the claim.

Using Proposition 3, there is a function $W : X \to \mathbb{R}$ satisfying the Weak Pareto and Anonymity axioms, where $X = Y^{\mathbb{N}}$. This result is mentioned without proof in Basu and Mitra (2007b, footnote 9, p.83).

Example 3.3:

Define $A = \{-1/n\}_{n\in\mathbb{N}}$, $B = \{1/n\}_{n\in\mathbb{N}}$ and $Y = A \cup B, X = Y^{\mathbb{N}}$. We claim that $Y(<)$ is not of order type μ. For if $Y(<)$ is of order type μ, then Y contains a non-empty subset Y' such that $Y'(<)$ is of order type σ.

Define $A' = A \cap Y'$ and $B' = B \cap Y'$. If B' is non-empty, then B' is a non-empty subset of B, and therefore $B'(<)$ has a last element (see Example 2 above), call it b. If A' is empty, then $Y' = B'$ and $Y'(<)$ has a last element, contradicting Lemma 1. If A' is non-empty, then for every $y \in A'$, we have $y < b$. Thus, b is a last element of $Y'(<)$, contradicting Lemma 1 again.

If B' is empty, then $Y' = A'$. Further, A' is a non-empty subset of A, and therefore has a first element (see Example 2 above). Thus, $Y'(<)$ must have a first element, contradicting Lemma 1.

The above cases exhaust all logical possibilities, and therefore our claim is established. Using Proposition 3, there is a function $W : X \to \mathbb{R}$ satisfying the Weak Pareto and Anonymity axioms, where $X = Y^{\mathbb{N}}$.

4 The Impossibility Result

We will first present the impossibility part of the result in Theorem 1 for the domain $Y = \mathbb{I}$, the set of positive and negative integers. Clearly $\mathbb{I}(<)$ is of type σ and therefore of type μ. This enables us to illustrate our approach to the impossibility result in the most transparent way. We will then use Proposition 2 to show that when an arbitrary non-empty subset, Y, of the reals is such that $Y(<)$ is of type μ, there is no social welfare function satisfying the Weak Pareto and Anonymity axioms.

Proposition 4 *Let $Y = \mathbb{I}$. Then there is no social welfare function $W : X \to \mathbb{R}$ satisfying the Weak Pareto and Anonymity axioms (where $X = Y^{\mathbb{N}}$).*

Proof. Suppose on the contrary that there is a social welfare function $W : X \to \mathbb{R}$ satisfying the Weak Pareto and Anonymity axioms (where $X \equiv Y^{\mathbb{N}} = \mathbb{I}^{\mathbb{N}}$).

Let Q be a fixed enumeration of the rationals in $(0,1)$. Then, we can write:

$$Q = \{q_1, q_2, q_3, ...\}$$

For any real number $t \in (0,1)$, there are infinitely many rational numbers from Q in $(0,t)$ and in $[t,1)$.

For each real number $t \in (0,1)$, we can then define the set $M(t) = \{n \in \mathbb{N} : q_n \in (0,t)\}$ and the sequence $\langle m_s(t)\rangle$ as follows:

$$m_1(t) = \min\{n \in \mathbb{N} : q_n \in (0,t)\}$$

and for $s \in \mathbb{N}$, $s > 1$,

$$m_s(t) = \min\{n \in \mathbb{N} \sim \{m_1(t), ..., m_{s-1}(t)\} : q_n \in (0,t)\}$$

The sequence $\langle m_s(t)\rangle$ is then well-defined, and:

$$m_1(t) < m_2(t) < m_3(t)...$$

and $M(t) = \{m_1(t), m_2(t), ...\}$.

For each real number $t \in (0,1)$, we can define the set $P(t) = \{n \in \mathbb{N} : q_n \in [t,1)\}$ and the sequence $\langle p_r(t)\rangle$ as follows:

$$p_1(t) = \min\{n \in \mathbb{N} : q_n \in [t,1)\}$$

and for $r \in \mathbb{N}$, $r > 1$,

$$p_r(t) = \min\{n \in \mathbb{N} \sim \{p_1(t), ..., p_{r-1}(t)\} : q_n \in [t,1)\}$$

The sequence $\langle p_r(t)\rangle$ is then well-defined, and:

$$p_1(t) < p_2(t) < p_3(t)...$$

and $P(t) = \{p_1(t), p_2(t), ...\}$.

