ISSN 1936-5098

CAE Working Paper #07-11

Default Estimation and Expert Information: All Likely Dataset Analysis and Robust Validation

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July 2007

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January 31, 2007

[1] *Disclaimer: The statements made and views expressed herein are solely those of the author, and do not necessarily represent official policies, statements or views of the Office of the Comptroller of the Currency or its staff.* *Acknowledgement: I thank Jeffrey Brown, Mike Carhill, Hwansik Choi, Erik Larson, Mark Levonian, Mitch Stengel, and seminar participants at Cornell University and the OCC. These thanks come with gratitude but without any implication of agreement with my views. This paper was presented as an invited lecture at the Far Eastern Meetings of the Econometric Society in Beijing, August 2006.*

Abstract

Default is a rare event, even in segments in the midrange of a bank's portfolio. Inference about default rates is essential for risk management and for compliance with the requirements of Basel II. Most commercial loans are in the middle-risk categories and are to unrated companies. Expert information is crucial in inference about defaults. A Bayesian approach is proposed and illustrated using a prior distribution assessed from an industry expert. The method of All Likely Datasets, based on sufficient statistics and expert information, is used to characterize likely datasets for analysis. A check of robustness is illustrated with an $\epsilon-$ mixture of priors.

Keywords: Bayesian inference, robustness, expert information, Basel II, risk management, prior assessment

1 Introduction

Estimation of default probabilities (PD), loss given default (LGD, a fraction) and exposure at default (EAD) for portfolio segments containing reasonably homogeneous assets is essential to prudent risk management. It is also crucial for compliance with Basel II rules for banks using the IRB approach to determine capital requirements (Basel Committee on Banking Supervision (2004)). Estimation of small probabilities is tricky, and this paper will focus on estimating PD. The emphasis is on segments in the middle of the risk profile of the portfolio. Although the risk is in the middle of the asset mix, the probability of default is still "small." It is in fact likely to be about 0.01; defaults, though seen, are rare. The bulk of a typical bank's commercial loans are concentrated in these segments (segments differ across banks). Very low risk institutions are relatively few in number and they have access to capital through many avenues in addition to commercial loans. Very high risk loans are largely avoided and when present are often due to the reclassification of a safer loan as conditions change. To put this in perspective, the middle-quality loans are approximately S&P Baa or Moody's BBB. In practice the bulk of these loans are to unrated companies and the bank has done its own rating to assign the loans to risk "buckets." The focus of this paper is on estimation of the default probability for such a risk bucket on the basis of historical information and expert knowledge. We introduce the "All Likely Data" (ALD) approach, using sufficient statistics to define dataset types characterized by the number of defaults for a particular sample size. The number of types is linear in the sample size, while the number of distinct datasets is exponential. This affords considerable simplification. Next, we use expert information to identify likely types, and then run analyses for all likely types - a set of types corresponding to the most likely datasets. Since defaults are expected to be rare events, a small number of types characterize the likely samples. Finally, we conduct a robustness analysis, in the spirit of validation exercises required of banks under Basel II.

Throughout the paper we take a probability approach to the quantitative description of uncertainty. There are many arguments that uncertainty is best described in terms of probabilities. The classic axiomatic treatment is Savage (1954). In the case of default modeling, where measuring and controlling risk is the aim, it is natural to focus on anticipating defaults, or at least anticipating the aggregate number of defaults. Suppose there are a number of default configurations, and we wish to assign numbers to these events and to use these numbers to describe the likelihood of the events. Simple arguments based on scoring rules (for example minimizing squared prediction error) or odds posting (avoiding certain losses) imply that these numbers must combine like probabilities. For constructions see De Finetti (1974). Lindley (1982b) elaborates on the development using scoring rules, Heath and Sudderth (1978) and Berger (1980) on betting. The probability approach to describing and modeling uncertainty is central to risk management and to the requirements of Basel II. There is no serious argument that the probability approach is wrong or inappropriate for modeling uncertain future defaults as well as other unknowns. The fact that probabilities combine in accordance with convexity, additivity and multiplication is central for moving from probabilities of default on an asset, to default rates in a segment, to rates in a portfolio, and to a default probability for the bank. Economists and risk managers do not need convincing that probabilistic reasoning is appropriate for modeling. It is less well appreciated, especially in the applied community, that uncertainty about the unknown default probability can be usefully modeled in exactly the same way as uncertainty about unknown defaults, for exactly the same reasons.

Reasoning about probabilities is not easy. There is a long literature beginning with Kahneman and Tversky (1974) demonstrating that people in practice find it difficult to think about probabilities consistently. Theoretical alternatives to probabilistic reasoning include possibility measures, plausibility measures, etc. These are reviewed and evaluated by Halpern (2003). Although these practical and theoretical objections to probability are often used to criticize the Bayesian approach, they apply equally to the likelihood specification and the modeling approach to risk management. While recognizing these objections, this paper will use the probability approach, noting that alternatives invariably lead to incoherence.

2 A Statistical Model for Defaults

The simplest and most common probability model for defaults of assets in a homogeneous segment of a portfolio is the Binomial, in which the defaults are independent across assets and over time, and defaults occur with common probability θ. Note that specification of this model requires expert judgement, that is, information. Denote the expert information by e. The role of expert judgement is not usually explicitly indicated at this stage, so it is worthwhile to point to its contribution. First, consider the statistical model. The independent Bernoulli model is not the only possibility. Certainly independence is a strong assumption and would have to be considered carefully. External factors not explicitly modeled, for example general economic conditions, could induce correlation. There is evidence that default probabilities vary over the business cycle (for example Nickell, Perraudin, and Varotto (2000)); we return to this topic below. The Basel prescription is for a marginal annual default probability, however, and correlation among defaults is accommodated separately in the formula for the capital requirement. Thus, many discussions of the inference issue have focussed on the binomial model and the associated frequency estimator. Second, are the observations really identically distributed? Perhaps the default probabilities differ across assets, even within the group. Can this be modeled, perhaps on the basis of asset characteristics? The requirements demand an annual default probability, estimated over a sample long enough to cover a full cycle of economic conditions. Thus the probability should be marginal with respect to external conditions. For specificity we will continue with the Binomial specification. Let d_i indicate whether the ith observation was a default ($d_i = 1$) or not ($d_i = 0$). The Bernoulli model (a single Binomial trial) for the distribution of $d_i$is $p(d_i|\theta, e) = \theta^{d_i}(1-\theta)^{1-d_i}$. Let $D = \{d_i, i = 1, ..., n\}$ denote the whole data set and $r = r(D) = \sum_i d_i$ the count of defaults. Then the joint distribution of the data is

$$\begin{aligned} p(D|\theta, e) &= \prod \theta^{d_i}(1-\theta)^{1-d_i} \\ &= \theta^r (1-\theta)^{n-r} \end{aligned} \tag{2.1}$$

As a function of θ for given data D this is the likelihood function $L(\theta|D, e)$. Since this distribution depends on the data D only through r (n is regarded as fixed), the sufficiency principle implies that we can concentrate attention on the distribution of r

$$p(r|\theta, e) = \binom{n}{r}\theta^r(1-\theta)^{n-r} \tag{2.2}$$

a Binomial(n,θ) distribution. This is so well known that it is easy to underappreciate the simplification obtained by passing from 2.1 to 2.2. Instead of separate treatment for each of the 2^n datasets possible, it is sufficient to restrict attention to n+1 data set types, characterized by the value of r. This theory of types can be made the basis of a theory of asymptotic inference. See Cover and Thomas (1991). In our application, the set of likely values of r is small, and an analysis can be done for each of these values of r, rather than for the $\binom{n}{r}$ distinct datasets corresponding to each value of r. Thus, by analyzing a few likely data set types, we analyze in effect all of the most likely data realizations. We refer to this approach as the method of all likely datasets, or ALD.

