ISSN 1936-5098

CAE Working Paper #07-04

A Dynamic Theory of Public Spending, Taxation and Debt

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April 2007

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This revision: February 2007

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Abstract

This paper presents a dynamic political economy theory of public spending, taxation and debt. Policy choices are made by a legislature consisting of representatives elected by geographically-defined districts. The legislature can raise revenues via a distortionary income tax and by borrowing. These revenues can be used to finance a national public good and district-specific transfers (interpreted as pork-barrel spending). The value of the public good is stochastic, reflecting shocks such as wars or natural disasters. In equilibrium, policy-making cycles between two distinct regimes: "business-as-usual" in which legislators bargain over the allocation of pork, and "responsible-policy-making" in which policies maximize the collective good. Transitions between the two regimes are brought about by shocks in the value of the public good. In the long run, equilibrium tax rates are too high and too volatile, public good provision is too low, and debt levels are too high. In some environments, a balanced budget requirement can improve citizen welfare.

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*For detailed comments, we thank Jon Eguia, Mark Gertler, Aleh Tsyvinski and two anonymous referees. We also thank Andrew Atkeson, Marina Azzimonti, Roland Benabou, Tim Besley, Faruk Gul, Per Krusell and seminar participants at Berkeley, Cal Tech, Cambridge, Chicago, Cornell, Georgetown, Harvard, LSE, Michigan, Munich, NBER, Northwestern, NYU, Ohio State, Oxford, Pittsburgh, Princeton, Queen's, Rochester, Stanford, UCLA, Virginia, and Wisconsin for helpful feedback. Marco Battaglini gratefully acknowledges financial support from a NSF CAREER Award (SES 0547748).

1 Introduction

This paper presents a dynamic political economy theory of public spending, taxation and debt. The theory builds on the well-known tax smoothing approach to fiscal policy pioneered by Barro (1979). This approach predicts that governments will use budget surpluses and deficits as a buffer to prevent tax rates from changing too sharply. Thus, governments will run deficits in times of high government spending needs and surpluses when needs are low. Underlying the approach are the assumptions that governments are benevolent, that government spending needs fluctuate over time, and that the deadweight costs of income taxes are a convex function of the tax rate. The economic environment underlying our theory is similar to that in the tax smoothing literature. Our key departure is that policy decisions are made by a legislature rather than a benevolent planner. Moreover, we introduce the friction that legislators can distribute revenues back to their districts via pork-barrel spending.

More specifically, our theory assumes that policy choices are made by a legislature comprised of representatives elected by single-member, geographically-defined districts. The legislature can raise revenues in two ways: via a proportional tax on labor income and by borrowing in the capital market. Borrowing takes the form of issuing one period bonds. The legislature can also purchase bonds and use the interest earnings to help finance future public spending if it so chooses. Public revenues are used to finance the provision of a public good that benefits all citizens and to provide targeted district-specific transfers, which are interpreted as pork-barrel spending. The value of the public good to citizens is stochastic, reflecting shocks such as wars or natural disasters. The legislature makes policy decisions by majority (or super-majority) rule and legislative policy-making in each period is modelled using the legislative bargaining approach of Baron and Ferejohn (1989). The level of public debt acts as a state variable, creating a dynamic linkage across policy-making periods.

Incorporating political decision making in this way resolves an important theoretical difficulty with the tax smoothing approach first pointed out by Aiyagari et al (2002). While Barro's original analysis assumes that the government perfectly anticipates its fluctuating spending needs, Aiyagari et al tackle the more relevant case of uncertainty. They demonstrate that the tax smoothing logic does not necessarily imply the counter-cyclical theory of deficits and surpluses that it had been presumed to. In some environments, the optimal policy is for the government to gradually acquire sufficient bond holdings so as to eventually be able to finance any level of spending with the interest earnings from these holdings. This permits the financing of government spending without distortionary taxation. Interest earnings in excess of spending needs are rebated back to citizens via lump-sum transfers. Obviously, the prediction of a steady state with huge government asset accumulation and zero taxes is unsatisfactory. This prediction is avoided if exogenous limits on the amount of debt that the government can hold are imposed, but Aiyagari et al rightly criticize these as "ad hoc".[1]

Intuitively, it seems likely that legislators entrusted with a large stock of government assets would run it down and distribute the proceeds back to their districts and this is precisely the force that our theory captures. Thus, despite the fact that there are no ad hoc debt limits, the long run level of government bond holdings in political equilibrium is below the efficient level. Moreover, equilibrium policies display the dynamic pattern suggested by Barro; namely, debt goes up when the value of public goods is high and down when it is low. In addition, debt serves to smooth taxes.