In order to make the exposition transparent, we now break up the proof into four steps.

Step 1 (Defining the sequence $\langle x(t) \rangle$)

For each real number $t \in (0,1)$, we note that $M(t) \cap P(t) = \emptyset$, and $M(t) \cup P(t) = \mathbb{N}$. Then, we can define a sequence $\langle x(t) \rangle$ as follows:

$$x_n(t) = \begin{cases} 2s - 1 & \text{if } n = m_s \text{ for some } s \in \mathbb{N} \\ -2r - 1 & \text{if } n = p_r \text{ for some } r \in \mathbb{N} \end{cases} \tag{15}$$

Note that the sequence $\langle x_n(t) \rangle$ will contain, by (15), all the positive odd numbers in increasing order of magnitude with n, and all the negative odd numbers less than (-1) in decreasing order of magnitude with n.

Step 2 (Comparing $\langle x(\alpha) \rangle$ with $\langle x(\beta) \rangle$)

Let α, β be arbitrary real numbers in $(0,1)$, with $\alpha < \beta$. Note that if $n \in M(\alpha)$, then $n \in M(\beta)$, and if $n \in P(\beta)$ then $n \in P(\alpha)$. Since there are an infinite number of rationals from Q in $[\alpha, \beta)$, there will be an infinite number of distinct elements of $\mathbb{N}$ in:

$$L(\alpha, \beta) = M(\beta) \cap P(\alpha) = \{n \in \mathbb{N} : q_n \in [\alpha, \beta)\}$$

For any $n \in L(\alpha, \beta)$, we have $n \in M(\beta)$ but $n \notin M(\alpha)$. That is, by (15), for each $n \in L(\alpha, \beta)$ it must be the case that $x_n(\alpha) < 0$ but $x_n(\beta) > 0$. Consequently, one has:

$$x_n(\beta) \geq x_n(\alpha) \quad \text{for all } n \in \mathbb{N} \tag{16}$$

Informally, these observations may be expressed as follows. In comparing the sequence $\langle x(\alpha) \rangle$ with $\langle x(\beta) \rangle$, whenever $\langle x(\alpha) \rangle$ has a positive entry for some co-ordinate, there must be a positive entry for that co-ordinate in $\langle x(\beta) \rangle$. There will be an infinite number of co-ordinates (switches) for which $\langle x(\alpha) \rangle$ will have a negative entry, but for which $\langle x(\beta) \rangle$ will have a positive entry. For the remaining co-ordinates, both $\langle x(\alpha) \rangle$ and $\langle x(\beta) \rangle$ will have negative entries. Because of the switches, $\langle x(\beta) \rangle$ uses up the sub-indices in $M(\beta)$ earlier and postpones using the sub-indices in $P(\beta)$ till later compared to $\langle x(\alpha) \rangle$, leading to (16).

One can strengthen the conclusion in (16) as follows. This also formalizes the informal observations given above. Define:

$$N = \min\{n \in \mathbb{N} : n \in L(\alpha, \beta)\}$$

Then, by (15), we have $x_N(\alpha) < 0, x_N(\beta) > 0$, and:

$$x_N(\beta) - x_N(\alpha) \geq 2 \tag{17}$$

Consider any $n \in \mathbb{N}$ with $n > N$. We have either (i) $n \in M(\alpha)$ or (ii) $n \in P(\alpha)$. Case (ii) can be subdivided as follows: (a) $n \in P(\alpha)$ and $n \in P(\beta)$, (b) $n \in P(\alpha)$ and $n \notin P(\beta)$.