Regarded as a function of θ for fixed r, 2.2 is the likelihood function. Figure 1 shows the likelihood functions for n=500, our reference data set size, and r={0,2,4,6,8}.

Figure 1

3 Uncertain Default Probabilities

Equation 2.2 is a statistical model. It generates probabilities for all default configurations as a function of a single parameter θ which remains unspecified. The default probability θ is an unknown, but that doesn't mean that nothing is known about its value. In fact, defaults are widely studied and risk managers, modelers, validators, and supervisors have detailed knowledge on values of θ for particular portfolio segments. The point is that θ is unknown in the same sense that the future default status of a particular asset is unknown. The fact that default is in the future is not important; the key is that it is unknown and the uncertainty can be described and quantified. We have seen how uncertain defaults can be modeled. The same methods can be used to model the uncertainty about θ. Continuing with the logic used to model default uncertainty, we see that uncertainty about values of θ are coherently described by probabilities. We assemble these probability assessments into a distribution describing the uncertainty about θ given the expert information e, $p(\theta|e)$.

The distribution $p(\theta|e)$ can be a quite general specification, reflecting in general the assessments of uncertainty in an infinity of possible events. This is in contrast with the case of default configurations, in which there are only a finite (though usually large) number of possible default configurations. However, this should not present an insurmountable problem. Note that we are quite willing to model the large number of probabilities associated with the possible different default configurations with a simple statistical model; in fact, a 1-parameter model. This involves an independence assumption, among other assumptions, but it simplifies the analysis and allows progress along empirical lines. The same can be done with the prior specification. That is, we can fit a few probability assessments by an expert to a suitable functional form and use that distribution to model prior uncertainty. There is some approximation involved, and care is necessary. In this regard, the situation is no different from that present in likelihood specification.

A convenient functional form is the beta distribution

$$p(\theta|\alpha,\beta)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \tag{3.1}$$

which has mean $\alpha/(\alpha+\beta)$ and variance $\alpha\beta/((\alpha+\beta)^2(1+\alpha+\beta))$. The special case of $\alpha=\beta=1$ is the uniform distribution on the unit interval. This is unlikely to represent information about default probabilities, since it assigns equal probabilities to each equal length interval in [0,1], but it is of great historical interest and is in common use as representing complete absence of information (it has maximal entropy among distributions on [0,1]). It will be useful in constructing a robust prior for a validation step in the analysis.

A particularly easy generalization is to specify the support $\theta\in[a,b]\subset[0,1]$.It is possible that some applications would require the support of θ. to consist of the union of disjoint subsets of $[0,1]$, but this seems fanciful in the current application. A simple starting point is the uniform $p(\theta|e)=1/(b-a)$. This prior would again sometimes be regarded as "uninformative," since it assigns equal probability to equal length subsets of $[a, b]$. It is informative in that it requires $\theta \in [a, b]$. The mean of this distribution is $(a + b)/2$. We may think that this specification is too restrictive, in that consideration might require that intervals near the most likely value should be more probable than intervals near the endpoints. A somewhat richer specification is the beta distribution 3.1 modified to have support $[a, b]$. Let t have the beta distribution and change variables to $\theta(t) = a + (b - a)t$ with inverse function $t(\theta) = (\theta - a)/(b - a)$ and Jacobian $dt(\theta)/d\theta = 1/(b - a)$. Then

$$p(\theta|\alpha, \beta, a, b) = \frac{\Gamma(\alpha + \beta)}{(b - a)\Gamma(\alpha)\Gamma(\beta)}((a - \theta)/(a - b))^{\alpha-1}((\theta - b)/(a - b))^{\beta-1} \tag{3.2}$$

over the range $\theta \in [a, b]$. This distribution has mean $E\theta = (b\alpha + a\beta)/(\alpha + \beta)$, allowing substantially more flexibility than the uniform. Examples of this distribution on the range [0,0.3] are graphed in Figure 2.

Figure 2

The four parameter Beta distribution allows flexibility within the range [a,b], but in some situations it may be too restrictive. For example it may not be flexible enough to allow combination of information from many experts. A simple generalization is the 7-parameter mixture of two 4-parameter Betas with common support. The additional parameters are the two new $\{\alpha, \beta\}$ parameters and the mixing parameter λ.

$$\begin{aligned} p(\theta|\alpha_1, \beta_1, \alpha_2, \beta_2, a, b) \quad = \quad & \frac{\lambda\Gamma(\alpha_1 + \beta_1)}{(b - a)\Gamma(\alpha_1)\Gamma(\beta_1)}((a - \theta)/(a - b))^{\alpha_1-1}((\theta - b)/(a - b))^{\beta_1-1} \\ & + \frac{(1 - \lambda)\Gamma(\alpha_2 + \beta_2)}{(b - a)\Gamma(\alpha_2)\Gamma(\beta_2)}((a - \theta)/(a - b))^{\alpha_2-1}((\theta - b)/(a - b))^{\beta_2-1} \end{aligned}$$

Computations with this mixture distribution are not substantially more complicated than computations with the 4-parameter Beta alone. If necessary, more mixture components with new parameters can be added, although it seems unlikely that expert information would be detailed and specific enough to require this complicated a representation. A useful further generalization is given by the 9-parameter mixture allowing different supports for the two mixture components. The prior family is then

$$p(\theta|\alpha_1, \beta_1, \alpha_2, \beta_2, a, b, c, d) =$$

$$\frac{I(\theta \in [a,b])\lambda\Gamma(\alpha_1+\beta_1)}{(b-a)\Gamma(\alpha_1)\Gamma(\beta_1)}((a-\theta)/(a-b))^{\alpha_1-1}((\theta-b)/(a-b))^{\beta_1-1}$$

$$+\frac{I(\theta \in [c,d])(1-\lambda)\Gamma(\alpha_2+\beta_2)}{(d-c)\Gamma(\alpha_2)\Gamma(\beta_2)}((c-\theta)/(c-d))^{\alpha_2-1}((\theta-d)/(c-d))^{\beta_2-1} \quad (3.3)$$

Here $[c, d]$ is the support set for the second mixture component and $I[x] = 1$ if condition x is true, 0 if false. As above, more than two mixture components could be added as needed, possibly with different support sets. By choosing enough Beta-mixture terms the approximation of an arbitrary continuous prior $p(\theta|e)$ for a Bernoulli parameter can be made arbitrarily accurate, in the sense that the sequence of approximations can be chosen to converge uniformly to $p(\theta|e)$. Note that there is nothing stochastic in this argument. The proof follows the proof of the Stone-Weierstrass approximation theorem for approximation of continuous functions by polynomials. See Diaconis and Ylvisaker (1985).

4 Inference

With the likelihood and prior at hand inference is a straightforward application of Bayes rule. Given the distribution $p(\theta|e)$, we obtain the joint distribution of r, the number of defaults, and θ :

$$p(r, \theta|e) = p(r|\theta, e)p(\theta|e)$$

from which we obtain the marginal (predictive) distribution of r,

$$p(r|e) = \int p(r, \theta|e)d\theta \tag{4.1}$$

If the value of the parameter θ is of main interest we divide to obtain the conditional (posterior) distribution of θ :

$$p(\theta|r, e) = p(r|\theta, e)p(\theta|e)/p(r|e) \tag{4.2}$$

which is Bayes rule. Since Basel II places more emphasis on the default probability than on the number of defaults in a given portfolio segment, we focus our discussion on $p(\theta|r, e)$.

