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A New and Easy-to-Use Measure of Literacy, Its Axiomatic Properties and an Application

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Abstract. It can be argued that just as there are different kinds of literacy, there are different kinds of illiteracy. A 'proximate illiterate,' i.e. an illiterate who has easy access to a literate person, is clearly better off than someone without such access. The existing literature that takes account of these differences (1) defines an illiterate person to be a proximate illiterate if he or she lives in a household with at least one literate person and (2) derives new measures of literacy which typically exceed the standard literacy rate. The latter risks generating policy complacency. The aim of this paper is to suggest a measure of literacy that is not limited by (1) and (2). The measure is axiomatically characterized and its use is illustrated with a numerical exercise for the provinces of South Africa.

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1. Introduction

That literacy has externalities is widely accepted. An illiterate, who is connected to a literate person or has easy access to one, gets many of the benefits of literacy. Such a person,'proximate illiterate,' is much better off than an 'isolated illiterate,' that is, an illiterate who lives amidst people who are all illiterates. A proximate illiterate is much less likely to make a mistake with a doctor's prescription or an agricultural extension worker's pamphlet, because he or she can refer to and consult a literate person. In comparison the life of an isolated illiterate can be one of alarming darkness. A weakness of the standard measures of literacy is that it does not distinguish between isolated and proximate illiteracy and this in turn can lead to policy distortions, whereby a government trying to raise the nation's literacy, concentrates on the easy target of educating the proximate illiterates, thereby leaving segments of the population in isolated illiteracy. Once one becomes sensitive to this problem and looks at literacy data from poor nations, one discovers that not only is the problem of illiteracy high (which is well-known), but the problem of isolated illiteracy is very large (which is less well-known). Most illiterate people in India, for instance, live in households in which everybody is illiterate.

Fortunately, there is now some literature on 'effective' measures of literacy, which try to take account of the externalities of literacy, by allowing for the fact that it is worse to be an isolated illiterate than a proximate illiterate (Almeyda-Duran [1]; Basu and Foster [2]; Basu, Narayan and Ravallion [3]; Chakravarty and Majumder [4]; Dutta [6]; Gibson [7]; Iversen and Palmer-Jones [8]; Lee [10]; Maddox [11]; Mitra [12]; Subramanian [14]; and Valenti [16])[2]. These measures can, however, be subjected to two important criticisms. First, all these new measures end up making an 'upward' correction on the standard literacy rate. That is, for most nations, if the standard literacy rate is x, the effective literacy measure is $x+\epsilon$, where $\epsilon > 0$. This in itself is not a weakness, since, from a policy point of view, it is the ordinal property of the measure that is significant. Nevertheless, we are creatures of habit and, when we suddenly see a higher number denoting the extent of literacy in a familiar region, it creates the illusion of some of the problems of illiteracy being solved. Further, as will be evident later, it is not clear

[2] Once we move away from the individualistic framework of standard economics it is possible to take many different lines. Subramanian [15], for instance, points out how externalities can be negative, since it can be distressing to be the only illiterate (or among the few illiterates) in a group of otherwise literate people. Kell [9] talks about how literacy can be a "distributed capacity" among a group of individuals each of whom may not be literate in any conventional sense.

why the ‘normalization’ of our new measure should be such that effective literacy is *higher* than the standard literacy rate.

Second, there is some ethnographic research (see Bryan Maddox [11]) which emphasizes that the idea of proximate illiteracy must not be equated with there being a literate member in the household. There are other forms of association and network links with literate persons not in the illiterate person’s household that enable the illiterate to get some of the benefits of literacy. As Maddox’s anthropological research in Bangladesh shows, a literate in one’s natal home can be a valuable link for an illiterate woman, with a large influence on her life.

This paper corrects both these weaknesses — the former by creating a new measure of literacy and the latter by altering the domain on which literacy is defined, to wit by treating the assignment of isolated and proximate illiteracy as a primitive. The user of the measure is then let free to decide if an illiterate person who lives in a household with all illiterates but in a village with some literates or has literate relatives in one’s natal home ought to be classified as an isolated or a proximate illiterate. The aim of developing this new measure is to provide a method that is easy to use and offer an externality-sensitive index that provides an alternative normalization to the one used by Basu and Foster [2]. We nevertheless view this as work in progress since there are many further directions to go. In this work we treat literacy as a well-defined concept — the ability to read and write. But in reality it is a complex idea involving gradations and unusual manifestations (see, for instance, Denny [5]). Kell [9] tell us about Winnie, in South Africa, “who could speak three languages and had knowledge of another four languages, had never spent a day in school and saw herself as totally ‘illiterate’.” In future work we will want measures that can accommodate these alternative conceptions of literacy, but the aim here is to make a small but practically-useful contribution. Hence, a new measure is suggested, its axiomatic properties rigorously examined and a simple illustration with South African data is provided of how it may be used.

2. The Domain and the Measure

A society consists of a collection of people partitioned into the literates, the isolated illiterates, or *isolates*, and the proximate illiterates, or *proximates*[3], with the restriction that, if there exists a proximate illiterate, there

[3] When the word ‘proximate’ is used as a noun, as is being suggested here, we recommend that it be pronounced ‘proxi-mayt,’ both because that rhymes nicely with isolate, and distinguishes it from the standard adjective, pronounced ‘proximit.’

must be at least one literate person, since the externality which the proximate illiterate benefits from must emanate from somewhere. Formally, a *society* is a triple of non-negative integers (n, r, i), where $n > 0$, $n \geq r + i$ and if $r = 0$, then $n = i$. The interpretation is as follows: n is the number of adults in the society, r the number of literates and i the number of isolated illiterates. By definition, the number of proximate illiterates is $n - (r + i)$. The last condition above (namely, $r = 0 \Rightarrow n = i$) is explained by the fact that to be a proximate illiterate there has to exist at least one literate person.