In case (i), we have $n \in M(\alpha)$ and so $n \in M(\beta)$. But since an additional element of $M(\beta)$ has been used up for index N, compared with $M(\alpha)$, if $n = m_k(\alpha)$, we must have $n = m_{k+j}(\beta)$ for some $j \in \mathbb{N}$. Thus, by (15), we must have:

$$x_n(\beta) - x_n(\alpha) \geq 2 \tag{18}$$

In case (ii)(a), we have $n \in P(\alpha)$ and $n \in P(\beta)$. But since an additional element of $P(\alpha)$ has been used up for index N, compared with $P(\beta)$, if $n = p_r(\alpha)$, we must have $n = p_{r-j}(\beta)$ for some $j \in \mathbb{N}$. Thus, by (15), we must have:

$$x_n(\beta) - x_n(\alpha) \geq 2 \tag{19}$$

In case (ii)(b), $n \in P(\alpha)$ and $n \notin P(\beta)$, so that $n \in M(\beta)$. That is, $n \in L(\alpha, \beta)$. Thus, by (15), we have $x_n(\alpha) < 0, x_n(\beta) > 0$, and:

$$x_n(\beta) - x_n(\alpha) \geq 2 \tag{20}$$

To summarize, for all $n \geq N$, we have:

$$x_n(\beta) - x_n(\alpha) \geq 2 \tag{21}$$

For $n \in \mathbb{N}$ with $n < N$ (if any), we have:

$$x_n(\beta) = x_n(\alpha) \tag{22}$$

Step 3 (Comparing $x(\alpha)$ with a finite permutation of $x(\beta)$)

Based on (21) and (22), we cannot say that $W(\langle x_n(\alpha)\rangle) < W(\langle x_n(\beta)\rangle)$, by invoking the Weak Pareto Axiom, except if $N = 1$, where (by (21)):

$$x_n(\beta) - x_n(\alpha) \geq 2 \text{ for all } n \in \mathbb{N} \tag{23}$$

We consider now the case in which $N > 1$. We will show that (21) and (22) can be used to obtain:

$$x'_n(\beta) - x_n(\alpha) \geq 2 \text{ for all } n \in \mathbb{N}$$

where $\langle x'(\beta)\rangle$ is a certain finite permutation of $\langle x(\beta)\rangle$.

Let $n_1, ..., n_{N-1}$ be the $(N-1)$ smallest elements of $\mathbb{N}$ (with $n_1 < \cdots < n_{N-1}$) for which $x_{n_i}(\alpha) < 0$ and $x_{n_i}(\beta) > 0$ for $i \in \{1, ..., N-1\}$. Note that

$N = n_1$. Then, define $\langle x'_n(\beta) \rangle$ to be the sequence obtained by interchanging the i th entry of $\langle x_n(\beta) \rangle$ with the n_i th entry of $\langle x_n(\beta) \rangle$ for $i = 1, ..., N-1$, and leaving all other entries unchanged.

If $i \in \{1, ..., N-1\}$, and $x_i(\alpha) < 0$, then:

$$x'_i(\beta) = x_{n_i}(\beta) > 0 \geq x_i(\alpha) + 2$$

and if $x_i(\alpha) > 0$, then by (16),

$$x'_i(\beta) = x_{n_i}(\beta) \geq x_i(\beta) + 2 \geq x_i(\alpha) + 2$$

That is, in either case,

$$x'_i(\beta) \geq x_i(\alpha) + 2 \quad \text{for all } i \in \{1, ..., N-1\} \tag{24}$$

If $i \in \{1, ..., N-1\}$, and $x_i(\beta) > 0$, then since $x_{n_i}(\alpha) < 0$, we have:

$$x'_{n_i}(\beta) = x_i(\beta) > 0 \geq x_{n_i}(\alpha) + 2$$

Also, if $x_i(\beta) < 0$, then by (22),

$$x'_{n_i}(\beta) = x_i(\beta) = x_i(\alpha) \geq x_{n_i}(\alpha) + 2$$

using the fact that $x_{n_i}(\alpha) < 0$ and $i < n_i$.That is, in either case,

$$x'_{n_i}(\beta) \geq x_{n_i}(\alpha) + 2 \quad \text{for all } i \in \{1, ..., N-1\} \tag{25}$$