5 Prior Distribution

I have asked an expert to specify a portfolio and give me some aspects of his beliefs about the unknown default probability. The portfolio consists of loans that might be in the middle of a bank's portfolio. These are typically commercial loans, mostly to unrated companies. If rated, these might be about S&P Baa or Moody's BBB. The method included a specification of the problem and some specific questions followed by a discussion. General discussions of the elicitation of prior distributions are given by Kadane, Dickey, Winkler, Smith, and Peters (1980), Garthwaite, Kadane, and O'Hagan (2005) and Kadane and Wolfson (1998). An example assessing a prior for a Bernoulli parameter is Chaloner and Duncan (1983). Chaloner and Duncan follow Kadane et al in suggesting that assessments be done not directly on the probabilities concerning the parameters, but on the predictive distribution. That is, questions should be asked about observables, to bring the expert's thoughts closer to familiar ground. In the case of a Bernoulli parameter and a 2-parameter beta prior, Chaloner and Duncan suggest first eliciting the mode of the predictive distribution for a given n (an integer), then assessing the relative probability of the adjacent values ("dropoffs"). Graphical feedback is provided for refinement of the specification. Examples consider n=20. Gavasakar (1988) suggests an alternative method, based on assessing modes of predictive distributions but not on dropoffs. Instead, changes in the mode in response to hypothetical samples are elicited and an explicit model of elicitation errors is proposed. The method is evaluated in the n=20 case and appears competitive. The suggestion to interrogate experts on what they would expect to see in data, rather than what they would expect of parameter values, is appealing and I have to some extent pursued this with our expert. This approach may be less attractive in the case of large sample sizes and small probabilities, and in our particular application, where the expert was sophisticated about probabilities. Our expert found it easier to think in terms of the probabilities directly than in terms of defaults in a hypothetical sample.

The sample period should be currently relevant, but should include a cycle, so that it is marginal with respect to business conditions. It could be argued that a recent period including the 2001-2002 period of mild downturn covers a modern cycle. A period that included the 1980's would yield higher default probabilities but these are probably not currently relevant. The default probability of interest is the current and immediate future value, not a guess at what past estimates might be. The precise definition of default is also at issue. In the economic theory of the firm, default occurs when debt payments are missed and ownership and control of the firm passes from existing owners (shareholders in the case of a corporation) to debtholders. As a lesser criterion, loans that are assigned to "nonaccrual" may be considered defaulted. We simply note the importance of using consistent definitions in the assessment of expert information and in data definition.

We did the elicitation assuming a sample of 500 asset-years. For our application, we also considered a "small" sample of 100 observations and a "large" sample of 1000 observations, and occasionally an enormous sample of 10000 observations. Considering first the predictive distribution on 500 observations, the modal value was five defaults. Upon being asked to consider the relative probabilities of five or four defaults, conditional on four or five defaults occurring (the conditioning does not matter here, for the probability ratio, but it is thought to be easier to think about when posed in this fashion), the expert expressed some trepidation as it is difficult to think about such rare events. Ultimately, the expert gave probability ratios not achievable by the binomial model even with known probability. This experience supports the implication of Gavasakar (1988) that dropoff probabilities are problematic. The expert was quite happy in thinking about probabilities over probabilities however. This may not be so uncommon in this technical area, as practitioners are accustomed to working with probabilities. The mean value was 0.01. The minimum value for the default probability was 0.0001 (one basis point). The expert reported that a value above 0.035 would occur with probability less than 10%, and an absolute upper bound was 0.3. The upper bound was discussed: the expert thought probabilities in the upper tail of his distribution were extremely unlikely, but he did not want to rule out the possibility that the rates were much higher than anticipated (prudence?). Quartiles were assessed by asking the expert to consider the value at which larger or smaller values would be equiprobable given the value was less than the median, then given the value was more than the median. The median value was 0.01. The former was 0.0075. The latter, the .75 quartile, was assessed at .0125. The expert seemed to be thinking in terms of a normal distribution, perhaps using informally a central limit theorem combined with long experience with this category of assets.

This set of answers is more than enough information to determine a 4-parameter Beta distribution. I used a method of moments to fit parametric probability statements to the expert assessments. The moments I used were squared differences relative to the target values, for example $((a - 0.0001)/0.0001)^2$. The support points were quite well-determined for a range of $\{\alpha, \beta\}$pairs at the assessed values $\{a, b\} = [0.0001, 0.3]$. These were allowed to vary (the parameter set is overdetermined) but the optimization routine did not change them beyond the 7th decimal place. Thus, the expert was able to determine these parameter values consistently with his probability assessments. Further, changing the weights did not matter much either. Probably this is due to the fact that there is almost no probability in the upper tail, so changing the upper bound made almost no difference in the assessed probabilities. Thus the rather high (?) value of b reflects the long tail apparently desired by the expert. The $\{\alpha, \beta\}$ parameters were rather less well-determined (the sum of squares function was fairly flat) and I settled on the values (7.9, 224.8) as best describing the expert's information. The resulting prior distribution $p(\theta|e)$ is graphed in Figure 3.

Figure 3

It is apparent that there is virtually no probability on the long right tail. A closer view of the relevant part of the prior is graphed in Figure 4.

Figure 4

The median of this distribution is 0.00988, the mean is 0.0103 and the standard deviation is 0.00355. In practice, after the information is aggregated into an estimated probability distribution, then additional properties of the distribution would be calculated and the expert would be consulted again to see if any changes were in order before proceeding to data analysis Lindley (1982a). This process would be repeated as necessary. In the present application there was one round of feedback, valuable since the expert had had time to consider the probabilities involved. The characteristics reported are from the second round of elicitation. An application within a bank might do additional rounds with the expert, or consider alternative experts and a combined prior.

The predictive distribution 4.1 corresponding to this prior is given in Figure 5 for n=500.

Figure 5

With our specification, the expected value of $r, E(r|e) = \sum_{k=0}^{n} kp(k|e)$ is 5.1 for n=500. Total defaults numbering 0-9 characterize 92% of expected data sets. Thus, carrying out our analysis for these 10 data types, comprising about 2^{62} distinct datasets, a trivial fraction of the 2^{500} possible datasets, actually covers 92% of the expected realizations. Defaults are expected to be rare events. This is the key to the ALD approach: we are not analyzing one particular dataset, rather we provide results applicable to 92% of the likely datasets.

6 Posterior Analysis

The posterior distribution, $p(\theta|r,e)$, is graphed in Figure 6 for r = 0, 2, 4, 6, and 8 and n=500. The corresponding likelihood functions, for comparison, were given in figures 1 and 2. Note the substantial differences in location. Comparison with the likelihood functions graphed in Figure 1 and the prior distribution graphed in Figure 3 reveals that the expert provides much more information to the analysis than do the data.

Figure 6

Given the distribution $p(\theta|r,e)$, we might ask for a summary statistic, a suitable estimator for plugging into the required capital formulas as envisioned by Basel Committee on Banking Supervision (2004). A natural value to use is the posterior expectation, $\overline{\theta} = E(\theta|r,e)$.The expectation is an optimal estimator under quadratic loss and is asymptotically an optimal estimator under a wide variety of loss functions. An alternative, by analogy with the maximum likelihood estimator $\widehat{\theta}$, is the posterior mode $\dot{\theta}$. As a summary measure of our confidence we would use the posterior standard deviation $\sigma_\theta = \sqrt{E(\theta - \overline{\theta})^2}$. By comparison, the usual approximation to the standard deviation of the maximum likelihood estimator is $\sigma_{\widehat{\theta}} = \sqrt{\widehat{\theta}(1-\widehat{\theta})/n}$.These quantities are given in Table 1 for r=0-9 and r=20, 50, 100, 200. As noted, the r=0-9 case covers the 2^{62} most likely datasets out of the possible 2^{500}. Together, these comprise analyses of 92% of likely datasets. The r=20 case is an extremely low probability outcome - less than 0.0001 - and is included to show the results in this case. There are approximately 2^{118} datasets corresponding to r=20. The rows for r=50, 100, and 200 are included as a further "stress test" and will be discussed below. Their combined prior probability of occurrence is less than 10^{-14}.

