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On the Phelps-Koopmans Theorem

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Abstract

We examine whether the Phelps-Koopmans theorem is valid in models with nonconvex production technologies. We show by example that a nonstationary path that converges to a capital stock above the smallest golden rule may indeed be efficient. This finding has the important implication that "capital overaccumulation" need not always imply inefficiency. We provide general conditions on the production function under which all paths that have a limit in excess of the smallest golden rule must be efficient, which proves a version of the theorem in the nonconvex case. Finally, we show by example that a nonconvergent path with limiting capital stocks bounded above (and away from) the smallest golden rule can be efficient, even if the model admits a unique golden rule. Thus the Phelps-Koopmans theorem in its general form fails to be valid.

Journal of Economic Literature Classification Numbers: C61, O41.

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1. Introduction

The phenomenon of inefficiency of intertemporal consumption streams has been traditionally identified with the overaccumulation of capital. In fact, this message is strongly conveyed in two famous papers on efficiency by Malinvaud (1953) and Cass (1972).[1]

In the standard aggregative model of economic growth, the Phelps-Koopmans theorem provides one of the most well-known sufficient conditions for inefficiency.[2] This result was conjectured by Phelps (1962), and its proof, based on a proof provided by Koopmans, appeared in Phelps (1965). It states that if the capital stock of a path is above, and bounded away from, the golden rule stock, from a certain time onward, then the path is inefficient.[3]

The purpose of this paper is to examine the validity of the Phelps-Koopmans theorem in aggregative models which allow for non-convexity of the production set.[4] Of course, nonconvexity is no impediment to the existence of a golden rule provided that suitable end-point conditions hold (which we shall assume). Indeed, there may be several; we will refer to the smallest of them as the *minimal* golden rule. The Phelps-Koopmans theorem can then be restated in three progressively stronger formats:

I. Every stationary path with capital stock in excess of the minimal golden rule is inefficient.

II. A path is inefficient if it converges to a limit capital stock in excess of the minimal golden rule.

III. A path is inefficient if it is lies above (and bounded away from) the minimal golden rule from a certain time onwards.

Obviously, version III nests II, which in turn nests version I.

It is very easy to see that the weakest version I of the Phelps-Koopmans theorem must be true. Our first result (Proposition 1) shows that version II of the theorem is generally false. We present an example of a path that converges to a limit stock that exceeds the minimal golden rule, which is nevertheless efficient. This has the important implication that the phenomenon of "overaccumulation of capital" need not always imply inefficiency.

Since this finding might appear somewhat surprising, we try to convey an intuition for the result. Consider a setting with multiple golden rule stocks, and construct a path whose capital stock converges to some *non-minimal* (and therefore, by version I, inefficient) golden rule stock from above in such a way that at each period, the consumption level on the

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[1]In fact, one might make a case that this message can be inferred from the titles of the two papers.
[2]In awarding the Prize in Economic Sciences in Memory of Alfred Nobel for 2006 to Edmund Phelps, the Royal Swedish Academy of Sciences referred to this result as follows: "Phelps . . . showed that all generations may, under certain conditions, gain from changes in the savings rate."
[3]The expression "overaccumulation of capital" in this literature refers therefore to accumulation of capital in excess of the golden rule capital stock in this precise sense. Thus, any stationary path with capital stock in excess of the golden rule capital stock, overaccumulates capital and is inefficient. The Phelps-Koopmans theorem generalizes this result to non-stationary paths.
[4]See Mitra and Ray (1984) for a description of the setting, which does not assume smoothness of the production function, and does not place restrictions on the types of non-concavities allowed.

path in every period exceeds golden rule consumption.[5] If the path were inefficient, then there would be a path starting from the same initial stock, which dominates it in terms of consumption (in the efficiency ordering). This forces the capial stock of the dominating path to go below (and stay below) the inefficient golden golden rule stock after a finite number of periods. This is where the non-convexity in the production set comes into play.

Suppose that the production function is such that the only golden rule stock below our inefficient golden rule stock is the minimal golden rule stock. In other words, capital is rather "unproductive" in this range between the two golden rules (although production is still increasing in capital). Then, in order to keep consuming at higher than golden rule consumption levels, the capital stock of the dominating path is forced to go below the minimal golden rule stock after a finite number of periods. Now, the standard theory applies: any path starting from below the minimal golden rule stock, and consuming at least the golden rule consumption level every period becomes infeasible after a finite number of periods. Thus, no dominating path can exist, and the constructed path must be efficient.

In view of the example it is natural to enquire whether there are general conditions on the production function under which version II of the Phelps-Koopmans theorem can be shown to be valid. Certainly, we would like to allow for situations in which multiple golden rule stocks can exist,[6] and we are specially interested in providing a testable condition on the production function that guarantees version II without further qualifications.