Finally, for $n \in \mathbb{N}$ with $n \notin \{1, ..., N-1\} \cup \{n_1, ..., n_{N-1}\}$, we have $x'_n(\beta) = x_n(\beta)$, and so by (21),

$$x'_n(\beta) \geq x_n(\alpha) + 2$$

Thus, we have established that:

$$x'_n(\beta) \geq x_n(\alpha) + 2 > x_n(\alpha) + 1 \quad \text{for all } n \in \mathbb{N}$$

Using the Anonymity and Weak Pareto Axioms, we have:

$$W(\langle x_n(\beta) \rangle) = W(\langle x'_n(\beta) \rangle) > W(\langle x_n(\alpha) + 1 \rangle) \tag{26}$$

Step 4 (Non-overlapping intervals for distinct real numbers in (0,1))

Define, for each $t \in (0,1)$, a sequence $\langle z_n(t) \rangle$ by:

$$z_n(t) = x_n(t) + 1 \text{ for all } n \in \mathbb{N} \tag{27}$$

Note that the sequence $\langle z_n(t) \rangle$ is in X, and by the Weak Pareto axiom:

$$W(\langle z_n(t) \rangle) > W(\langle x_n(t) \rangle)$$

Thus, for each $t \in (0,1)$,

$$I(t) = [W(\langle x_n(t) \rangle), W(\langle z_n(t) \rangle)] \tag{28}$$

is a non-degenerate closed interval in $\mathbb{R}$.

Let α, β be arbitrary real numbers in $(0,1)$, with $\alpha < \beta$. Then, by (26),

$$W(\langle x_n(\beta) \rangle) > W(\langle z_n(\alpha) \rangle) \tag{29}$$

Thus, the interval $I(\beta)$ lies entirely to the right of the interval $I(\alpha)$ on the real line.

That is, for arbitrary real numbers α, β in $(0,1)$, with $\alpha \neq \beta$, the intervals $I(\alpha)$ and $I(\beta)$ are disjoint. Thus, we have a one-to-one correspondence between the real numbers in $(0,1)$ (which is an uncountable set) and a set of non-degenerate, pairwise disjoint closed intervals of the real line (which is countable). This contradiction establishes the Proposition. ■

We can now state the impossibility result for general domains of order type μ.

Proposition 5 *Let Y be a non-empty subset of $\mathbb{R}$ such that $Y(<)$ is of order type μ. Then there is no social welfare function $W : X \to \mathbb{R}$ satisfying the Weak Pareto and Anonymity axioms (where $X = Y^{\mathbb{N}}$).*

Proof. Suppose on the contrary that there is a social welfare function $W : X \to \mathbb{R}$ satisfying the Weak Pareto and Anonymity axioms (where $X = Y^{\mathbb{N}}$). Since $Y(<)$ is of order type μ, Y contains a non-empty subset Y' such that $Y'(<)$ is of order type σ. That is, there is a one-to-one mapping, g, from $\mathbb{I}$ onto Y' such that:

$$a_1, a_2 \in \mathbb{I} \text{ and } a_1 < a_2 \Longrightarrow g(a_1) < g(a_2)$$

Thus, g is an increasing function from $\mathbb{I}$ to Y. Using Proposition 2, there is a function $V : J \to \mathbb{R}$ satisfying the Weak Pareto and Anonymity axioms, where $J = \mathbb{I}^{\mathbb{N}}$. But this contradicts the result proved in Proposition 4, and establishes the result. ■

We now discuss examples which illustrate the usefulness of Proposition 5.

Example 4.1:

Let $Y = A \cup B$, where $A = \{-n/(1+n)\}_{n\in\mathbb{N}}$ and $B = \{n/(1+n)\}_{n\in\mathbb{N}}$ and let $X = Y^{\mathbb{N}}$. Define $f : \mathbb{I} \to \mathbb{R}$ by:

$$f(y) = \frac{y}{1+|y|} \quad \text{for all } y \in \mathbb{I}$$

where $\mathbb{I} \equiv \{n\}_{n\in\mathbb{N}} \cup \{-n\}_{n\in\mathbb{N}}$. Then, f is an increasing function from $\mathbb{I}$ onto Y. Thus, $Y(<)$ is similar to $\mathbb{I}(<)$ and is therefore of order type σ. By Proposition 5, there is no function $W : X \to \mathbb{R}$ satisfying the Anonymity and Weak Pareto axioms.