We shall use Δ to denote the set of all societies. A *literacy mapping* is a function $f : \Delta \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Note that the *standard literacy rate*, denoted here by R is a literacy mapping defined as follows. For any society (n, r, i), $R(n, r, i) = r/n$.

So note that in the domain who is an isolate and who proximate is treated as a primitive. This is unlike in the existing papers such as, for instance, Basu and Foster [2]; Dutta [6]; and Subramanian [14].

The new literacy mapping that we propose here — called the e-*literacy mapping*, the e being a mnemonic of its externality sensitivity — belongs to a family of literacy mappings defined as follows. $\mathcal{L} : \Delta \to \mathbb{R}$ is called an e-*literacy mapping* if $\exists\, \alpha \in (0, 1)$ such that, $\forall\, (n, r, i) \in \Delta$,

$$\mathcal{L}(n, r, i) = \frac{(1 - \alpha)r}{(1 - \alpha)n + \alpha i}. \tag{2.1}$$

We shall call $\mathcal{L}(n, r, i)$ the e-literacy rate of society (n, r, i). It is interesting to note that as $\alpha \to 0$, $\mathcal{L}(n, r, i) \to R(n, r, i)$. If $i = 0$, $\mathcal{L}$ is always R.

By dividing the numerator and denominator by n, we can write the e-*literacy mapping* as

$$\frac{(1 - \alpha)r/n}{(1 - \alpha) + \alpha i/n} \equiv \frac{(1 - \alpha)R}{(1 - \alpha) + \alpha I}.$$

Note that R is the standard literacy rate and I will be called the *isolated illiterate rate*. Hence, what we have just shown is that the e-literacy rate of a society can be derived from the society's literacy rate, R, and isolated illiteracy rate, I.

3. Characterizing E-Literacy mappings

To critically evaluate the e-literacy rate that we are proposing as a good measure for representing a society's extent of literacy, it is useful to factorize it into axioms, each of which can then be assessed separately. That is precisely what we do in the present section. In fact, we provide a full axiomatic characterization of the family of e-literacy mappings, defined in 2.1.

So let us write down some plausible axioms that we would like a literacy mapping to satisfy. Consider first the strong normalization axiom, the word ‘strong’ being a reminder that this is stronger than the version used routinely in this literature.

Axiom N (Normalization): For all $(n, r, i) \in \Delta$, if $n = i$, then $f(n, r, i) = 0$, and if $i = 0$, then $f(n, r, i) = R(n, r, i)$.

The first part of this axiom is standard. If everybody in a society is an isolated illiterate or, equivalently, if nobody is literate, then this society must be described as having zero literacy. The second part is what makes it ‘strong.’ Most standard measures of literacy (Basu and Foster [2], Dutta [6], and Gibson [7] for instance) do not satisfy this; the only exception is Subramanian [14]. What this part of the axiom says is that, if there are no isolated illiterates in a society, then the literacy of that society is equal to the standard literacy rate. This, coupled with the next axiom, means that our measure of literacy will always be less than or equal to the literacy rate. And here is the next axiom.

Axiom M (Monotonicity): For all $(n, r, i) \in \Delta$ such that $(n, r+1, i) \in \Delta$ and $(n, r, i-1) \in \Delta$, $f(n, r+1, i) > f(n, r, i)$, and $f(n, r, i-1) > f(n, r, i)$.

This axiom says that literacy in a society rises not only when a formerly illiterate person becomes literate but also when an isolate becomes a proximate. This captures the central idea of educational externality. Axioms N and M really define the agenda of this paper.

Let us now turn to decomposability, which essentially says the following. Suppose we have three societies where the last is formed by a concatenation of the first two. Hence, the three societies are $(n_1, r_1, i_1) \equiv x_1$, $(n_2, r_2, i_2) \equiv x_2$, and $(n_1 + n_2, r_1 + r_2, i_1 + i_2) \equiv x_3$. Then, decomposability assures us that the literacy in x_3 is the weighted average of the literacies in societies x_1 and x_2, where the weights are given by population size. That is,

$$f(x_3) = \frac{n_1}{n_1 + n_2} f(x_1) + \frac{n_2}{n_1 + n_2} f(x_2).$$

However, it is easy to see that decomposability, along with axioms M and N, give us an impossibility result. Consider three societies $x_1 = (2, 1, 0), x_2 =$

$(1,0,1)$, and $x_3 = (3,1,1)$. Note

$$\begin{aligned} f(x_3) &= \frac{2}{3}f(x_1) + \frac{1}{3}f(x_2), \text{ by decomposability} \\ &= \frac{2}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot 0, \text{ by axiom N} \\ &= \frac{1}{3}. \end{aligned}$$

Next note $f(3,1,0) = 1/3$ by axiom N. And $f(x_3) < f(3,1,0)$ by axiom M. This is a contradiction. Hence, we have a small result: N, M and decomposability are together incompatible.

If we want to use some kind of decomposability, we therefore have no option but to use a weaker version of the standard decomposability axiom. What follows is one such axiom, which seems quite unexceptionable to us.

Axiom D (Weak Decomposability): If $(n_1, r_1, i_1), (n_2, r_2, i_2) \in \Delta$ are such that $f(n_1, r_1, i_1) = f(n_2, r_2, i_2)$, then $f(n_1 + n_2, r_1 + r_2, i_1 + i_2) = f(n_1, r_1, i_1)$.

Axiom D has a useful implication stated in the next lemma.