Proposition 2 provides such a condition, which involves the comparative local behavior of the production function across multiple golden rules. Loosely stated, the condition requires that the marginal product of capital fall more slowly at the minimal golden rule than at any of the other golden rules. It is therefore a condition which compares the local *curvatures* of the production function at various golden rules. This condition is always satisfied when the production function is concave, which is the focus of the traditional Phelps-Koopmans theory.[7]

Finally, we examine version III of the theorem, which is the Phelps-Koopmans result in its strongest form. We show that this version of the theorem is generally false with or without the sufficient condition used to establish version II (Propositions 3 and 4). Indeed, we prove that the version III is generally false even when there exists a unique golden rule. An interesting research question is to describe conditions under which version III is valid. We suspect that such conditions will involve strong restrictions on the production technology. Whether those conditions usefully expand the subset of convex technologies remains an open question.

[5]The consumption levels must, of course, converge to the golden rule consumption level over time.

[6]We know that in the case of an S-shaped production function, the theorem is valid (see Majumdar and Mitra (1982)). However, in that setting, there is a unique golden rule stock, which occurs in the concave region of the production function, so that the traditional argument (used in models with concave production functions) applies.

[7]More precisely, the traditional Phelps-Koopmans theory assumes that the production function is *strictly* concave, so that there is a unique golden rule. But the condition nevertheless holds for production functions which are weakly concave.

2. Preliminaries

We begin by describing an aggregative model of growth. At every date, capital k_t produces output $f(k_t)$, where $f : \mathbb{R} \to \mathbb{R}$ is the production function. We assume throughout that f satisfies the following restrictions:

[F.1] f is increasing and continuous, with $f(0) = 0$.

[F.2] There is $K \in (0, \infty)$ such that $f(x) > x$ for all $x \in (0, K)$ and $f(x) < x$ for all $x > K$.

We refer to K as the *maximum sustainable stock*. Observe that f is permitted to be nonconcave.

A *feasible path* from $\kappa \geq 0$ is a sequence of *capital stocks* $\{k_t\}$ with

$$k_0 = \kappa \text{ and } 0 \leq k_{t+1} \leq f(k_t)$$

for all $t \geq 0$. Associated with the feasible path $\{k_t\}$ from κ is a *consumption sequence* $\{c_t\}$, defined by

$$c_t = f(k_{t-1}) - k_t \text{ for } t \geq 1.$$

It is obvious that for every feasible path $\{k_t\}$ from κ, both k_t and c_{t+1} are bounded above by $\max\{K, \kappa\}$. With no real loss of generality, we presume that $\kappa \in [0, K]$, so that for every feasible path $\{k_t\}$ from κ,

$$k_t \leq K \text{ for } t \geq 0 \text{ and } c_t \leq K \text{ for } t \geq 1.$$

A feasible path $\{k'_t\}$ from κ *dominates* a feasible path $\{k_t\}$ from κ if

$$c'_t \geq c_t \text{ for all } t \geq 1,$$

with strict inequality for some t.

A feasible path $\{k_t\}$ from κ is *inefficient* if there is a feasible path $\{k'_t\}$ from κ which dominates it. It is *efficient* if it is not inefficient. A capital stock $k \in [0, K]$ will similarly be called inefficient if the corresponding stationary feasible path with $k_t = k$ for all t is inefficient; otherwise it is efficient.

Under [F.1] and [F.2] there is $z \in (0, K)$ such that

$$f(z) - z \geq f(x) - x \text{ for all } x \geq 0.$$

Then we call z a *golden rule stock*, or simply a *golden rule*. Certainly, there can be several golden rule stocks, all in $(0, K)$. Let G be the set of all golden rules. Obviously, G is nonempty and compact and so has a minimal element, which we denote by γ. *Golden rule consumption* is, of course, the same for all golden rules; it is given by $[f(z) - z]$ for $z \in G$, and is denoted by c.

It is easy to prove that the minimal golden rule is efficient. It is also easy to see that any capital stock that exceeds the minimal golden rule is inefficient. So version I of the Phelps-Koopmans theorem (see Introduction) must be true.

3. Phelps-Koopmans Version II: An Example

In this section, we present an example in which (i) there is an inefficient stock that exceeds the minimal golden rule, but (ii) there is an *efficient* path along which the capital stock converges to this inefficient stock. This example shows that that it is possible to have higher capital stocks for all time periods compared to the capital stocks of an inefficient path, and still be efficient. Thus, version II of the Phelps-Koopmans theorem (see Introduction) breaks down, and the overaccumulation of capital does not translate into consumption inefficiency.