Example 4.2:

Let Y be the set of rationals in $\mathbb{R}$, and let $X = Y^{\mathbb{N}}$. Then, since $\mathbb{I} \equiv \{n\}_{n\in\mathbb{N}} \cup \{-n\}_{n\in\mathbb{N}}$ is a subset of Y, and $\mathbb{I}(<)$ is of order type σ, $Y(<)$ is of order type μ. Thus, by Proposition 5, there is no function $W : X \to \mathbb{R}$ satisfying the Anonymity and Weak Pareto axioms.

Example 4.3:

Let Y be the set of positive rationals in $\mathbb{R}$, and let $X = Y^{\mathbb{N}}$. Define $Y' = \{1/n\}_{n\in\mathbb{N}} \cup \{n\}_{n\in\mathbb{N}}$, and $f : \mathbb{I} \to \mathbb{R}$ by:

$$f(y) = \begin{cases} y & \text{if } y \in B \\ 1/|y| & \text{if } y \in A \end{cases}$$

where $\mathbb{I} \equiv A \cup B$, with $A = \{-n\}_{n\in\mathbb{N}}$, and $B = \{n\}_{n\in\mathbb{N}}$. Then, f is an increasing function from $\mathbb{I}$ onto Y'. Thus, $Y'(<)$ is similar to $\mathbb{I}(<)$ and is therefore of order type σ. Since $Y' \subset Y$, $Y(<)$ is of order type μ. By Proposition 5, there is no function $W : X \to \mathbb{R}$ satisfying the Anonymity and Weak Pareto axioms.

Example 4.4:

Let Y be the closed interval $[0,1]$ in $\mathbb{R}$, and let and $X = Y^{\mathbb{N}}$. Define $Z = \{1/n\}_{n\in\mathbb{N}} \cup \{n\}_{n\in\mathbb{N}}$, Y' to be the set of rationals in $(0,1)$, and $f : Z \to \mathbb{R}$ by:

$$f(y) = \frac{y}{1+y} \quad \text{for all } y \in Z$$

Then, f is an increasing function from Z into Y'. Thus, $f(Z)(<)$ is similar to $Z(<)$, which is of type σ (by Example 4.3). Since Y' contains $f(Z)$, $Y'(<)$ is of type μ. Since Y contains Y', Y contains $f(Z)$, and so $Y(<)$ is of type μ.

By Proposition 5, there is no function $W : X \to \mathbb{R}$ satisfying the Anonymity and Weak Pareto axioms. Our discussion of Example 4.4 provides an alternative proof for the impossibility theorem of Basu and Mitra (2007b, Theorem 4, p.78).

5 A Reformulation of the Main Result

We have demonstrated that the complete characterization result in Theorem 1 can be applied to provide possibility and impossibility results for a variety of domains. Nevertheless, it will not have escaped the reader's attention that checking the criterion involves checking all possible subsets of Y and determining whether any of these subsets is of the order type σ, the order type of the set of positive and negative integers. Checking whether a set in $\mathbb{R}$ is of order type σ is relatively easy, given Lemma 1, but checking this for all possible subsets of Y may not be.

With this in mind, we devote this final section to a reformulation of the main result in terms of a criterion which involves looking at the *accumulation points* of Y.

For what follows, Y will be taken to be a non-empty subset of $[0, 1]$. If for an application, one encounters a non-empty subset Y of $\mathbb{R}$ which is not a subset of $[0, 1]$, one can always make a change of variable in the domain (through a monotone increasing function) so that the new domain Y' is a non-empty subset of $[0, 1]$. We have, in fact, done this already in discussing examples in Sections 3 and 4.