Table 1: Default Probabilities - Location and Precision, n=500

r	$\overline{\theta}$	$\dot{\theta}$	$\widehat{\theta}$	σ_θ	$\sigma_{\widehat{\theta}}$
0	0.0063	0.0081	0.000	0.0022	0 (!).
1	0.0071	0.0092	0.002	0.0023	0.0020
2	0.0079	0.0103	0.004	0.0025	0.0028
3	0.0086	0.0114	0.006	0.0026	0.0035
4	0.0094	0.0125	0.008	0.0027	0.0040
5	0.0102	0.0136	0.010	0.0028	0.0044
6	0.0109	0.0147	0.012	0.0029	0.0049
7	0.0117	0.0158	0.014	0.0030	0.0053
8	0.0125	0.0169	0.016	0.0031	0.0056
9	0.0132	0.0180	0.018	0.0032	0.0060
20	0.0215	0.0296	0.040	0.0040	0.0088
50	0.0431	0.0425	0.100	0.0053	0.0134
100	0.0753	0.0749	0.200	0.0065	0.0179
200	0.1267	0.1266	0.400	0.0069	0.0219

For values of r below its expected value the posterior mean is greater than the MLE, for values above the posterior is less than the MLE, as expected. As is well-known and widely discussed, the MLE is unsatisfactory when there are no observed defaults (Basel Committee on Banking Supervision (2005), Pluto and Tasche (2005), BBA, LIBA, and ISDA (2005), Kiefer (2006a)) The Bayesian approach provides a coherent resolution of the inference problem without resort to desperation (sudden reclassification of defaulted assets, technical gimmicks).

Expert information will have larger weight in smaller sample sizes, and smaller relative weight for larger sample sizes. For n=1000, for example, r=5-15 reflects 76% of the most likely datasets; r=0-20 represents 97%. To put this in perspective, the cases r=0-20 correspond to approximately 2^{138} datasets out of a possible 2^{1000}. Thus, 97% of the likely observations are contained in the small fraction 2^{-862} of the possible datasets, or 0.0021 of the possible types. A substantial simplification results from concentrating on the distribution of the sufficient statistic and use of expert judgement to characterize possible samples. Naturally, this simplification depends critically on the use of expert judgement in specification of the likelihood function (our choice admits a sufficient statistic) and in specification of the prior distribution. Rather than resorting to extensive tabulation, we report ALD results for 97% of likely samples in Figure 7. The error bands, dotted for the MLE and dashed for the prior mean, are plus/minus one standard deviation.

Figure 7

Turning now to an extremely large sample, in which inference is not quite so problematic, as the likelihood can be expected to dominate the prior, we find a lessened role for the expert. With n=10000, r=45-155 covers 88% of all datasets. In these cases the likelihood and Bayesian analyses essentially coincide. Estimators and associated error bands for the 88% ALD analysis are shown in Figure 8.

Figure 8

There is still a clear difference for extremely unlikely values of r; thus for $r = 0, E\theta = 0.00083$, while the MLE is zero. For large or very likely datasets, the posterior mean and MLE will nearly coincide. For example, in a sample of corporate bonds from KMV (North American Non Financial) over 1993 to 2004 from an aggregated mid-portfolio segment (roughly BBB+ through B-) we observe 7272 asset-years and 73 defaults. For details on the data see Kiefer and Larson (2006). Here the data set is completely as expected. The probability using $p(r|e)$ that r =73 is 0.015 (out of 7273 possible values); the probability that $65 \leq r \leq 80$ is 0.234, that

$55 \le r \le 90$ is 0.502. Thus this data set would be included in any reasonable ALD analysis. The posterior mean and standard deviation are 0.01006 and 0.00111. The MLE and its standard error are 0.01004 and 0.00117.

7 Robustness - the cautious Bayesian

Suppose we are rather less sure of our expert than he is of the default probability. Or, more politely, how can we assess just how important the tightly-held views of the expert are in determining our estimates? Table 1 gives one answer by comparing the MLE and the posterior location measures. Another answer was proposed by Kiefer (2006b) , who considered a less-certain expert with a prior with the same location but substantially higher variance than the actual expert. An alternative approach, more formal and based on the literature on Bayesian robustness (Berger and Berliner (1986)) is to mix the actual expert's prior with an alternative prior, and see exactly how seriously the inferences are affected by changes in the mixing parameter. Berger and Berliner (1986) in fact suggested mixing in a class of distributions, corresponding to different amounts or directions of uncertainty in the prior elicitation. In this spirt, we will mix the expert's 4-parameter beta distribution with a uniform distribution. Here, there are two clear possibilities. One is to mix with the uniform on [a,b], accepting the expert's bounds but examining robustness to alpha and beta. The second is to mix with the uniform on [0,1], allowing all theoretically feasible values of θ. We choose the latter approach. This is not a completely comfortable approach. Although the uniform is commonly interpreted as an uninformative prior, it in fact has a mean of 1/2, not a likely value for our default probability by any reasonable prior. An alternative might be to mix with a prior with the same mean as our expert's distribution, but maximum variance. We do not pursue this here. Our results suggest that it would not make much difference; the key is to mix in a distribution with full support, so that likelihood surprises can appear. We choose to mix the expert's prior with a uniform on all of [0,1]. This allows input from the likelihood if the likelihood happens to be concentrated above b (or below a). The mixture distribution is

$$p(\theta|e,\epsilon) = (1-\epsilon)p(\theta|\alpha,\beta,a,b)I(\theta \in [a,b]) + \epsilon \tag{7.1}$$

for $\theta \in [0,1]$. The approach can be used whatever prior is specified, not just the 4-parameter beta. Our robust prior is in the 9-parameter mixture family 3.3, consisting of our expert's 4-parameter beta mixed with the 4-parameter beta with parameters $\{\alpha,\beta,a,b\} = \{1,1,0,1\}$and mixing parameter ϵ. Table 2 shows the posterior means for the mixture priors for $\epsilon = \{0.01, 0.1,\ 0.2, 0.3, 0.4\}$.

Table 2: Robustness - Posterior means for mixture priors, n=500

r	$\overline{\theta}; \epsilon = .01$	$\overline{\theta}; \epsilon = .1$	$\overline{\theta}; \epsilon = .2$	$\overline{\theta}; \epsilon = .3$	$\overline{\theta}; \epsilon = .4$
0	0.0063	0.0063	0.0062	0.0061	0.0061
1	0.0071	0.0071	0.0071	0.0071	0.0070
2	0.0079	0.079	0.0079	0.0079	0.0078
3	0.0086	0.0086	0.0086	0.0086	0.0086
4	0.0094	0.0094	0.0094	0.0094	0.0094
5	0.0102	0.0102	0.0102	0.0102	0.0102
6	0.0109	0.0109	0.0110	0.0110	0.0110
7	0.0117	0.0117	0.0117	0.0118	0.0118
8	0.0125	0.0125	0.0125	0.0125	0.0126
9	0.0132	0.0133	0.0133	0.0134	0.0134
20	0.0358	0.0358	0.0386	0.0398	0.0405
50	0.1016	0.1016	0.1016	0.1016	0.1016
100	0.2012	0.2012	0.2012	0.2012	0.2012
200	0.4004	0.4004	0.4004	0.4004	0.4004

Mixing the expert's prior with the uniform prior makes essentially no difference to the posterior mean for data in the likely part of the set of potential samples. For r=20, unlikely but not outrageous, using the robust prior makes a substantial difference. For the extremely unlikely values, 50, 100, 200, the differences are dramatic. The actual value of ϵ makes almost no difference. The numbers for $\epsilon = 0.001$, not shown in the table, give virtually the same mean for all r. For comparison, we recall the values of $\overline{\theta}$ for r={20,50,100,200} from Table 1. These are {0.0215,0.0431,0.0753,0.1267}. Figure 9 shows the posterior distributions for our expert's prior, $p(\theta|r, e)$ for r=50, 100, and 200. It is clear that the prior plays a huge role here, as the likelihood mass is concentrated near .1, .2 and .4, while the prior gives only trivial weight to values greater than about .03, see Figures 1 and 3. On the other hand, Figure 10 shows the posterior corresponding to 7.1 with 1% mixing ($\epsilon = 0.01$). Here, the likelihood dominates, as the likelihood value near the expert's prior is vanishingly small relative to the likelihood in the tail area of the mixing prior.