Lemma 1. *Assume the literacy mapping $f : \Delta \to \mathbb{R}$ satisfies axiom D. If $(n_1, r_1, i_1), (n_2, r_2, i_2) \in \Delta$ and $r_1/n_1 = r_2/n_2$ and $i_1/n_1 = i_2/n_2$, then $f(n_1, r_1, i_1) = f(n_2, r_2, i_2)$.*

Proof. Assume the hypothesis of the lemma is true. Let n be the least common multiple of n_1 and n_2. Then there are positive integers k_1 and k_2 such that $k_1 n_1 = k_2 n_2 = n$. By this equality and the assumption that $r_1/n_1 = r_2/n_2$, it follows that

$$k_1 r_1 = k_1 r_1 \frac{n_1}{n_1} = n\frac{r_1}{n_1} = n\frac{r_2}{n_2} = k_2 r_2 \frac{n_2}{n_2} = k_2 r_2.$$

Similarly $k_1 i_1 = k_2 i_2$. Then, if f satisfies axiom D, it must be that

$$f(n_1, r_1, i_1) = f(k_1 n_1, k_1 r_1, k_1 i_1) = f(k_2 n_2, k_2 r_2, k_2 i_2) = f(n_2, r_2, i_2),$$

where the first and last equalities follow by repeated application of axiom D. $\square$

Define $\overline{\Delta}$ to be the set of ordered pairs, (p, q), of nonnegative rational numbers such that $(n, pn, qn) \in \Delta$ for some positive integer n. Hence, by the definition of Δ, both pn and qn are nonnegative integers, $n \geq pn + qn$, and $q = 1$ if $p = 0$.

In the light of Lemma 1, we can define a function $\overline{f} : \overline{\Delta} \to \mathbb{R}$ such that $\forall(n, r, i) \in \Delta$,

$$f(n, r, i) = \overline{f}(r/n, i/n) \equiv \overline{f}(R, I). \tag{3.2}$$

In other words, if we know the literacy rate $r/n \equiv R$ and the isolated illiteracy rate $i/n \equiv I$ of a nation and if f satisfies axiom D, then we can use the function, $\overline{f}$, to find the literacy of the nation. This is given by $\overline{f}(R, I)$.

The domain of $\overline{f}$, defined as $\overline{\Delta}$, is illustrated in Figure 1. $\overline{\Delta}$ consists of all points with rational coordinates within a triangular region of (R, I)-space bounded by the x- and y-axes and the line $I = 1 - R$. The boundary of this region consists of the line segment from $\langle 0, 0\rangle$ to $\langle 0, 1\rangle$ along the y-axis, where only the point $\langle 0, 1\rangle$ is feasible (since proximate illiteracy is impossible in a society without literate members); the line segment from $\langle 0, 0\rangle$ to $\langle 1, 0\rangle$ along the x-axis, of which all points except $\langle 0, 0\rangle$ are feasible; and finally the line segment from $\langle 0, 1\rangle$ to $\langle 1, 0\rangle$, of which all points are feasible.

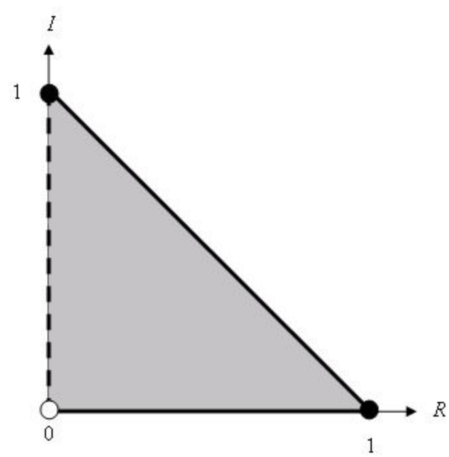

FIGURE 1. Domain of $\overline{f}(R, I)$

It is now obvious that, once axiom D is given, any axiom imposed on f can equivalently be thought of as an axiom on $\overline{f}$ and vice versa. In writing our next and final axiom, it is useful to work with $\overline{f}$. We will first show that there is a natural and unique way to extend $\overline{f}$ from the domain $\overline{\Delta}$ to the closure of $\overline{\Delta}$. For this we need to establish the continuity of $\overline{f}$. The next lemma is a preliminary step.

Lemma 2. *Assume the literacy mapping $f : \Delta \to \mathbb{R}$ satisfies axiom D, and $\overline{f} : \overline{\Delta} \to \mathbb{R}$ is defined according to equation 3.2. If $\overline{f}(R_1, I_1) = \overline{f}(R_2, I_2) = a$ and $\langle R, I \rangle$ is a point with rational coordinates on the line segment between $\langle R_1, I_1 \rangle$ and $\langle R_2, I_2 \rangle$, then $\overline{f}(R, I) = a$.*

Proof. Suppose $\langle R, I \rangle = \lambda \langle R_1, I_1 \rangle + (1 - \lambda) \langle R_2, I_2 \rangle$, where $0 < \lambda < 1$ is a rational number. Write $R_j = r_j/n_j$ and $I_j = i_j/n_j$ for $j = 1, 2$. By converting to the least common denominator, we may assume, without loss of generality, that $n_1 = n_2$. For simplicity, write $n = n_1 = n_2$. Thus $f(n, r_1, i_1) = f(n, r_2, i_2) = a$, and by axiom D, $f(2n, r_1 + r_2, i_1 + i_2) = a$. Equivalently $\overline{f}(R_1/2 + R_2/2, I_1/2 + I_2/2) = a$. Repeat this argument to obtain

$$\overline{f}\left(\frac{2}{3}R_1 + \frac{1}{3}R_2, \frac{2}{3}I_1 + \frac{1}{3}I_2\right) = \overline{f}\left(\frac{1}{3}R_1 + \frac{2}{3}R_2, \frac{1}{3}I_1 + \frac{2}{3}I_2\right) = a.$$

Continuing in this way, one arrives at the general result

$$\overline{f}\left(\frac{k}{m}R_1 + \frac{m-k}{m}R_2, \frac{k}{m}I_1 + \frac{m-k}{m}I_2\right) = a$$

for all $m = 1, 2, \ldots$ and $k = 0, 1, \ldots, m$. Since λ is a rational number strictly between 0 and 1, this formula establishes that $\overline{f}(R, I) = a$. $\square$

As a consequence of this lemma, the iso-literacy curves for any measure of literacy satisfying axioms D and M are lines (see figure 2). And now to the continuity of $\overline{f}$.