It should be clear (and will become obvious in the analysis below) that the inefficient stock in any such example *must* itself be a golden rule.

Proposition **1.** *There exists a production function satisfying Conditions* [F.1] *and* [F.2] *and an efficient path with capital stocks that converge to a limit strictly in excess of the minimal golden rule for that function.*

Proof. Consider the production function given by

$$f(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1 \\ 2 + (x-1)^2 & \text{for } 1 < x \leq 2 \\ 3 + (x-2) & \text{for } 2 < x \leq 3 \\ 4 + 0.5(x-3) & \text{for } 3 < x \end{cases} \tag{1}$$

Clearly, f satisfies [F.1] and [F.2], and $K = 5$ (see Figure 1). The set of golden rule stocks is given by

$$G = \{1, [2, 3]\} \tag{2}$$

Golden rule consumption c is 1. As we've already observed, the minimal golden rule γ (equal to 1 in this example) is efficient. All other golden rules are inefficient.

Consider the sequence $\{k_t\}$ defined by

$$k(t) = 2 + [1/(t+1)] \text{ for all } t \geq 0$$

Then

$$\begin{aligned} f(k_t) - k_{t+1} &= 3 + [1/(t+1)] - 2 - [1/(t+2)] \\ &= 1 + [1/(t+1)(t+2)] \end{aligned}$$

for all $t \geq 0$, so that $\{k_t\}$ is a feasible path from $\kappa = 3$. Associated consumption is given by

$$c_{t+1} = 1 + [1/(t+1)(t+2)] \text{ for all } t \geq 0. \tag{3}$$

We claim that $\{k_t\}$ is efficient. Suppose, on the contrary, that there is a feasible path $\{k'_t\}$ from $\kappa = 3$ that dominates $\{k_t\}$. Define, for any $k \geq 0$,

$$\beta(k) = [f(\gamma) - \gamma] - [f(k) - k]$$

Of course, $\beta(k) \geq 0$; this is the "value loss" from operating at k.

Notice that k_t is itself a golden rule at every t, so that

$$c_{t+1} + k_{t+1} - k_t = f(k_t) - k_t = c, \tag{4}$$

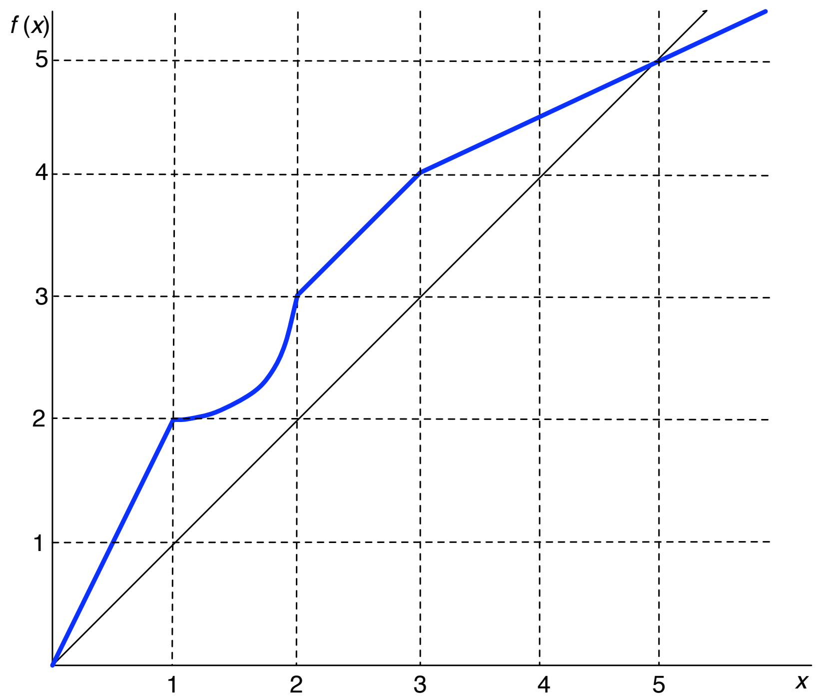

FIGURE 1. THE FUNCTION f DEFINED IN EQUATION (1)

while for $\{k'_t\}$, we see that

$$c'_{t+1} + k'_{t+1} - k'_t = f(k'_t) - k'_t = c - \beta_t \tag{5}$$

where $\beta_t \equiv \beta(k'_t)$ for $t \geq 0$.