Figure 9

Figure 10

Thus, the robust analysis with even a very small nonzero mixing fraction can reveal disagreements between the data and the expert opinion which are perhaps masked by the formal analysis. This robust analysis may have a role to play in the validation phase.

I what sense is the robust analysis useful? We are really bringing something outside the model, namely the uniform distribution representing no one's beliefs, into the analysis as a formal tool for diagnostic analysis. The spirit is the same as usual procedures associated with good statistical practice - residual analysis, out of sample fits, forecast monitoring, or comparison with alternative models. All of these procedures involve stepping away from the specified model and its analysis, and asking, post estimation, does the specification make sense? Post-estimation model evaluation techniques are often informal, sometimes problem specific, and require sound statistical judgement OCC (2006). The analysis of robustness via an artificial prior is an attempt to merge the formal analysis with the informal post-estimation model checking. A related method, checking for irrelevant data using a mixture distribution, is proposed by Ritov (1985) and this might have a role as well.

8 Heterogeneity

It is clearly contemplated in the Basel II guidance that heterogeneity is mitigated by the classification of assets into homogeneous groups before estimation of the group-specific default probability. However, there may be remaining heterogeneity, due to asset characteristics or to changing macroeconomic conditions. In fact, the Basel II prescription is for a default probability that is averaged over a cycle. This seems to indicate that the default probability varies over the cycle, and perhaps a model that takes this possibility into account would be appropriate. The variation is unlikely to affect inference about the marginal probability in an important way (though confidence may be overstated), but if there is a cycle effect, it would be valuable to know its magnitude. Clearly, if there is not much variation in these cyclical variables, the current analysis would apply over rather short sample periods. A natural first step therefore would be to group assets according to upturn years, downturn years, and stable years and run separate analyses. For low-default portfolios there is unlikely to be enough data to sort out differences between these years. However, there is evidence from other markets that default probabilities vary over the cycle Nickell, Perraudin, and Varotto (2000). The abstract problem is to specify a model in which the default probability varies with market conditions, indicated by the variable x_t. A specification which has proved useful is $\theta_t/(1-\theta_t) = \exp\{\alpha + \beta x_t\}$. The prior is then taken on the parameter set $\{\alpha, \beta\}$. With this specification α and β are unrestricted and a normal prior distribution may be suitable. Assessment of the combination $\alpha + \beta\overline{x}$ can follow the procedure above. Additional thought is required to sort out the likely effect of x, that is, appropriate values for β. Both assessment and validation are crucial here since data evidence is inherently sparse. Estimation of this model is now straightforward, building on the early work of Albert and Chib (1993) using Monte Carlo Markov Chain (MCMC) and related procedures (see Robert and Casella (2004) and Geweke (2005)).

9 Conclusion

I have considered inference about the default probability for a midrange portfolio segment on the basis of data information and expert judgement. Examples focus on the sample size of 500; results are also presented for the large sample sizes of 1000 and 10000 observations, not unreasonable for large banks in this risk range. These analyses are relevant to hypothetical portfolios of middle-risk commercial loans. These are predominantly to unrated companies; if rated these would be approximately S&P Baa or Moody's BBB. I have also represented the judgement of an expert in the form of a probability distribution, for combination with the likelihood function. The expert is a practitioner experienced in risk management in well-run banks. The 4-parameter Beta distribution seems to reflect expert opinion fairly well. Errors, which would be corrected through additional feedback and respecification in practice, are likely to introduce more certainty into the distribution rather than less. Using the ALD approach, it is possible to study the posterior distributions for all of the most likely configurations of defaults in the samples. Using ALD, we consider the possible realizations of the sufficient statistic for the specified statistical model. In the default case, the number of realizations is linear in the sample size (while the number of potential distinct samples is exponential). Using the expert information, it is possible to isolate the most likely realizations. In the sample of 500, five defaults are expected. In this case, our analysis of 0 through 9 defaults covers 92% of expected datasets. Our analyses of samples of 1000 and 10000 covered 97% and 88% of the likely realizations respectively.

At the validation stage, modelers can be expected to have to justify the likelihood specification and the representation of expert information. Analysis of the sensitivity of the results to the prior should be a part of this validation procedure. We propose using a mixture of the expert's prior and an alternative, less informative prior. In our case, we mix the prior with a uniform distribution on the unit interval. While it is not likely that the uniform describes any expert's opinion on the default probability, mixing in the uniform allows unexpected disagreement between the prior and the data to appear vividly. An example shows that even a trivially small weight on the alternative will do. Of course, within the context of the model, the decision based on the expert's posterior is correct. A broader view might suggest something wrong with the specification - of either the likelihood or the prior. Perhaps these do not refer to the same risk class, or perhaps the default definitions are inconsistent. The situation is not unlike that arising in ordinary validation exercises in which the model is evaluated in terms of residual analysis or out-of-sample fits. These involve considerations which are relevant but which are outside the formal model. As a result there are a number of different methods in use, corresponding to different ways in which models can fail, and expert judgement remains crucial in this less formal context as well as in the formal specification of the likelihood and the prior. For further discussion, see OCC (2006).

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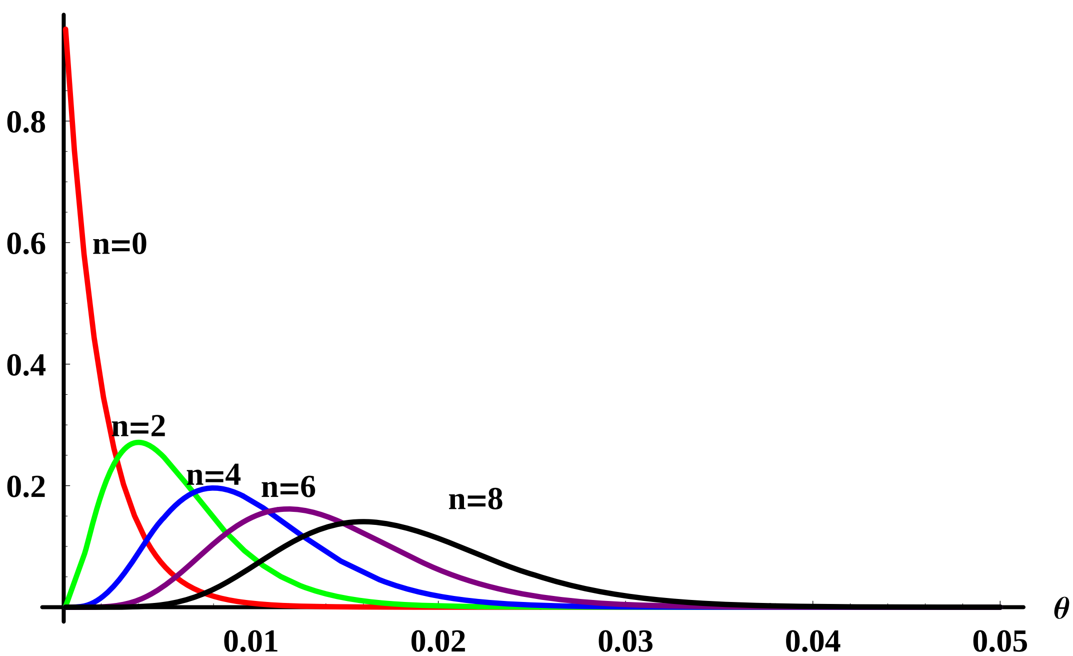
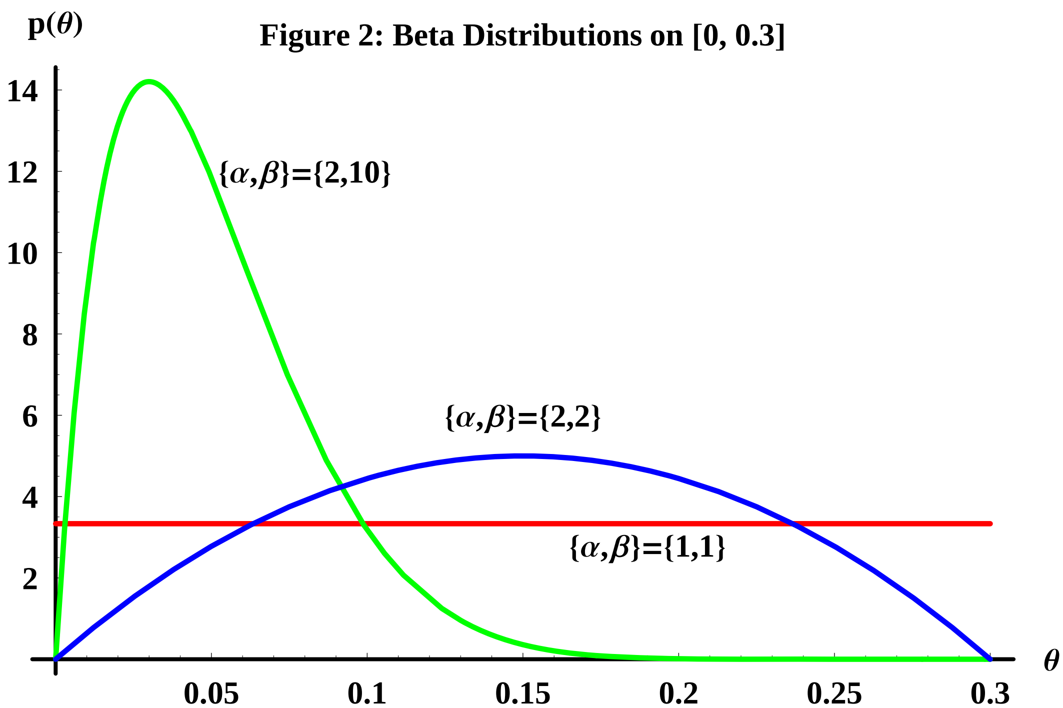