Our theory also offers a number of other advantages over the basic tax smoothing approach. First, it allows for the possibility that the government can be in perpetual debt. Second, it provides predictions concerning the dynamics of legislative policy-making and on the mix of public spending between pork and public goods. Third, it provides a sharp account of how political decision-making "distorts" public policies. Fourth, the theory permits a welfare analysis of fiscal restraints such as balanced budget rules.

That pork-barrel spending gives rise to inefficiencies in legislative decision making is a core idea of political economy. Moreover, it is now well understood that in dynamic environments redistributive considerations can lead legislatures to be present-biased. What is novel about our paper is that we incorporate these ideas into a dynamic general equilibrium model that incorporates the key assumptions of the tax smoothing literature. This allows us to better integrate the political economy and tax smoothing literatures. In particular, we can study how the political forces favoring present bias in legislative policy-making interact with the economic forces favoring the use of debt for tax smoothing purposes. The interplay between these forces gives rise to what is, in our judgement, a richer and more satisfying theory of fiscal policy.

[1] Shin (2006) shows that this prediction can be avoided if citizens face idiosyncratic and uninsurable productivity shocks.

Our basic approach to incorporating legislative decision making into a dynamic general equilibrium model follows our earlier work in Battaglini and Coate (2007). In that paper, we explored how pork-barrel spending impacts the overall size of government and distorts investment in public capital goods. We analyzed an environment in which in each period the legislature can raise revenues via a distortionary income tax and these revenues can be used to finance investment in a public good and pork-barrel spending. The environment we study in this paper differs in three key ways. First, the government can borrow as well as levy income taxes. Second, the public good is not an investment good. Third, the value of the public good is stochastic. This makes for a very different application, with the key dynamic linkage across periods created by the level of debt rather than the stock of public good.

The organization of the remainder of the paper is as follows. In the next section we present the model. Section 3 provides a benchmark by describing the planning solution for the economy. Section 4 characterizes the political equilibrium and develops the positive predictions of the theory. Section 5 explains precisely how political decision making distorts the efficient solution. Section 6 discusses the empirical implications of the theory and section 7 applies the theory to analyze the desirability of a balanced budget requirement. Section 8 discusses the related political economy literature and section 9 offers a brief conclusion. The Appendix contains the proofs of the propositions.

2 The model

2.1 The economic environment

A continuum of infinitely-lived citizens live in n identical districts indexed by $i = 1, ..., n$. The size of the population in each district is normalized to be one. There is a single (nonstorable) consumption good, denoted by z, that is produced using a single factor, labor, denoted by l, with the linear technology $z = wl$. There is also a public good, denoted by g, that can be produced from the consumption good according to the linear technology $g = z/p$.

Citizens consume the consumption good, benefit from the public good, and supply labor. Each citizen's per period utility function is

$$z + Ag^{\alpha} - \frac{l^{(1+1/\varepsilon)}}{\varepsilon + 1}, \tag{1}$$

where $\alpha \in (0, 1)$ and $\varepsilon > 0$. The parameter A measures the value of the public good to the citizens.

Citizens discount future per period utilities at rate δ.

The value of the public good varies across periods in a random way, reflecting shocks to the society such as wars and natural disasters. Specifically, in each period, A is the realization of a random variable with range $[\underline{A}, \overline{A}]$ (where $0 < \underline{A} < \overline{A}$) and cumulative distribution function $G(A)$. The function G is continuously differentiable and its associated density is bounded uniformly below by some positive constant $\xi > 0$, so that for any pair of realizations such that $A < A'$, the difference $G(A') - G(A)$ is at least as big as $\xi(A' - A)$. Thus, G assigns positive probability to all nondegenerate sub-intervals of $[\underline{A}, \overline{A}]$.