Expression (4) tells us that for all $T \geq 0$,

$$\sum_{t=0}^{T} [c_{t+1} - c] = \kappa - k_{T+1} \tag{6}$$

while (5) similarly informs us that

$$\sum_{t=0}^{T} [c'_{t+1} - c] = \kappa - k'_{T+1} - \sum_{t=0}^{T} \beta_t \tag{7}$$

Combining (6) and (7) and recalling that $\{k'_t\}$ dominates $\{k_t\}$, we conclude that there is $N \geq 0$ and $\alpha > 0$ such that

$$k'_{T+1} \leq k_{T+1} - \alpha \text{ for all } T \geq N. \tag{8}$$

Because $k_t \to 2$ as $t \to \infty$, (8) implies the existence of $N' \geq 0$ such that

$$k'_t < 2 \text{ for all } t \geq N'. \tag{9}$$

For such dates t, we must have

$$k'_{t+1} = f(k'_t) - c'_{t+1} = [f(k'_t) - k'_t] + k'_t - c'_{t+1} \leq 1 + k'_t - c'_{t+1} < k'_t$$

where the first inequality follows from the fact that golden rule consumption equals 1, and the second inequality from the fact that $c'_{t+1} \geq c_{t+1} > 1$. So k'_t decreases over time for $t \geq N'$, and so must converge to some $k' \in [0, 2)$, with associated c'_{t+1} converging to $f(k') - k'$. But $c'_{t+1} > 1$ for all t, so $f(k') - k' \geq 1$. There is only one value of $k \in [0, 2)$ for which this is true, and that is $\gamma = 1$. We must therefore conclude that there exists $N'' > N$ such that

(10) $$1 < k'_t < 1.5 \text{ for all } t \geq N''.$$

For $t \geq N''$, define $\epsilon_t \equiv k'(t) - 1$. Let $\delta_t \equiv 1/(t+1)(t+2)$. Then we know from (3) that $c'_t \geq c_t = 1 + \delta_t$, so that for $t \geq N''$,

(11) $$k'_{t+1} = f(k'_t) - c'_{t+1} \leq 2 + \epsilon_t^2 - 1 - \delta_t.$$

It follows that

(12) $$\epsilon_{t+1} \leq \epsilon_t^2 \text{ for all } t \geq N''.$$

Let $q \equiv \epsilon(N'')$. Then (10) informs us that $q \in (0, 0.5)$, and iteration on (12) yields

(13) $$\epsilon_t \leq q^{2(t-N'')} \text{ for all } t \geq N''.$$

Using (13) in (11), we see that for $t \geq N''$,

(14) $$\epsilon_{t+1} \leq \epsilon_t^2 - \delta_t \leq q^{4(t-N'')} - \delta_t.$$

Now $\delta_t \geq 1/(t+2)^2$ and $q^{4t} \leq 1/2^{4t}$ for all $t \geq 0$, so that for $t \geq N''$,

(15) $$1/(t+2)^2 \leq \delta_t \leq (q^{4t}/A) \leq 1/(2^{4t})A$$

where $A \equiv q^{4N''}$. This implies that for all $t \geq N''$,

(16) $$A \leq \frac{(t+2)^2}{2^{4t}}$$

But the right hand side of (16) converges to zero as $t \to \infty$. This contradiction establishes our claim. ∎

4. Phelps-Koopmans Version II: A Theorem

In view of the example discussed in the previous section, it is natural to enquire whether there are general conditions on the production function under which version II of the Phelps-Koopmans theorem can be shown to be valid. We want to allow for frameworks in which multiple golden rule stocks can exist, and we are specifically interested in identifying a class of production functions for which any convergent path with limit higher than the minimal golden rule is necessarily inefficient. This motivates the following condition:

[C] For any golden rule $k > \gamma$, there is a golden rule $k' < k$ and $a > 0$ such that

(17) $$f(k' + \epsilon) - f(k') \geq f(k + \epsilon) - f(k) \text{ for all } \epsilon \in (-a, a).$$

We describe two scenarios in which [C] holds.

1. *Concave production function.* If f is concave, and there is $k \in G$ with $k > \gamma$, then $[\gamma, k] \subset G$. Pick any $k' \in (\gamma, k)$, and pick $0 < a < \min\{k - k', k' - \gamma\}$. Then, for $\epsilon \in (-a, a)$, we have $k' + \epsilon \in (\gamma, k)$, so that $(k' + \epsilon) \in G$. Thus, $f(k' + \epsilon) - f(k') = f(k' + \epsilon) - (k' + \epsilon) + k' + \epsilon - f(k') = c - c + \epsilon = \epsilon$. On the other hand, $f(k + \epsilon) - f(k) = f(k + \epsilon) - (k + \epsilon) + k + \epsilon - f(k) \leq c - c + \epsilon = \epsilon$.