Lemma 3. *Assume the literacy mapping $f : \Delta \to \mathbb{R}$ satisfies axiom D, M, and N, and $\overline{f} : \overline{\Delta} \to \mathbb{R}$ is defined as in equation 3.2. Then $\overline{f} : \overline{\Delta} \to \mathbb{R}$ is continuous. Equivalently, given any $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(n_1, r_1, i_1) - f(n_2, r_2, i_2)| < \varepsilon$ whenever*

$$\sqrt{\left(\frac{r_1}{n_1} - \frac{r_2}{n_2}\right)^2 + \left(\frac{i_1}{n_1} - \frac{i_2}{n_2}\right)^2} < \delta.$$

Proof. First consider the point $\langle 1, 0 \rangle$. Given any $\varepsilon > 0$, let $R_0 = 1 - \varepsilon/2$. Then $\overline{f}(R_0, 0) = 1 - \varepsilon/2 > 1 - \varepsilon$ by axiom N. Furthermore there is an iso-literacy line, call it l_0, through $\langle R_0, 0 \rangle$. By the normality axiom, the slope of l_0 is strictly positive, and l_0 is not a vertical line by the monotonicity axiom. Hence l_0 meets the right boundary of the feasible set $\overline{\Delta}$, namely the line $I = 1 - R$, at a point $\langle R_1, 1 - R_1 \rangle$, where $R_0 < R_1 < 1$. The distance from $\langle 1, 0 \rangle$ to the line l_0 is strictly positive so there exists some $\delta > 0$ such that the δ-neighborhood of $\langle 1, 0 \rangle$ lies entirely to the right of the iso-literacy

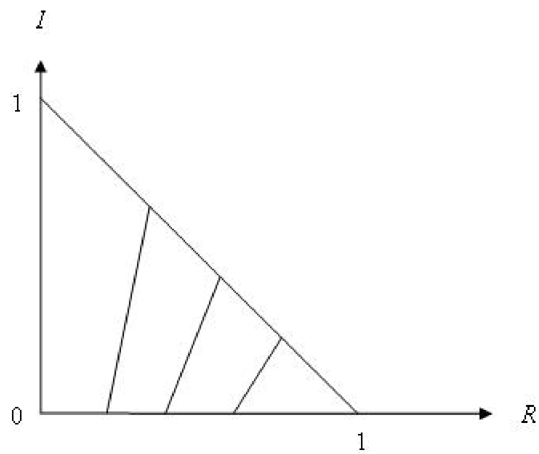

FIGURE 2. Iso-literacy curves are lines

line for $\overline{f} = 1 - \varepsilon/2$. Thus $\overline{f}$ exceeds $1 - \varepsilon/2$ on this neighborhood by axiom M, and the measure is continuous at $\langle 1, 0 \rangle$.

Second consider the point $\langle 0, 1 \rangle$ and any $\varepsilon > 0$. In this case, a very similar argument gives the existence of a δ-neighborhood of $\langle 0, 1 \rangle$, for some $\delta > 0$, which lies entirely to the left of the iso-literacy line, again call it l_0, for $\overline{f} = \varepsilon/2$. Specifically l_0 is the line from the point $\langle \varepsilon/2, 0 \rangle$ to a point $\langle R_1, 1 - R_1 \rangle$, where $R_1 > \varepsilon/2$ and $f(R_1, 1 - R_1) = \varepsilon/2$.

Next consider a point $\langle R_0, 0 \rangle$ where $0 < R_0 < 1$, and let $\varepsilon > 0$. By the normality axiom, $\overline{f}(R_0, 0) = R_0$. The iso-literacy line, l_0, through this point meets the line $I = 1 - R$ at a point $\langle R_1, 1 - R_1 \rangle$ and has strictly positive (and finite) slope

$$m = \frac{1 - R_1}{R_1 - R_0}.$$

Let $\underline{R} = R_0 - \varepsilon/2$ and $\overline{R} = R_0 + \varepsilon/2$, and denote by $\underline{l}$ and $\bar{l}$, respectively, the iso-literacy lines through $\langle \underline{R}, 0 \rangle$ and $\langle \overline{R}, 0 \rangle$. The slope, $\underline{m}$, of $\underline{l}$ is finite, strictly positive and satisfies

$$\underline{m} > \frac{1 - R_1}{R_1 - \underline{R}},$$

since $\underline{l}$ meets the line $I = 1 - R$ at a point strictly left of $\langle R_1, 1 - R_1\rangle$. Similarly the slope, $\overline{m}$, of $\overline{l}$ is finite, strictly positive, and

$$\overline{m} < \frac{1 - R_1}{R_1 - \overline{R}}.$$

Since $\underline{m}$ and $\overline{m}$ are bounded in this way, we can choose $\delta > 0$ such that δ is less than the minimum of the distance between $\langle R_0, 0\rangle$ and $\underline{l}$ and the distance between $\langle R_0, 0\rangle$ and $\overline{l}$. Then the entire δ-neighborhood of $\langle R_0, 0\rangle$ lies between $\underline{l}$ and $\overline{l}$. In other words, at every point in this δ-neighborhood, the inequality $R_0 - \varepsilon < \overline{\mathcal{L}} < R_0 + \varepsilon$ is satisfied. Hence the measure is continuous at $\langle R_0, 0\rangle$.