Because $Y \subset [0, 1]$ is a subset of $\mathbb{R}$, it is possible to define *right accumulation points* and *left accumulation points* in the same spirit as right hand limits and left hand limits.

We will say that $z \in \mathbb{R}$ is a *right accumulation point* of Y if given any $\delta > 0$, there is $y \in Y$ such that:

$$0 < y - z < \delta$$

Similarly, we will say that $z \in \mathbb{R}$ is a *left accumulation point* of Y if given any $\delta > 0$, there is $y \in Y$ such that:

$$0 < z - y < \delta$$

Denote by R the set of right accumulation points of Y and by L the set of left accumulation points of Y. If Y has an infinite number of elements, then (being bounded) it will have an accumulation point.[8] Any accumulation point of Y will be either a right accumulation point or a left accumulation point or both. Further a right or left accumulation point of Y is also an accumulation point of Y.

Let us denote:

$$\rho \equiv \inf R \quad \text{and} \quad \lambda \equiv \sup L$$

with the convention that if R is empty, then $\rho = \infty$, and if L is empty then $\lambda = -\infty$.

We can now state our characterization result as follows.

Theorem 2 *Let Y be a non-empty subset of $[0,1]$. There exists a social welfare function $W : X \to \mathbb{R}$ (where $X = Y^{\mathbb{N}}$) satisfying the Weak Pareto and Anonymity axioms if and only if:*

$$\rho \equiv \inf R \geq \sup L \equiv \lambda \tag{30}$$

Proof. (Necessity) Suppose condition (30) is violated; that is:

$$\inf R < \sup L \tag{31}$$

Given the convention adopted, this means that ρ, λ are in $\mathbb{R}$ and $\rho < \lambda$. It follows that there is a right accumulation point ρ' of Y and a left accumulation point λ' of Y such that $\rho' < \lambda'$.

Choose $c \in (\rho', \lambda')$. Then, we can find $c < y_1 < y_2 < y_3 < \cdots$ with $y_k \in Y$ for $k \in \mathbb{N}$ such that $y_k \uparrow \lambda'$ as $k \uparrow \infty$ (since λ' is a left accumulation point of Y). Similarly, we can find $c > y'_1 > y'_2 > y'_3 > \cdots$ with $y'_r \in Y$ for $r \in \mathbb{N}$ such that $y'_r \downarrow \rho'$ as $r \uparrow \infty$ (since ρ' is a right accumulation point of Y). That is, we have:

$$\rho' < \cdots y'_3 < y'_2 < y'_1 < c < y_1 < y_2 < y_3 < \cdots < \lambda' \tag{32}$$

Consider the set $Y' = \{y_1 < y_2 < y_3 < \cdots\} \cup \{y'_1 < y'_2 < y'_3 < \cdots\}$. Clearly, Y' is a subset of Y and because of (32), we note that (A) Y' has neither a maximum nor a minimum, and (B) for every cut $[Y'_1, Y'_2]$ of Y', the set Y'_1 has a last element and the set Y'_2 has a first element. Thus, by Lemma 1, $Y'(<)$ is of order type σ. This means $Y(<)$ is of order type μ, and by Theorem 1, there is no social welfare function $W : X \to \mathbb{R}$ (where $X = Y^{\mathbb{N}}$) satisfying the Weak Pareto and Anonymity axioms.

[8] For the standard definition of an accumulation point of a set, see Royden (1988, p.46).

(Sufficiency) Suppose (30) holds. We claim that $Y(<)$ is not of order type μ. For if it is of order type μ, there is a non-empty subset $Y' \subset Y$, such that $Y'(<)$ is of order type σ. Since $Y' \subset [0,1]$, it has a greatest lower bound, a, and a least upper bound, b. Clearly $a \leq b$.

Since $Y'(<)$ is of order type σ, it does not have a maximum. So, b cannot be in Y'. Since b is a least upper bound of Y', we can find $y_1 < y_2 < y_3 < \cdots$ with $y_k \in Y'$ for $k \in \mathbb{N}$ such that $y_k \uparrow b$ as $k \uparrow \infty$. Then, $b \in L$, and so $b \leq \lambda$.