2. *Smooth production function*[8] *with* $[-f''(\gamma)] < [-f''(k)]$ *for every* $k \in G$ *with* $k > \gamma$. Observe that $f''(k) \leq 0$ at every golden rule k. So the condition described here means that the rate at which the marginal product of capital is falling at the minimal golden rule is smaller than the corresponding rate at any of the other golden rules.

We may verify [C] as follows. Pick any golden rule $k > \gamma$. There exists $a > 0$ such that

$$[-f''(x)] < [-f''(z)] \text{ for all } x \in B(k, a) \text{ and all } z \in B(k', a), \tag{18}$$

where $B(y, \epsilon)$ is the open ball of radius ϵ around y. Then, for $\epsilon \in (-a, a)$, we have

$$f(\gamma + \epsilon) - f(\gamma) = f'(\gamma)\epsilon + (1/2)f''(\xi)\epsilon^2 = \epsilon + (1/2)f''(\xi)\epsilon^2 \tag{19}$$

and

$$f(k + \epsilon) - f(k) = f'(k)\epsilon + (1/2)f''(\zeta)\epsilon^2 = \epsilon + (1/2)f''(\zeta)\epsilon^2, \tag{20}$$

where $\xi \in B(\gamma, \epsilon)$, and $\zeta \in B(k, \epsilon)$, as given by Taylor's theorem. Since $\xi \in B(\gamma, a)$ and $\zeta \in B(k, a)$ as well, we can use (19) and (20) to conclude that

$$f(\gamma + \epsilon) - f(\gamma) = \epsilon + (1/2)f''(\xi)\epsilon^2 > \epsilon + (1/2)f''(\zeta)\epsilon^2 = f(k + \epsilon) - f(k),$$

which establishes (17).

Note that if f is C^2 *and* concave, the condition $[-f''(\gamma)] < [-f''(k)]$ *cannot* hold for every golden rule $k \neq \gamma$. For if f is concave, every stock in $[\gamma, k]$ must be a golden rule stock as well. It follows that for every $k' \in (\gamma, k)$, we have $f''(k') = 0$, while $f''(\gamma) \leq 0$, by definition of a golden rule. Therefore, the two scenarios described above do not overlap when there are multiple golden rules.

We can now proceed to show that under Condition C, version II of the Phelps-Koopmans theorem holds.

Proposition **2.** *Suppose that* [C] *holds. If* $\{k_t\}$ *is a feasible path from* κ *with* $\lim_{t\to\infty} k_t > \gamma$, *then* $\{k_t\}$ *is inefficient.*

Proof. Define $k \equiv \lim_{t\to\infty} k_t$. First suppose that k lies in G.

By [C], there is a golden rule $k' < k$ and $a > 0$ such that (17) holds. Denote $(k - k')$ by δ, $\min\{a, k'\}$ by b, and $(k_t - k)$ by ϵ_t for $t \geq 0$. Then, one can find $T \geq 0$ such that $\epsilon_t \in (-b, b)$ for all $t > T$. Define $k'_t = k_t$ for $0 \leq t \leq T$, and $k'_t = k' + \epsilon_t$ for $t > T$. Then, we have $k'_t \geq 0$ for all $t \geq 0$, and $c'_{t+1} = f(k'_t) - k'_{t+1} = f(k_t) - k_{t+1} = c_{t+1}$ for all $0 \leq t \leq T - 1$. Moreover, $c'_{t+1} = f(k'_t) - k'_{t+1} = f(k_t) - k_{t+1} + \delta = c_{t+1} + \delta > c_{t+1}$ for $t = T$. And for $t > T$, we have

$$\begin{aligned} c'_{t+1} &= f(k'_t) - k'_{t+1} = f(k_t - \delta) - (k_{t+1} - \delta) \\ &= f(k_t - \delta) - f(k_t) + f(k_t) - k_{t+1} + \delta \\ &= f(k_t - \delta) - f(k_t) + c_{t+1} + \delta \end{aligned}$$

Thus, it is enough to show that $f(k_t - \delta) - f(k_t) + \delta \geq 0$ for all $t > T$.

[8] To be exact, f is C^2.

Note that for $t > T$, we have $\epsilon_t \in (-b, b)$, so:

$$\begin{aligned} f(k_t - \delta) - f(k_t) + \delta &= f(k' + \epsilon_t) - f(k') + f(k') - f(k_t) + (k - k') \\ &= f(k' + \epsilon_t) - f(k') + c - f(k + \epsilon_t) + k \\ &= f(k' + \epsilon_t) - f(k') + f(k) - f(k + \epsilon_t) \\ &\geq 0 \end{aligned}$$

the last inequality following from (17).

This establishes the inefficiency of $\{k_t\}$ when $k \in G$.