There is a competitive labor market and competitive production of the public good. Thus, the wage rate is equal to w and the price of the public good is p. There is also a market in risk-free one period bonds. The assumption of a constant marginal utility of consumption implies that the equilibrium interest rate on these bonds must be $\rho = 1/\delta - 1$. At this interest rate, citizens will be indifferent as to their allocation of consumption across time.

2.2 Government policies

The public good is provided by the government. The government can raise revenue by levying a proportional tax on labor income. It can also borrow and lend in the bond market by selling and buying risk-free one period bonds.[2] Revenues can not only be used to finance the provision of the public good but can also be diverted to finance targeted district-specific transfers which are interpreted as (non-distortionary) pork-barrel spending.

Government policy in any period is described by an $n + 3$-tuple $\{r, g, x, s_1, ..., s_n\}$, where r is the income tax rate; g is the amount of the public good provided; x is the amount of bonds sold; and s_i is the proposed transfer to district i's residents. When x is negative, the government is buying bonds. In each period, the government must also repay any bonds that it sold in the previous period. Thus, if it sold b bonds in the previous period, it must repay $(1 + \rho)b$ in the current period. The government's initial debt level in period 1 is given exogenously and is denoted by b_0.

In a period in which government policy is $\{r, g, x, s_1, ..., s_n\}$, each citizen will supply an amount

[2] Thus we do not consider state-contingent debt as in Lucas and Stokey (1983). We believe that this is the appropriate assumption for a positive analysis. We also do not consider debt with different maturity structures. While Angeletos (2002) has argued that maturity structures can substitute for state-contingent debt, his argument does not apply in our model because the interest rate is constant.

of labor

$$l^*(w(1-r)) = \arg\max_l \{w(1-r)l - \frac{l^{(1+1/\varepsilon)}}{\varepsilon+1}\}. \tag{2}$$

It is straightforward to show that $l^*(w(1-r)) = (\varepsilon w(1-r))^\varepsilon$, so that ε is the elasticity of labor supply. A citizen in district i who simply consumes his net of tax earnings and his transfer will obtain a per period utility of $u(w(1-r), g; A) + s_i$, where

$$u(w(1-r), g; A) = \frac{\varepsilon^\varepsilon (w(1-r))^{\varepsilon+1}}{\varepsilon+1} + Ag^\alpha. \tag{3}$$

Since citizens are indifferent as to their allocation of consumption across time, their lifetime expected utility will equal the value of their initial bond holdings plus the payoff they would obtain if they simply consumed their net earnings and transfers in each period.

Government policies must satisfy three feasibility constraints. The first is that revenues must be sufficient to cover expenditures. To see what this implies, consider a period in which the initial level of government debt is b and the policy choice is $\{r, g, x, s_1, ..., s_n\}$. Expenditure on public goods and debt repayment is $pg + (1+\rho)b$. Tax revenue is

$$R(r) = nrwl^*(w(1-r)) = nrw(\varepsilon w(1-r))^\varepsilon \tag{4}$$

and revenue from bond sales is x. Letting the *net of transfer surplus* (i.e., the difference between revenues and spending on public goods and debt repayment) be denoted by

$$B(r, g, x; b) = R(r) - pg + x - (1+\rho)b, \tag{5}$$

the constraint requires that $B(r, g, x; b) \geq \sum_i s_i$.

The second constraint is that the district specific transfers must be non-negative (i.e., $s_i \geq 0$ for all i). This rules out financing public spending via district-specific lump sum taxes. With lump sum taxes, there would be no need to impose the distortionary labor tax and hence no tax smoothing problem.

The third and final constraint is that the amount of government borrowing must be feasible. In particular, there is an upper limit $\overline{x}$ on the amount of bonds the government can sell. This is motivated by the unwillingness of borrowers to hold bonds that they know will not be repaid. If the government were borrowing an amount x such that the interest payments exceeded the maximum possible tax revenues; i.e., $\rho x > \max_r R(r)$, then it would be unable to repay the debt *even if it provided no public goods or transfers.* Thus, the maximum level of debt is certainly less than this level, implying that $\overline{x} \leq \max_r R(r)/\rho$. In fact, we will assume that $\overline{x}$ is slightly smaller than $\max_r R(r)/\rho$. This is because if $\overline{x}$ equals $\max_r R(r)/\rho$ then if government debt ever reached $\overline{x}$ it would stay there forever, because the legislature could never pay it off. For our dynamic results, it is convenient to assume away this (relatively uninteresting) possibility.