Since $Y'(<)$ is of order type σ, it does not have a minimum. So, a cannot be in Y'. Since a is a greatest lower bound of Y', we can find $y'_1 > y'_2 > y'_3 > \cdots$ with $y'_r \in Y'$ for $r \in \mathbb{N}$ such that $y'_r \downarrow a$ as $r \uparrow \infty$. Then, $a \in R$, and so $a \geq \rho$.

Thus, we have:

$$a \geq \rho \geq \lambda \geq b \geq a$$

so that $a = b$. But then Y' must be a singleton, and therefore $Y'(<)$ cannot be of order type σ. This contradiction establishes our claim. Now, applying Theorem 1, there exists a social welfare function $W : X \to \mathbb{R}$ (where $X = Y^{\mathbb{N}}$) satisfying the Weak Pareto and Anonymity axioms. ■

Remarks:

We can now re-examine the examples in Sections 3 and 4 to see the applicability of Theorem 2 in deciding on possibility and impossibility results.

In the examples in Section 3, one can check that $\rho \geq \lambda$, so by Theorem 2 there exists a social welfare function $W : X \to \mathbb{R}$ (where $X = Y^{\mathbb{N}}$) satisfying the Weak Pareto and Anonymity axioms.

In Example 3.1, $Y = \mathbb{N}$ which is similar to $Y' = \{n/(1+n)\}_{n \in \mathbb{N}}$. So it is enough to examine Y' which is a subset of $[0,1]$. There is one left accumulation point (namely 1) and no right accumulation point. So $\rho = \infty$ while $\lambda = 1$, yielding $\rho \geq \lambda$.

In Example 3.2, $Y = \{1/n\}_{n \in \mathbb{N}}$, so there is one right accumulation point (namely 0) and no left accumulation point. Thus $\rho = 0$ while $\lambda = -\infty$, yielding $\rho \geq \lambda$.

In Example 3.3, $Y = \{1/n\}_{n \in \mathbb{N}} \cup \{-1/n\}_{n \in \mathbb{N}}$, so there is one right accumulation point (namely 0) and one left accumulation point (namely 0). Thus $\rho = 0 = \lambda$.

In the examples in Section 4, one can check that $\rho < \lambda$ (for a non-empty set Y' similar to Y) so by Theorem 2 there does not exist any social welfare function $W : X \to \mathbb{R}$ (where $X = Y^{\mathbb{N}}$) satisfying the Weak Pareto and Anonymity axioms.

In Example 4.1, $Y = A \cup B$, where $A = \{-n/(1+n)\}_{n\in\mathbb{N}}$ and $B = \{n/(1+n)\}_{n\in\mathbb{N}}$. Then Y is similar to $Y' = f(Y)$, where f is given by:

$$f(y) = (\frac{1}{2})(1+y) \text{ for all } y \in \mathbb{Y}$$

Then Y' is a non-empty subset of $[0, 1]$. It has a right accumulation point at 0 and a left accumulation point at 1. Thus $\rho = 0 < 1 = \lambda$.

In Example 4.2, Y is the set of rationals in $\mathbb{R}$. Then Y is similar to $Y' = f(Y)$, where f is given by (14). Then, Y' is a non-empty subset of $[0, 1]$, coinciding with set of rationals in $(0, 1)$. Thus, every point in $[0, 1)$ is a right accumulation point, and every point in $(0, 1]$ is a left accumulation point of Y'. Thus, $\rho = 0 < 1 = \lambda$.

In Example 4.3, Y is the set of positive rationals in $\mathbb{R}$. Then, Y is similar to $Y' = f(Y)$, where f is given by:

$$f(y) = \frac{y}{1+y} \text{ for all } y \in \mathbb{Y}$$

Then, Y' coincides with set of rationals in $(0, 1)$, and (as in Example 4.2), we have $\rho < \lambda$.

In Example 4.4, $Y = [0, 1]$. Then, every point in $[0, 1)$ is a right accumulation point, and every point in $(0, 1]$ is a left accumulation point of Y. Thus, $\rho = 0 < 1 = \lambda$.

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