To finish the proof of continuity along the boundary, suppose $\varepsilon > 0$ and $0 < R_1 < 1$. To see that $\overline{f}$ is continuous at the point $\langle R_1, 1 - R_1\rangle$, let l_0 be the iso-literacy line from $\langle R_1, 1 - R_1\rangle$ to the point $\langle R_0, 0\rangle$ where $R_0 = \overline{f}(R_0, 0) = \overline{f}(R_1, 1 - R_1)$. By axioms M and N,

$$0 = \overline{f}(0, 1) < \overline{f}(R_1, 1 - R_1) < \overline{f}(1, 0) = 1,$$

and in particular $0 < R_0 < 1$. Let $\underline{R} = R_0 - \varepsilon/2$ and $\overline{R} = R_0 + \varepsilon/2$, and denote the respective iso-literacy lines by $\underline{l}$ and $\overline{l}$. That $\overline{f}$ is continuous at $\langle R_1, 1 - R_1\rangle$ follows, as before, from the observation that there is a strictly positive distance from this point to either of the lines $\underline{l}$ or $\overline{l}$.

Finally let $\langle \hat{R}, \hat{I}\rangle$ be a point in the interior of $\overline{\Delta}$ and choose any $\varepsilon > 0$. $\langle \hat{R}, \hat{I}\rangle$ lies on an iso-literacy line, l_0, through points $\langle R_0, 0\rangle$ and $\langle R_1, 1 - R_1\rangle$. Define $\underline{R}$, $\overline{R}$, $\underline{l}$, and $\overline{l}$ as before. Since $\underline{l}$ (resp. $\overline{l}$) meets the line $I = 1 - R$ at a point strictly left (right) of $\langle R_1, 1 - R_1\rangle$, there is a strictly positive distance between $\langle \hat{R}, \hat{I}\rangle$ and either of the lines $\underline{l}$ or $\overline{l}$. Hence $\overline{f}$ is continuous at $\langle \hat{R}, \hat{I}\rangle$. □

As a result of this lemma, there is a unique extension of the literacy mapping, $\overline{f}$, to a continuous function on the set of all real-valued points in the feasible region, that is, on the closure of $\overline{\Delta}$. With a slight misuse of language, we shall refer to the continuous extension of $\overline{f}$ by $\overline{f}$. It is easy to see that the extension of $\overline{f}$ satisfies axioms M, N, and D, whenever $\overline{f}$ does. Indeed the result of lemma 2 holds for points with real coordinates as well, so the iso-literacy curves of $\overline{f}$ remain lines.

Before introducing the next axiom we need to define some more terms. Define a *normal* society as one in which no one is an isolate; and a *perverse society* as one in which there are isolates and no one is a proximate. Every society is normal, perverse, or neither.

Consider a normal society with literacy rate R. Let $\phi(R)$ be the literacy rate of a perverse society which has the same effective literacy rate as the normal society with literacy rate R. In other words,

$$\overline{f}(R,0) = \overline{f}(\phi(R), 1-\phi(R)).$$

Clearly, by axiom N, for all $R < 1$, $R < \phi(R) < 1$. The question is: how large should $\phi(R)$ be? One way to deal with this question is to be relatively non-committal. Since $\phi(R)$ lies between 1 and R, note that the proportion of its deficiency from 1 and the proportion of if rise from R are given, respectively, by $(1-\phi(R))/1$ and $(\phi(R)-R)/R$. A simple requirement is to demand that these two proportions be balanced. The next axiom states this 'balancedness' condition. Since 'invariability' is treated in the same cluster of words as 'balancedness' in Roget's Thesaurus, we shall call this axiom 'Invariability,' which gives us an axiom that begins with a vowel, thereby rendering the set of axioms 'pronounceable.'

Axiom I (Invariability): There exists a $\beta > 0$ such that $\forall\, 0 < r < n$, $\phi(r)$ is such that

$$1 - \phi(R) = \beta \left[\frac{\phi(R) - R}{R}\right].$$

Theorem 1. *A literacy mapping f satisfies axioms M, I, N, and D if and only if it is an* e*-literacy mapping for some α satisfying* $0 < \alpha < 1$.

Proof. It is easy to verify that the e-literacy mapping $\mathcal{L}(n, r, i)$, which could equivalently be defined by

$$\overline{\mathcal{L}}(R, I) = \frac{(1-\alpha)R}{(1-\alpha) + \alpha I},$$

where $R \equiv r/n$ and $I \equiv i/n$, satisfies axioms M and N. To see that $\overline{\mathcal{L}}$ satisfies axiom I, consider a normal society with R literates. Then $\phi(R)$ is defined by the equation $\overline{\mathcal{L}}(R,0) = \overline{\mathcal{L}}(\phi(R), 1-\phi(R))$. Solving this equation gives

$$\phi(R) = \frac{R}{(1-\alpha) + \alpha R}.$$

Checking that axiom I holds, then, reduces to the following verification:

$$\beta\frac{\phi(R) - R}{R} = \beta\frac{\alpha - \alpha R}{(1-\alpha) + \alpha R} = \frac{(1-\alpha) - (1-\alpha)R}{(1-\alpha) + \alpha R} = 1 - \phi(R),$$

where $\beta = (1-\alpha)/\alpha > 0$ since $\alpha \in (0, 1)$.