We avoid assuming that there is any "ad hoc" limit on the amount of bonds that the government can purchase (see Aiyagari et al (2002)). In particular, the government is allowed to hold sufficient bonds to permit it to always finance the Samuelson level of the public good from the interest earnings. This level of bonds is given by $\underline{x} = -pg_S(\overline{A})/\rho$, where $g_S(A)$ is the level of the public good that satisfies the *Samuelson Rule* when the value of the public good is A.[3] Since the government will never want to hold more bonds than this, there is no loss of generality in constraining the choice of debt to the interval $[\underline{x}, \overline{x}]$ and we will do this below.[4] We also assume that the initial level of government debt, b_0, belongs to the interval $[\underline{x}, \overline{x}]$.

2.3 The political process

Government policy decisions are made by a legislature consisting of representatives from each of the n districts. One citizen from each district is selected to be that district's representative. Since all citizens have the same policy preferences, the identity of the representative is immaterial and hence the selection process can be ignored.[5] The legislature meets at the beginning of each period. These meetings take only an insignificant amount of time, and representatives undertake private sector work in the rest of the period just like everybody else. The affirmative votes of $q < n$ representatives are required to enact any legislation.

To describe how legislative decision-making works, suppose the legislature is meeting at the beginning of a period in which the current level of public debt is b and the value of the public good

[3] The Samuelson Rule is that the sum of marginal benefits equal the marginal cost, which means that $g_S(A)$ satisfies the first order condition that $n\alpha A g^{\alpha-1} = p$.

[4] By assuming that the government can choose to borrow any amount in the interval $[\underline{x}, \overline{x}]$, we are implicitly assuming that the wage is sufficiently high that the amount spent on public goods is never higher than national income. To see this, imagine that the initial level of government debt is b and the government chooses the policy $\{r, g, x, s_1, ..., s_n\}$. Then, feasibility demands that the amount borrowed x must be less than the total amount of private sector income. The latter is given by $\sum_i s_i + (1+\rho)b + n(1-r)w(\varepsilon w(1-r))^{\varepsilon}$. Assuming government budget balance, we know that $\sum_i s_i + (1+\rho)b$ is equal to $x + R(r) - pg$. Substituting this in, the feasibility condition amounts to the requirement that $nw(\varepsilon w(1-r))^{\varepsilon}$ (which is national income) exceeds pg. In either the equilibrium or the planner's solution, national income always exceeds $nw(\varepsilon w(\frac{\varepsilon}{1+\varepsilon}))^{\varepsilon}$ and public good spending is always less than $pg_S(\overline{A})$. Thus, a sufficient condition is that $nw(\varepsilon w(\frac{\varepsilon}{1+\varepsilon}))^{\varepsilon} > pg_S(\overline{A})$. Of course, such a condition would not be required in the case of a small open economy which could borrow from abroad.

[5] While citizens may differ in their bond holdings, this has no impact on their policy preferences.

is A. One of the legislators is randomly selected to make the first proposal, with each representative having an equal chance of being recognized. A proposal is a policy $\{r, g, x, s_1, ..., s_n\}$ that satisfies the feasibility constraints. If the first proposal is accepted by q legislators, then it is implemented and the legislature adjourns until the beginning of the next period. At that time, the legislature meets again with the difference being that the initial level of public debt is x and there is a new realization of the value of public goods. If, on the other hand, the first proposal is not accepted, another legislator is chosen to make a proposal. There are $T \geq 2$ such proposal rounds, each of which takes a negligible amount of time. If the process continues until proposal round T, and the proposal made at that stage is rejected, then a legislator is appointed to choose a default policy. The only restrictions on the choice of a default policy are that it be feasible and that it involve a uniform district-specific transfer (i.e., $s_i = s_j$ for all i, j).