Figure 1: Likelihood Functions



p(θ)
Figure 2: Beta Distributions on [0, 0.3]
14
12
10
8
6
4
2
{α,β}={2,10}
{α,β}={2,2}
{α,β}={1,1}
0.05
0.1
0.15
0.2
0.25
0.3
θ

Figure 3: Expert Information

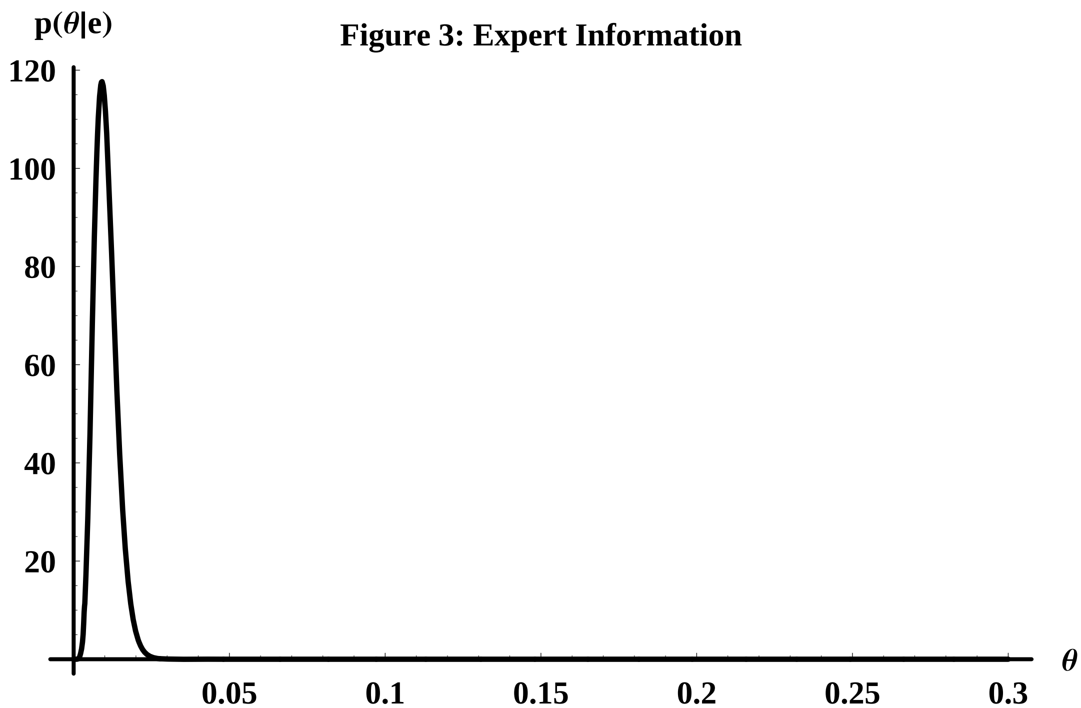
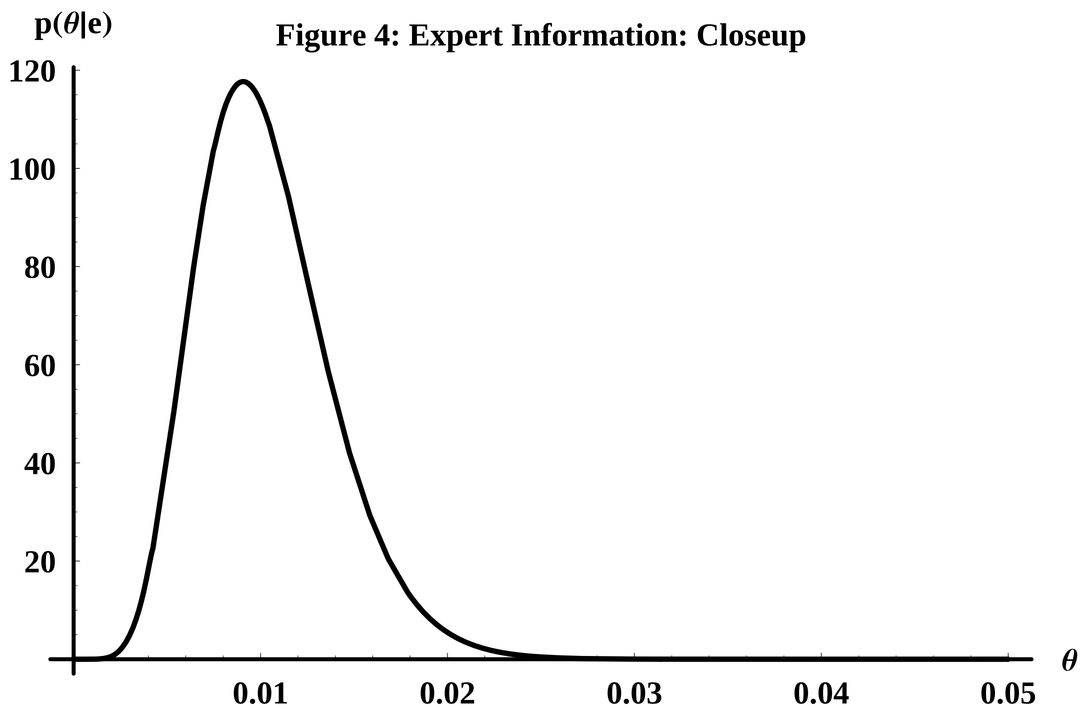