In order to verify axiom D, suppose $(n_1, r_1, i_1), (n_2, r_2, i_2) \in \Delta$ and $\mathcal{L}(n_1, r_1, i_1) = \mathcal{L}(n_2, r_2, i_2) = a \in \mathbb{R}$. Then

$$\begin{aligned}\mathcal{L}(n_1+n_2, r_1+r_2, i_1+i_2) &= \frac{(1-\alpha)(r_1+r_2)}{(1-\alpha)(n_1+n_2)+\alpha(i_1+i_2)} \\ &= \lambda\frac{(1-\alpha)r_1}{(1-\alpha)n_1+\alpha i_1} + (1-\lambda)\frac{(1-\alpha)r_2}{(1-\alpha)n_2+\alpha i_2} \\ &= \lambda a + (1-\lambda)a = a,\end{aligned}$$

where λ and $1-\lambda$ are given by

$$\lambda = \frac{(1-\alpha)n_1+\alpha i_1}{(1-\alpha)(n_1+n_2)+\alpha(i_1+i_2)} \text{ and}$$

$$1-\lambda = \frac{(1-\alpha)n_2+\alpha i_2}{(1-\alpha)(n_1+n_2)+\alpha(i_1+i_2)}.$$

What remains to be proved is that any literacy mapping, f, which satisfies axioms M, I, N, and D must be the e-literacy mapping $\mathcal{L}$ defined by equation 2.1 for some $\alpha \in (0,1)$.

So suppose f satisfies axioms M, I, N, and D, and define $\overline{f} : \overline{\Delta} \to \mathbb{R}$ by equation 3.2. Recall that the iso-literacy curves for $\overline{f}$ are straight lines by lemma 2. Fix a number $0 < R < 1$. By axiom N, there is a unique point $\mathcal{A}$ on the x-axis boundary of $\overline{\Delta}$ where $\overline{f}$ takes the value R. In fact the coordinates of $\mathcal{A}$ are $\langle R, 0\rangle$. Axiom N also implies $\overline{f}(1,0) = 1$ and $\overline{f}(0,1) = 0$. Since f is continuous, the intermediate value theorem gives a point (possibly more than one) along the line segment joining $\langle 0,1\rangle$ and $\langle 1,0\rangle$ where $\overline{f}$ takes the value R. The monotonicity axiom implies this point, call it $\mathcal{B}$, is unique. Label the coordinates of $\mathcal{B}$ as $\langle \phi(R), 1-\phi(R)\rangle$.

As we have seen, axiom D implies that $\overline{f}$ takes the value R at all points along the line segment joining $\mathcal{A}$ and $\mathcal{B}$ (see figure 3). This line segment is the entirety of the iso-literacy curve for the value R by axiom M. The slope $\mu(R)$ of this line is

$$\mu(R) = \frac{1-\phi(R)}{\phi(R)-R}.$$

By axiom I, this slope is $\mu(R) = \beta/R$. Furthermore this line meets the negative y-axis at the point $\langle 0, -\omega(R)\rangle$, where

$$\omega(R) = R\,\frac{1-\Phi(R)}{\Phi(R)-R}.$$

Again by axiom I, this point has coordinates $\langle 0, -\omega(R)\rangle = \langle 0, -\beta\rangle$.

Thus for any $0 < R < 1$, the iso-literacy curve for the value R is given by the intersection of $\overline{\Delta}$ with the ray from $\langle 0, -\beta\rangle$ through $\langle R, 0\rangle$. Given any

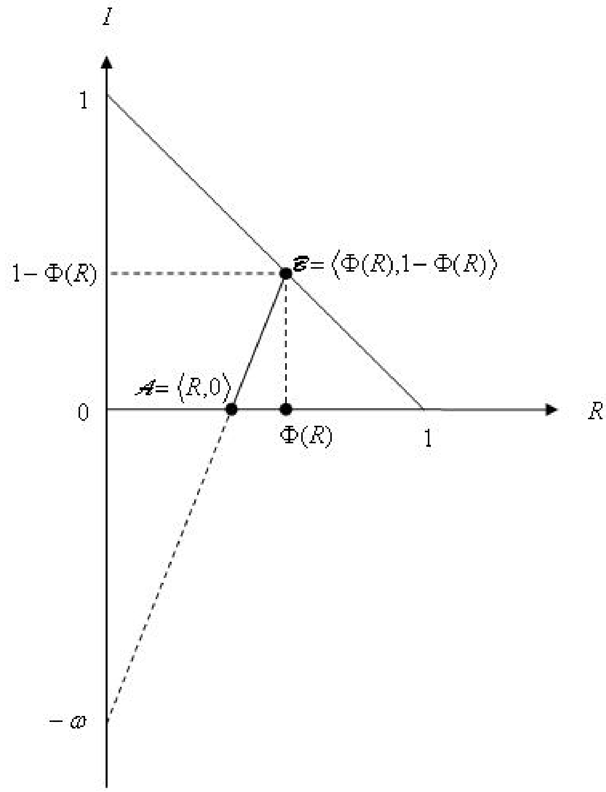

FIGURE 3. $\Phi(R)$ in Proof of Theorem 1

feasible pair $\langle R_0, I_0\rangle$, it is straightforward to compute $\overline{f}(R_0, I_0)$. If $R_0 = 0$, then $I_0 = 1$ and $\overline{f}(0, 1) = 0$ by the normality axiom. Likewise $\overline{f}(1, 0) = 1$. So we may assume $0 < R_0 < 1$. Then there is a unique line through $\langle 0, -\beta\rangle$ and $\langle R_0, I_0\rangle$, which meets the x-axis at the point $\langle \frac{\beta R_0}{\beta+I_0}, 0\rangle$. As this line is an iso-literacy curve,

$$\overline{f}(R_0, I_0) = \overline{f}\left(\frac{\beta R_0}{\beta + I_0}, 0\right) = \frac{\beta R_0}{\beta + I_0}.$$

By axiom I, $\beta > 0$ so there exists a unique $0 < \alpha < 1$ such that $\beta = (1-\alpha)/\alpha$, and, in terms of f,

$$f(n_0, r_0, i_0) = \overline{f}(r_0/n_0, i_0/n_0) = \frac{\beta r_0}{\beta n_0 + i_0} = \frac{(1-\alpha) r_0}{(1-\alpha) n_0 + \alpha i_0}.$$

Thus f is the e-literacy mapping with parameter α. As a last observation, note that, in terms of α, all iso-literacy lines meet at the point $\langle 0, -\omega\rangle = \langle 0, -(1-\alpha)/\alpha\rangle$. $\square$

It is easy to see that the iso-literacy curves in the (R, I)-space generated by the e-literacy mapping will be of the kind illustrated in Figure 4.