3 The social planner's solution

To establish a normative benchmark with which to compare the political equilibrium, we begin by describing the policies that would be chosen by a social planner whose objective was to maximize aggregate utility. This is basically the problem considered by Aiyagari et al (2002). However, we will derive the solution in a way that sets the stage for the more complicated analysis of the political equilibrium.[6]

The planner's problem can be formulated recursively. The *state* of the economy is summarized by the current level of public debt b and the value of the public good A. Let $v(b, A)$ denote maximal average citizen expected utility (net of the value of initial bond holdings) at the beginning of a period in which the state is (b, A).[7] Then, in a period in which the state is (b, A), the planner's problem is to choose a policy $\{r, g, x, s_1, ..., s_n\}$ to solve:

$$\begin{aligned} &\max u(w(1-r), g; A) + \frac{\sum_i s_i}{n} + \delta E v(x, A') \\ s.t. \quad &\sum_i s_i \leq B(r, g, x; b), \quad s_i \geq 0 \text{ for all } i, \text{ and } x \in [\underline{x}, \overline{x}]. \end{aligned} \tag{6}$$

The three constraints are the feasibility constraints described in section 2.2.

[6] Aiyagari et al allow for more general preferences, focusing on the quasi-linear case as a leading example. This complicates the model because interest rates are affected by government policy. These complications require them to use a less transparent solution method.

[7] Maximal average expected utility will be $b(1+\rho)/n + v(b, A)$.

This problem can be simplified by observing that if the net of transfer surplus $B(r, g, x; b)$ were positive, the planner would use it to finance transfers and hence $\sum_i s_i = B(r, g, x; b)$. Thus, we can eliminate the choice variables $(s_1, ..., s_n)$ and reformulate the problem as choosing a tax rate-public good-public debt triple (r, g, x) to solve:

$$\begin{aligned} &\max \ u(w(1-r), g; A) + \tfrac{B(r,g,x;b)}{n} + \delta E v(x, A') \\ &s.t. \ \ B(r, g, x; b) \geq 0 \text{ and } x \in [\underline{x}, \overline{x}]. \end{aligned} \tag{7}$$

The problem in this form is fairly standard. The planner's value function must satisfy the functional equation

$$v(b, A) = \max_{(r,g,x)} \{u(w(1-r), g; A) + \frac{B(r, g, x; b)}{n} + \delta E v(x, A') : B(r, g, x; b) \geq 0 \ \ \& \ x \in [\underline{x}, \overline{x}]\}. \tag{8}$$

Familiar arguments can be applied to show that such a value function exists and that $Ev(\cdot, A)$ is differentiable and strictly concave. From this, the properties of the optimal policies may be deduced.

3.1 The optimal policies

Using equations (3) and (4) and letting λ denote the multiplier on the budget constraint, we can write the first order conditions for the maximization problem in (8) as follows:

$$1 + \lambda = \frac{1-r}{1-r(1+\varepsilon)}, \tag{9}$$

$$n\alpha A g^{\alpha-1} = [\frac{1-r}{1-r(1+\varepsilon)}]p, \tag{10}$$

and

$$\frac{1-r}{1-r(1+\varepsilon)} \geq -\delta n E[\frac{\partial v(x, A')}{\partial x}] \ \ (= \text{ if } x < \overline{x}). \tag{11}$$

To interpret these, note that $(1-r)/(1-r(1+\varepsilon))$ measures the marginal cost of taxation - the social cost of raising an additional unit of revenue via a tax increase. It exceeds unity whenever the tax rate (r) is positive, because taxation is distortionary. For a given tax rate, the marginal cost of taxation is higher the more elastic is labor supply; that is, the higher is ε. Condition (9) therefore says that the benefit of raising an additional unit of revenue - which is measured by $1+\lambda$ - must equal the marginal cost of taxation. Condition (10) says that the marginal social benefit of the public good must equal its price times the marginal cost of taxation. This is basically the

Samuelson Rule modified to take into account the fact that taxation is distortionary. Condition (11) says that the benefit of increasing debt in terms of reducing taxes must equal the marginal cost of an increase in the debt level. This cost is that there is a higher initial level of debt next period. The condition can hold as an inequality, if the debt level is at its ceiling.

In any particular state (b, A), there are two possibilities. The first is that the planner is making transfers to the citizens in which case $\lambda = 0$. In this case, conditions (9) and (10) imply that the tax rate r must be zero and the level of the public good g must equal the Samuelson level $g_S(A)$. Intuitively, if r were positive, the planner would find it strictly optimal to simultaneously reduce transfers and the tax rate: this would reduce the deadweight loss of taxation and increase citizen welfare. Similarly, if the public good level were less than the Samuelson level, then the planner could reduce transfers and increase public good provision. The debt level in this case, which we denote by x^o, must satisfy the requirement that the expected marginal cost of borrowing equals 1. We will investigate what this implies below.