Figure 4: Expert Information: Closeup



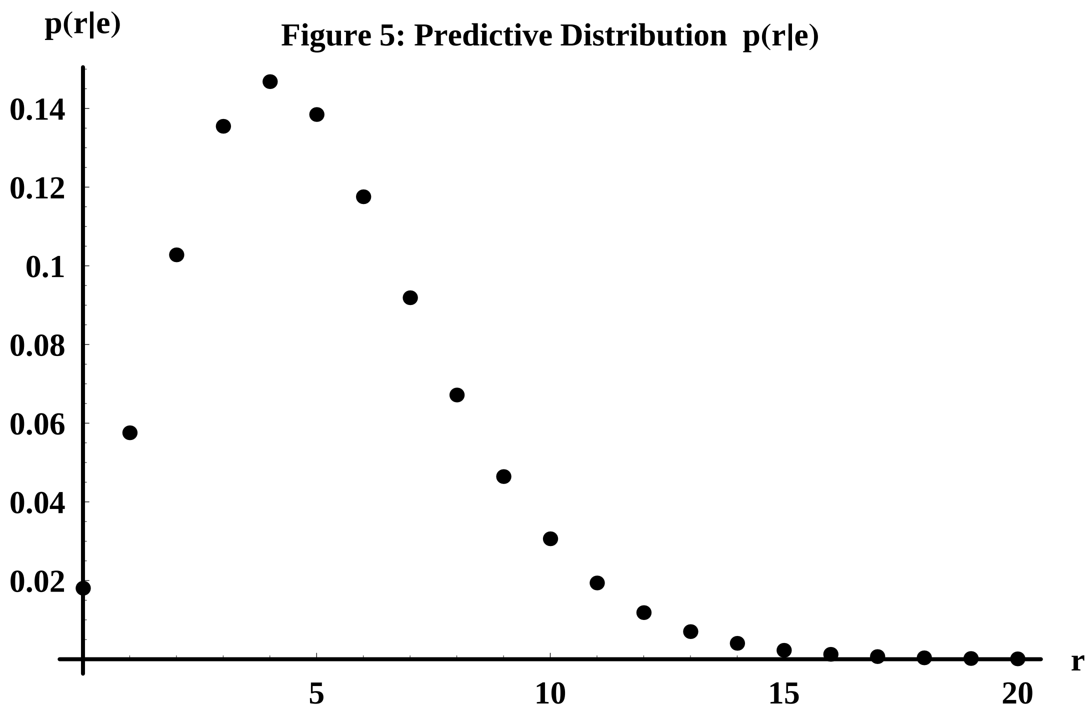
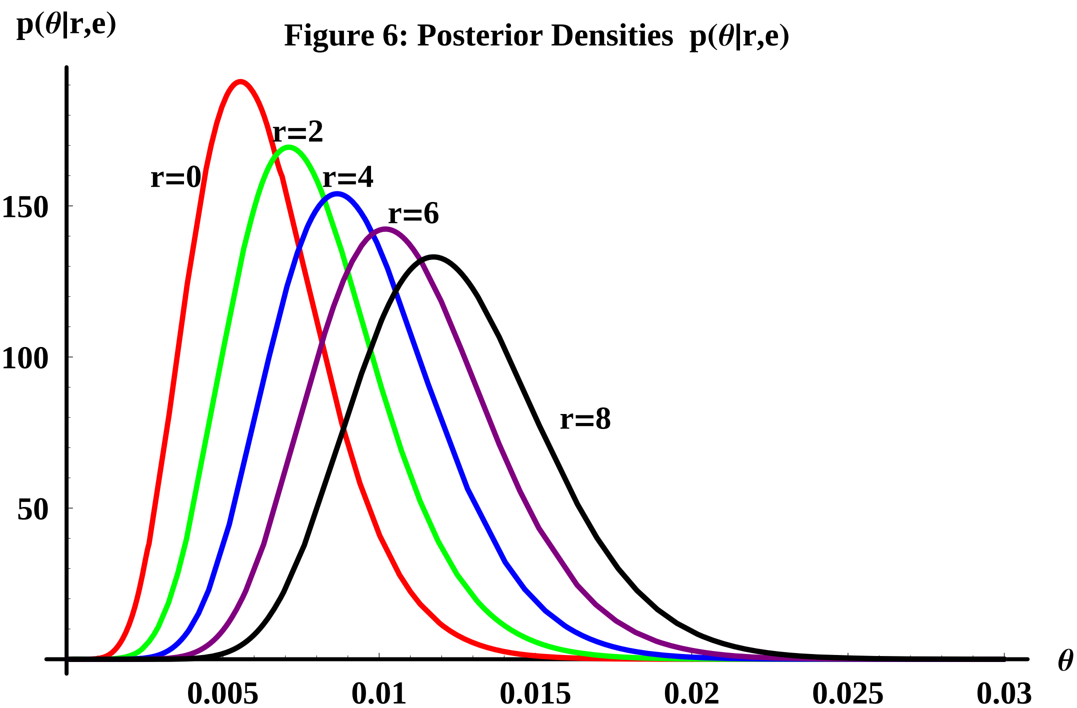
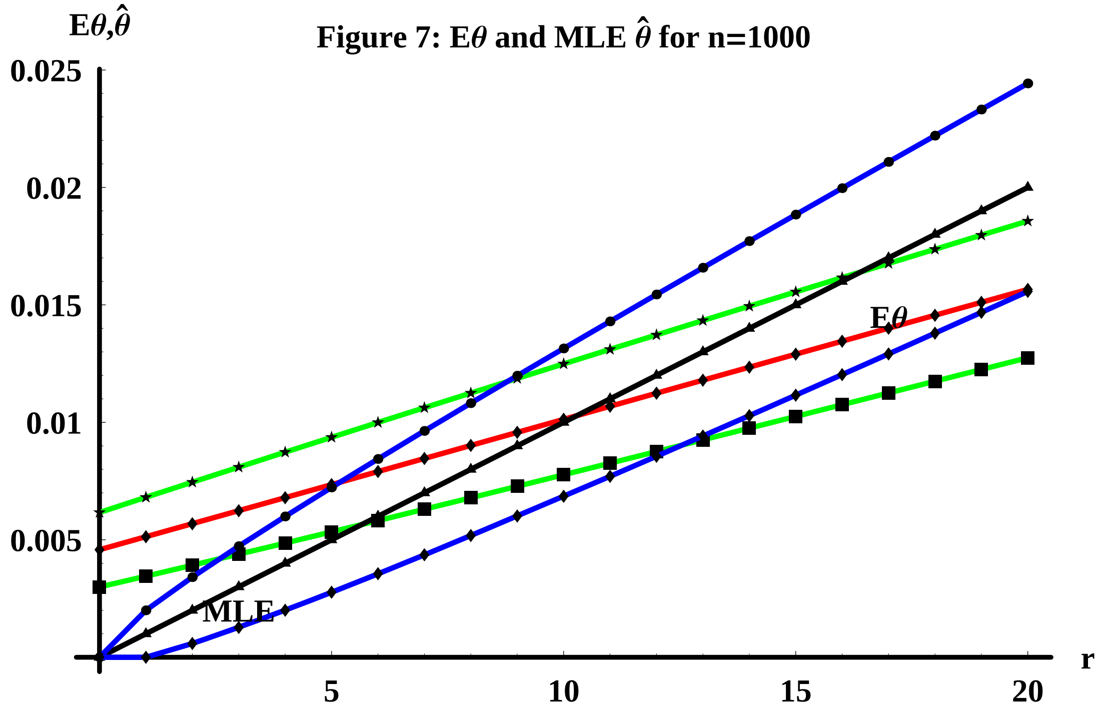


Figure 5: Predictive Distribution p(r|e)


p(θ|r,e)
Figure 6: Posterior Densities p(θ|r,e)
r=0
r=2
r=4
r=6
r=8
150
100
50
0.005
0.01
0.015
0.02
0.025
0.03
θ


Eθ, θ̂
Figure 7: Eθ and MLE θ̂ for n=1000
0.025
0.02
0.015
0.01
0.005
5
10
15
20
r
Eθ
MLE