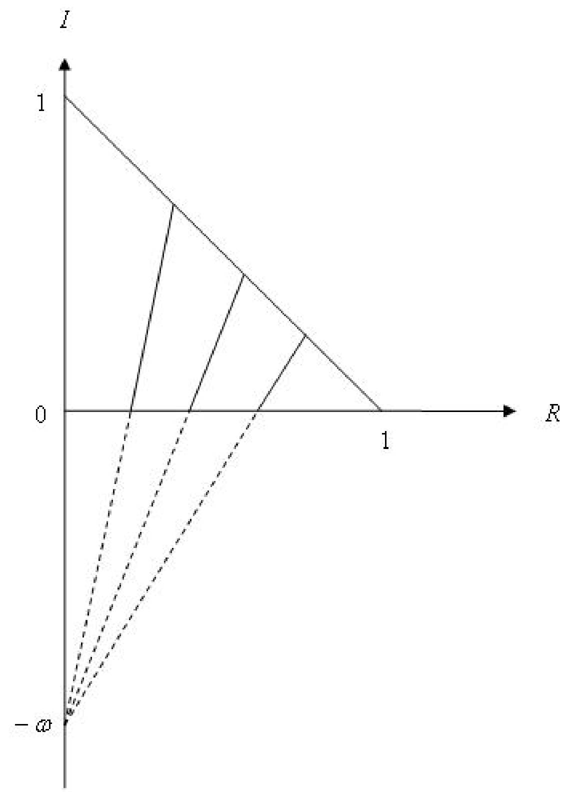

Figure 4. Iso-literacy lines

4. Application

Having discussed the theoretical underpinnings of this new measure of literacy, we now turn to an empirical application. For purposes of illustration, we present data on household literacy in South Africa. Our data comes from Statistics South Africa's General Household Survey July 2005, obtained via the South African Data Archive [13]. This data set provides province-wise cross-sectional data on adult literacy at the household level, thereby permitting computation of the standard literacy rate R, as well as proximate and isolate illiteracy rates, and hence our e-literacy rate[4]. For good measure, we make comparisons with the 'effective literacy rate,' $\mathcal{L}^*$, proposed in Basu and Foster [2].

The effective literacy rate is given by the expression $\mathcal{L}^* = R + \alpha' P$, where R and P are again the standard literacy and proximate illiteracy rates and α' is a parameter strictly between zero and one. This measure of literacy is greater than or equal to R and strictly greater in virtually any application. Furthermore it is bounded above by $1 - I$ for any feasible choice of the parameter α'.

Province	R	I	$\mathcal{L}_{0.25}$	$\mathcal{L}_{0.5}$	$\mathcal{L}_{0.75}$	$\mathcal{L}^*_{0.5}$	$\mathcal{L}^*_{0.75}$
Gauteng	79.6	2.05	79.1	78.0	75.0	88.8	93.4
Western Cape	74.0	1.72	73.6	72.8	70.4	86.1	92.2
Free State	73.2	2.03	72.8	71.8	69.0	85.6	91.8
Northern Cape	71.5	3.08	70.8	69.4	65.4	84.2	90.6
Kwazulu-Natal	71.1	2.35	70.6	69.5	66.5	84.4	91.0
North West	68.0	3.36	67.24	65.78	61.8	82.3	89.5
Eastern Cape	67.9	3.19	67.18	65.80	62.0	82.4	89.6
Limpopo	65.2	2.68	64.6	63.5	60.3	81.2	89.3
Mpumalanga	61.2	4.44	60.3	58.6	54.0	78.4	87.0
South Africa	70.4	2.67	69.8	68.5	65.2	83.9	90.6

Table 1. *E*-Literacy and Effective Rates for South African Provinces

Table 1 gives the e-literacy rates, expressed in percentage terms, for several values of α, for each of South Africa's provinces. The table also gives the standard literacy rate R, which is the e-literacy rate with parameter $\alpha = 0$. As is clear from the table, the e-literacy rate is less than R whenever $\alpha > 0$. As mentioned above, as α tends to one, $\mathcal{L}$ tends to a function which is equal to R, when $I = 0$, and zero otherwise. Thus unlike the effective literacy rate, the e-literacy rate does not have a well-identified bound below on any real-world data set. What we know is that, if isolated illiteracy is nonzero, then, as α goes to one, it goes to zero.

As discussed above, practitioners may deem the inequality $\mathcal{L} < R < \mathcal{L}^*$ significant. By adopting the effective literacy rate of Basu and Foster, one

[4]For present purposes, literacy is defined as an affirmative answer to both survey questions: "Can you read in at least one language?" and "Can you write in at least one language?".

can give the appearance of an improvement in literacy attainment, even when no real change has taken place. Take, for instance, the province of Mpumalanga. As Table 1 shows, this has a literacy rate of 61.2%. But by the measure of effective literacy it gets a score of 87%, when α happens to be .75, which is close to the empirical estimate of α, found by Gibson [7]. It will be very difficult for a local policy maker to use this measure and still consider the task ahead to be as large as it actually is. Our new measure deflates the standard rate R, while the effective literacy measure inflates it. This contrast is evident in the last two columns of Table 1.