If, on the other hand, $k \notin G$, then $f(k) - k < c$. Consequently, $k(t) \to f(k) - k < c$ as $t \to \infty$. Then one can easily dominate $\{k(t)\}$ by switching to the minimal golden rule γ sufficiently far in the future, and then staying at γ thereafter. ∎

In the example of the previous section, the set of golden rule stocks is $G = \{1, [2, 3]\}$. Choosing $k = 2$, we see that in order to verify [C], we must select $k' = \gamma$. However, for all $\epsilon \in (0, 1)$, we have $f(k' + \epsilon) - f(k') = \epsilon^2 < \epsilon$, while $f(k + \epsilon) - f(k) = \epsilon$. Thus, [C] is violated, as it must be if both Propositions 1 and 2 are correct.

5. Phelps-Koopmans Version III

5.1. **A Negative Result.** Consider a production function f that satisfies [F.1] and [F.2], and the following additional requirement:

[F.3] The function $f(f(k)) - k$ is uniquely maximized on $[0, K]$ at some value $a > \gamma$.

The following propositions show that version III of the Phelps-Koopmans theorem (see Introduction) does not extend to the case of nonconvex technologies, even if Condition C is satisfied.

Proposition **3.** *Whenever* [F.1]–[F.3] *are satisfied, there exists an efficient path* $\{k_t\}$ *from some initial stock, with* $\inf_t k_t > \gamma$.

Proof. Take a as given by [F.3], and define $b \equiv f(a)$. It is clear that $b > a > \gamma$. Define $d = f(f(a)) - a$.

Consider the path $\{k_t\}$ from initial stock b, given by $k_t = b$ for all t even, and $k_t = a$ for all t odd. Clearly, this path is feasible, and the associated consumption stream is given by $c_t = d$ for t odd, and $c_t = 0$ for t even. Note that $\inf_t k_t > \gamma$.

We claim that $\{k_t\}$ is efficient. Suppose not. Then there is a feasible path $\{k'_t\}$ from b with associated consumption stream $\{c'_t\}$ such that $c'_t \geq c_t$ for all t, with strict inequality for some t. Without loss of generality, we may presume that strict domination occurs in the very first consumption period, so that $c'_1 > c_1 = d$. Define $\theta \equiv c'_1 - c_1 > 0$. It is easy to see that at $t = 1$,

$$k'_t \leq a - \theta. \tag{21}$$

Let $d' \equiv \max f(f(x)) - x$ on the domain $0 \leq x \leq a - \theta$. Because a is the unique maximizer of this function on the fully unrestricted domain, it follows that $\epsilon \equiv d - d' > 0$.

We now claim that at any odd date t, if (21) holds, then

$$k'_{t+2} \le k'_t - \epsilon. \tag{22}$$

To prove this claim, note that $k'_{t+1} \le f(k'_t)$, so that

$$f(k'_{t+1}) \le f(f(k'_t)).$$

At the same time,

$$c'_{t+2} = f(k'_{t+1}) - k'_{t+2} \ge c_{t+2} = d,$$

so that combining these two pieces of information,

$$f(f(k'_t)) - k'_{t+2} \ge d. \tag{23}$$

Because $k'_t \le a - \theta$, we know that $f(f(k'_t)) - k'_t \le d' = d - \epsilon$. Combining this information with (23), we obtain (22).

Recall that (21) holds when $t = 1$. Thus applying the claim repeatedly, we see that capital stocks along $\{k'_t\}$ must become negative in finite time, which contradicts feasibility. ∎

It remains to show that the class described by [F.3] is nonempty. Indeed, we show below that there exist functions that satisfy [F.3] and have a *unique* golden rule. (In particular, Condition C is satisfied.)

Proposition **4.** *There exists a function f that satisfies* [F.1]—[F.3] *and besides, exhibits a unique golden rule stock.*

Proof. Pick numbers a_1, a_2, b_1 and b_2 such that the following conditions are met:

(i) $a_i > 1$ and $0 < b_i < 1$ for $i = 1, 2$.

(ii) $a_i b_j > 1$ for $i = 1, 2$ and $j = 1, 2$.

(iii) $a_1 > a_2$, but $(a_1 - 1)(1 + b_1) < (a_2 - 1)(1 + b_2)$.

It is easy to see that these conditions are mutually consistent (see remarks at the end of the proof).

Choose $\epsilon > 0$ and small enough so that the following conditions are satisfied:

$$\frac{a_1 - b_1}{1 - b_1} - \epsilon > a_1, \tag{24}$$

$$a_1 - 1 > (a_2 - 1) + (1 - b_1)\epsilon. \tag{25}$$

By condition (i), (24) holds when $\epsilon = 0$, and by condition (iii), (25) holds when $\epsilon = 0$. So there is $\epsilon > 0$ such that (24) and (25) both hold.