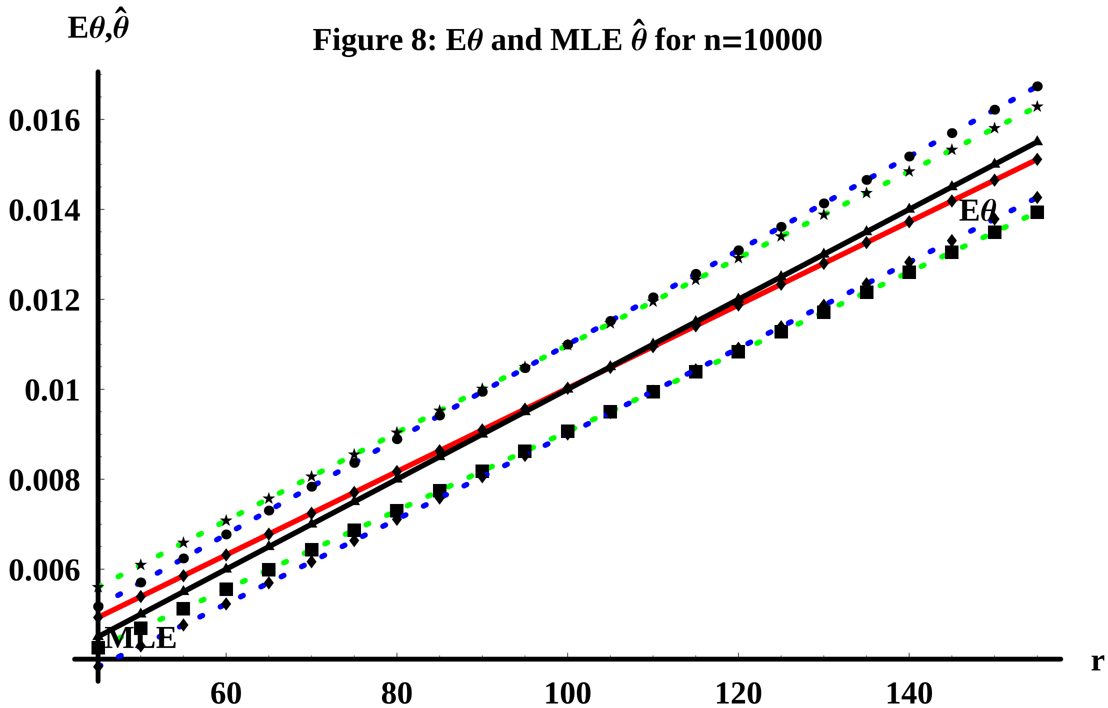


Figure 8: Eθ and MLE $\hat{\theta}$ for n=10000
Eθ,$\hat{\theta}$
0.016
0.014
0.012
0.01
0.008
0.006
60
80
100
120
140
r
MLE
Eθ

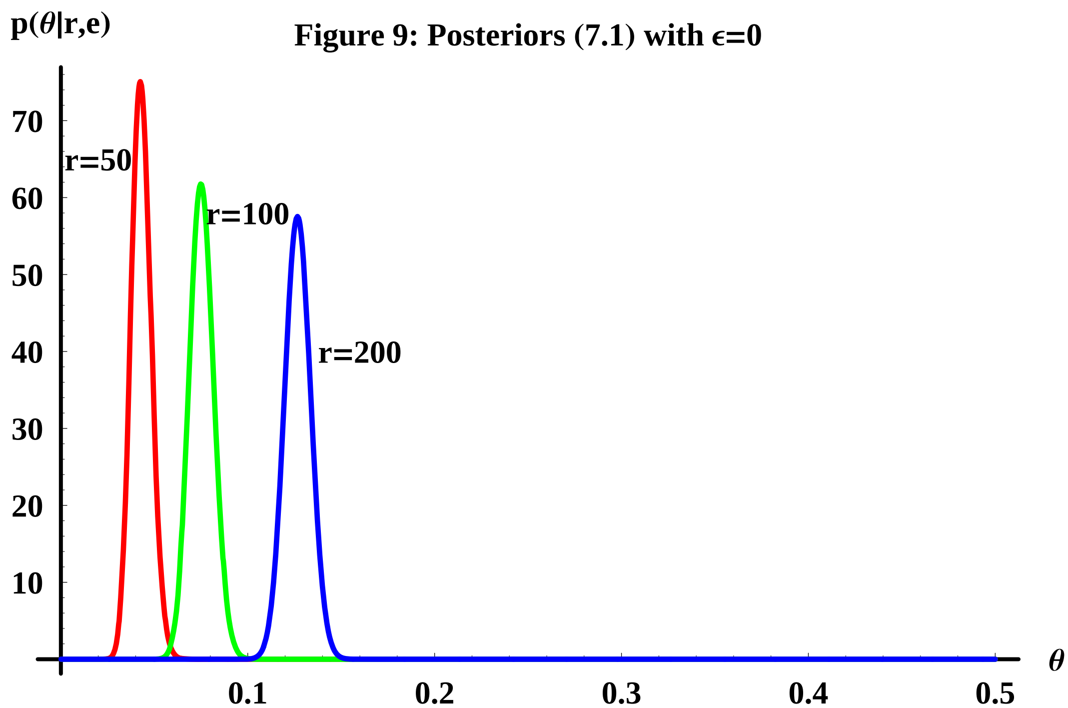


Figure 9: Posteriors (7.1) with ϵ=0

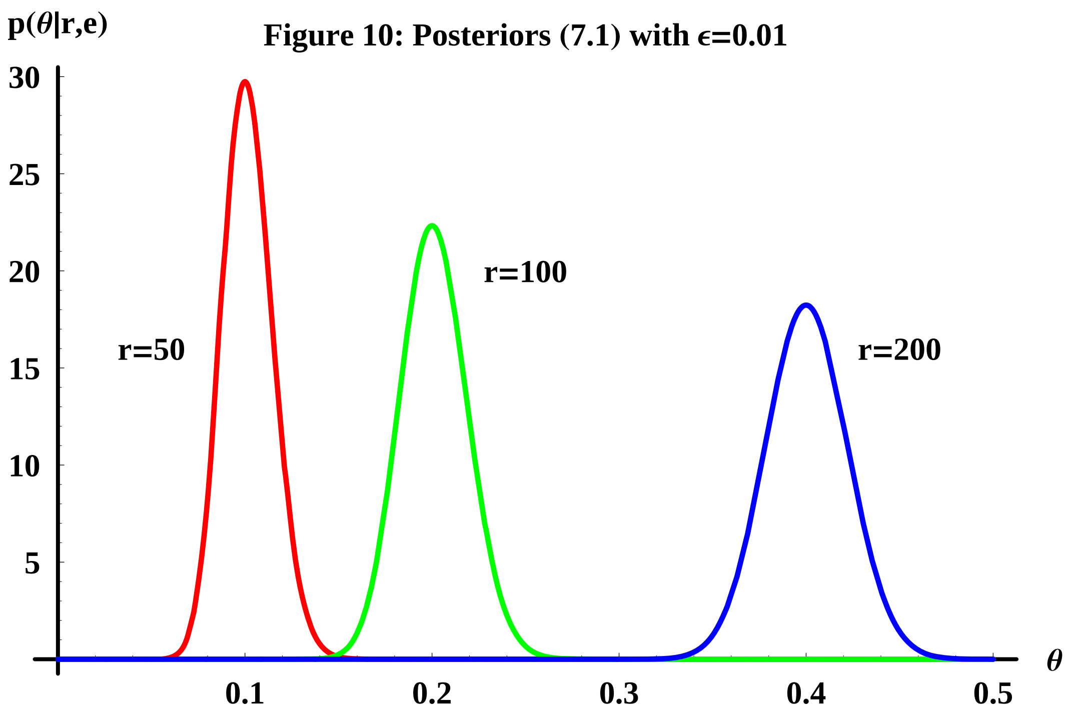


Figure 10: Posteriors (7.1) with ϵ=0.01