$\mathcal{L}_{0.75}$	$\mathcal{L}_{0.5}$	$\mathcal{L}_{0.25}$	R
Gauteng	Gauteng	Gauteng	Gauteng
Western Cape	Western Cape	Western Cape	Western Cape
Free State	Free State	Free State	Free State
Kwazulu-Natal	Kwazulu-Natal	Northern Cape	Northern Cape
Northern Cape	Northern Cape	Kwazulu-Natal	Kwazulu-Natal
Eastern Cape	Eastern Cape	North West	North West
North West	North West	Eastern Cape	Eastern Cape
Limpopo	Limpopo	Limpopo	Limpopo
Mpumalanga	Mpumalanga	Mpumalanga	Mpumalanga

TABLE 2. Ranking Provinces, Highest to Lowest E-Literacy

As Table 2 makes clear, measuring literacy via the new e-literacy rate can also change the rankings of the provinces, from most to least literate. In particular, there are two exchanges: for parameter values $\alpha = 0.5$ and $\alpha = 0.75$, Kwazulu-Natal moves above Northern Cape and Eastern Cape jumps above North West. It is a curiosity that the same reversals are evident in Table 1 for the effective literacy measure with parameter $\alpha' = 0.5$ and $\alpha' = 0.75$. In this case the alterations in ranking are small but it alerts us to the possibility that this can happen and by a larger measure in other applications and in particular when making comparisons of very disparate nations and regions.

It is worth noting that as α approaches one, the rankings change more substantially, as illustrated in Table 3. In fact for α close to one, the rankings are determined by the size of R/I (as can be verified by taking the derivative of $\mathcal{L}$ with respect to α). Provinces with greatest R/I ratio rise in the rankings, while those with least fall. The effective literacy rate exhibits a similar phenomenon, in that, as α' nears one, $\mathcal{L}^*$ approaches

$R + P$. Thus, for the largest feasible values of the parameter α', rankings are determined by the quantity $R + P$.

There is, however, a contrast in terms of how quickly $\mathcal{L}$ and $\mathcal{L}^*$ approach their limiting behavior, as $\alpha \to 1$ and $\alpha' \to 1$, respectively. Since the effective literacy measure is linear in its parameter, specifically $\mathcal{L}^* = R + \alpha' P$, values of α' moderately close to one will suffice for most data sets. The e-literacy rate $\mathcal{L}$, on the other hand, is nonlinear in α, and as a result, it is necessary to consider values of α quite close to one. Table 3 reports the rankings of South African provinces for such values. In fact, Column $\mathcal{L}_{0.99}$ of this table gives e-literacy rankings which are final in the sense that no more reshuffling of the provinces can be achieved by increasing α.

Province	R/I	$\mathcal{L}_{0.99}$	$\mathcal{L}_{0.98}$	$\mathcal{L}_{0.96}$
Western Cape	42.95	27.35	40.13	52.36
Gauteng	38.79	26.26	39.69	53.33
Free State	36.10	24.34	36.73	49.26
Kwazulu-Natal	30.33	21.42	33.10	45.52
Limpopo	24.31	17.83	28.16	39.65
Northern Cape	23.23	17.66	28.50	41.12
Eastern Cape	21.29	16.33	26.49	38.46
North West	20.22	15.70	25.68	37.62
Mpumalanga	13.79	11.35	19.28	29.65
South Africa	26.38	19.33	30.50	42.90

Table 3. Ranking Provinces, Highest to Lowest E-Literacy for α Close to One

Since there is a unique e-literacy measure for every $0 < \alpha < 1$, there is a great deal of choice involved in using the new measure. For example, as just mentioned, one could choose a value of α very near to one in order to achieve some 'finality,' in the sense that no further increases in the parameter could lead to reordering. On the other hand, for any given set of data, the degree of dispersion in the e-literacy rate rises as α increases, then falls. This pattern reflects the fact that $\mathcal{L} = R$ when $\alpha = 0$ and, as α tends to one, $\mathcal{L}$ approaches a function equal to R, when $I = 0$, and equal to zero otherwise. With our South African data set, for example, the standard deviation of the e-literacy rates for the nine provinces is maximized for α approximately equal to 0.04. Ultimately practitioners are free to choose values of α by whatever criteria they wish.

Subgroup	$\mathcal{L}_{0.75}$	$\mathcal{L}_{0.5}$	$\mathcal{L}_{0.25}$	R	$\mathcal{L}^*_{0.5}$	$\mathcal{L}^*_{0.75}$
African	62.5	66.1	67.4	68.1	82.6	89.8
Coloured	68.2	70.9	71.9	72.4	85.2	91.5
Indian/Asian	86.2	87.3	87.7	87.9	93.6	96.5
White	88.9	89.3	89.5	89.6	94.7	97.2
Total	65.2	68.5	69.8	70.4	83.9	90.6

TABLE 4. Literacy Rates for Population Subgroups

Finally Table 4 illustrates the e-literacy and effective literacy rates for major subgroups of the South African population, for a few different parameter values. The table illustrates the large African-White and Coloured-White gaps in literacy attainment. The tendency of $\mathcal{L}$ and $\mathcal{L}^*$ to deflate and inflate the standard literacy rate implies that the e-literacy rate exacerbates these gaps, while the effective literacy rate mollifies it. In fact, the African-White and Coloured-White standard literacy rate gaps are 21.5% and 17.2%, respectively. These gaps rise to 26.4% and 20.7%, under $\mathcal{L}_{0.75}$, and fall to 7.4% and 5.7%, under $\mathcal{L}^*_{0.75}$.

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