Define $\theta = [(a_1 - b_1)/(1 - b_1)] - \epsilon$. Note that by (24), $\theta > a_1 > 1$. Define a function f by

$$f(k) = \begin{cases} a_1 k & \text{for } 0 \le k \le 1 \text{(zone1)} \\ a_1 + b_1(k-1) & \text{for } 1 < k \le \theta \text{ (zone 2)} \\ f(\theta) + a_2(k - \theta) & \text{for } \theta < k \le \theta + 1 \text{(zone 3)} \\ f(\theta) + a_2 + b_2(k - \theta - 1) & \text{for } k > \theta + 1 \text{(zone 4)} \end{cases} \tag{26}$$

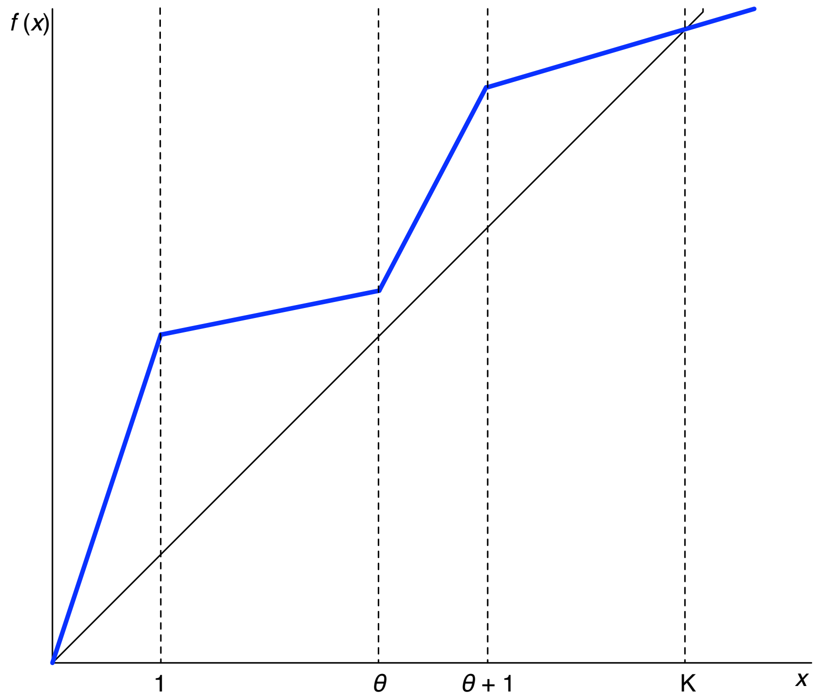

FIGURE 2. THE FUNCTION f IN THE PROOF OF PROPOSITION 3

(See Figure 2 for a diagrammatic depiction.)

By condition (i), there are only two candidates for a golden rule, the stocks 1 and $\theta + 1$. Evaluated at the former, $f(k) - k = a_1 - 1$. Evaluated at the latter, $f(k) - k = (a_2 - 1) + [f(\theta) - \theta] = (a_2 - 1) + (1 - b_1)\epsilon < a_1 - 1$, by (25). So the unique golden rule stock is $k = 1$.

Now we show that f satisfies [F.3]. Consider the problem: $\max_{k \geq 0} f(f(k)) - k$. In what follows, numbered "zones" refer to the capital stock regions demarcated in (26).

If k lies in zone 1, $f(k)$ must lie in zone 1 or in the interior of zone 2. [To prove this, simply observe that $f(1) = a_1 < \theta$, by (24).] Therefore, it is easy to see that $f(f(k)) = \min\{a_1^2 k, a_1 + b_1(a_1 k - 1)\} - k$, which by (i) and (ii) is strictly increasing on zone 1.

Next, suppose that k lies in zone 2. If $f(k)$ lies in the interior of zone 2, then $f(f(k)) - k = a_1 + b_1[a_1 + b_1(k-1) - 1] - k$, which is decreasing in k.

There are some stocks k in zone 2 for which $f(k)$ lies in zone 3. It is easy to verify that these stocks must lie in the subinterval of zone 2 given by $[\theta - \epsilon(1 - b_1)/b_1, \theta]$. Within this subinterval condition (ii) assures us that $f(f(k)) - k$ is increasing in k.[9]

If k lies in zone 3, then $f(k)$ must lie in zones 3 or 4. For k in zone 3, then,

$$f(f(k)) - k = f(\theta) + \min\{a_2[f(\theta) + a_2(k - \theta) - \theta], a_2 + b_2[f(\theta) + a_2(k - \theta) - \theta - 1]\} - k,$$

[9] In this zone, $f(f(k)) - k = f(\theta) + a_2[a_1 + b_1(k-1) - \theta] - k$, which is increasing in k because $a_2 b_1 > 1$, by (ii).

which is increasing in k by conditions (i) and (ii).

Finally, if k lies in zone 4 , so must $f(k)$, and it is easy to see, using condition (i), that $f(f(k))-k$ is decreasing in this zone.[10]

These arguments show that there are only two possible candidates that solve the problem $\max f(f(k)) - k$, and these are the stocks $k = 1$ and $k = \theta + 1$. Recall that $f(1) = a_1 < \theta$, so that

$$f(f(1)) - 1 = a_1 + b_1(a_1 - 1) - 1 = (a_1 - 1)(1 + b_1). \tag{27}$$

Similarly,

$$\begin{aligned} f(f(\theta+1)) - (\theta+1) &= f(\theta) + a_2 + b_2[f(\theta) + a_2 - \theta - 1] - (\theta+1) \\ &= \{f(\theta) - \theta\} + b_2\left[\{f(\theta) - \theta\} + (a_2 - 1)\right] + (a_2 - 1) \\ &= (1 - b_1)(1 + b_2)\epsilon + (a_2 - 1)(1 + b_2) \\ &> (a_2 - 1)(1 + b_2). \end{aligned} \tag{28}$$

Compare (27) and (28), and use condition (iii) to complete the proof. ∎

Remarks.

(i) We have assumed that $f(k) > k$ for all $k \in (0, K)$, where K is the maximum sustainable stock. If we are willing to weaken this assumption to $f(k) \geq k$ for all $k \in (0, K)$ (with strict inequality somewhere), then we can set $\epsilon = 0$ in the construction above and the argument is made much simpler.

(ii) The following values satisfy all the requirements in the proof of the proposition: $a_1 = 17$, $a_2 = 13$, $b_1 = 1/4$, $b_2 = 5/6$, and $\epsilon = 1$. Then $\theta = 64/3$, and $f(f(1)) - 1 = 20$, while $f(f(\theta + 1)) - (\theta + 1) \simeq 23.38$.

5.2. **A Positive Result for Nonconvergent Paths.**

Given the results of the preceding subsection, it appears difficult to make a general positive statement for nonconvergent paths. However, the following restatement of the Phelps-Koopmans theorem is valid even when the production set is nonconvex. This restatement is equivalent to the standard statement of the theorem when the production function is strictly concave.

In this section, we assume

[F.4] f is twice continuously differentiable, with $f'(k) > 0$ for all k.

Say that a feasible path $\{k_t\}$ from κ is *interior* if $\inf_{t \geq 0} k_t > 0$.

Proposition **5.** *Assume* [F.1], [F.2] *and* [F.4]. *Suppose that* $\{k_t\}$ *is an interior path from* $\kappa > 0$ *with*

$$\limsup_{t\to\infty} f'(k_t) < 1. \tag{29}$$

Then $\{k_t\}$ *is inefficient.*

[10] In this region, $f(f(k)) - k = f(\theta) + a_2 + b_2[f(\theta) + a_2 + b_2(k - \theta - 1) - \theta - 1] - k$, which is decreasing in k, by (i).

Proof. Given (1), we have for $t \geq 1$,

$$\sum_{t=1}^{\infty} \prod_{s=0}^{t-1} f'(k_t) < \infty,$$

so by following the method of Cass (1972, pp. 218–220), and noting that concavity of f is nowhere required, $\{k_t\}$ is inefficient. ∎

Remarks.

(i) This proof has been used in Majumdar and Mitra (1982, p.111,Theorem 3.2), under the assumption that f is "S-shaped".

(ii) Suppose f satisfies [F.1], F.2], and [F.4], and moreover is strictly concave. Then there is a unique golden-rule γ. If $\{k_t\}$ is a feasible path from $\kappa > 0$ with

$$\liminf_{t\to\infty} k_t > \gamma,$$

then $\{k_t\}$ is an interior path from $\kappa > 0$ which satisfies (29), so that $\{k_t\}$ is inefficient. This is the standard version of the Phelps-Koopmans theorem.

(ii) In the example of Proposition 2, $f'(k_t) = 1$ for all $t \geq 0$, so (29) does not hold. Thus Proposition 5 is not applicable to (a smoothened version of) the example.

(iv) When f is smooth, [F.3] implies that

$$f'(b)f'(a) = 1$$

where $b \equiv f(a)$. Consequently, a path which exhibits the period two cycle $(b, a, b, a, ...)$ must violate (29). Thus, the theorem above is not applicable in a smoothened version of the framework considered in Proposition 3.

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