The second possibility is that the planner is making no transfers. In this case, the optimal tax rate-public good-public debt triple is implicitly defined by equations (10), (11) and the requirement that the net of transfer surplus is zero; i.e.,

$$B(r, g, x; b) = 0. \tag{12}$$

A positive value of λ implies that the tax rate r must exceed zero and the level of the public good g is less than the Samuelson level $g_S(A)$. Moreover, the level of debt exceeds x^o. The tax rate and debt level are increasing in b and A, while the public good level is decreasing in b and increasing in A.[8] Intuitively, an increase in b makes the budget harder to satisfy forcing the planner to raise more revenues and skimp on the public good. An increase in A makes the public good more valuable and leads the planner to raise taxes and debt to finance more public spending.

In which states will the two possibilities arise? Let $A^o(b, x^o)$ be the largest value of A consistent with the triple $(0, g_S(A), x^o)$ satisfying the constraint that $B(0, g_S(A), x^o; b) \geq 0$.[9] Then, if the state (b, A) is such that $A < A^o(b, x^o)$, the optimal policy involves transfers, while if $A \geq A^o(b, x^o)$ it does not.

[8] These facts are established in the appendix.

[9] If $B(0, 0, x^o; b) < 0$, let $A^o(b, x^o) = 0$.

3.2 The debt level x^o

The next step is to characterize the debt level x^o the planner chooses when he makes transfers. Intuitively, if the planner is willing to rebate scarce revenues back to citizens then he must expect not to be imposing taxes in the next period otherwise he would be better off reducing transfers and acquiring more bonds. This suggests that the debt level x^o must be such that future taxes are equal to zero, implying that x^o equals $\underline{x}$. This is indeed the case but it is instructive to derive it formally.

Recall that x^o is such that the expected marginal cost of borrowing equals 1. Given the above discussion, we can write the value function as

$$v(x,A) = \begin{cases} \max_{\{r,g,z\}} \left\{ \begin{array}{c} u(w(1-r),g;A) + \frac{B(r,g,z;x)}{n} + \delta Ev(z,A) \\ B(r,g,z;x) \geq 0 \ \ \& \ \ z \in [\underline{x},\overline{x}] \end{array} \right\} & \text{if } A \geq A^o(x,x^o) \\ u(w,g_S(A);A) + \frac{B(0,g_S(A),x^o;x)}{n} + \delta Ev(x^o,A') & \text{if } A < A^o(x,x^o) \end{cases} . \tag{13}$$

Then, by the *Envelope Theorem*:

$$\frac{\partial v(x,A)}{\partial x} = \begin{cases} -(\frac{1-r^o(x,A)}{1-r^o(x,A)(1+\varepsilon)})(\frac{1+\rho}{n}) & \text{if } \ A \geq A^o(x,x^o) \\ -(\frac{1+\rho}{n}) & \text{if } \ A < A^o(x,x^o) \end{cases} , \tag{14}$$

where $r^o(x,A)$ is the optimal tax rate. Notice that this derivative is continuous at $A = A^o(x,x^o)$ since $r^o(x,A^o) = 0$. Taking expectations, we have that the expected marginal social cost of debt is

$$-\delta n E[\frac{\partial v(x,A)}{\partial x}] = G(A^o(x,x^o)) + \int_{A^o(x,x^o)}^{\overline{A}} (\frac{1-r^o(x,A)}{1-r^o(x,A)(1+\varepsilon)})dG(A). \tag{15}$$

Thus, the debt level x^o must satisfy the following equation

$$1 = G(A^o(x^o,x^o)) + \int_{A^o(x^o,x^o)}^{\overline{A}} (\frac{1-r^o(x^o,A)}{1-r^o(x^o,A)(1+\varepsilon)})dG(A). \tag{16}$$

This implies that $A^o(x^o,x^o) = \overline{A}$, which in turn means that $x^o = \underline{x}$.

3.3 Dynamics

The optimal policies determine a distribution of public debt levels in each period. In the long run, this sequence of debt distributions converges to the distribution that puts point mass on the debt level $\underline{x}$. To understand this, first note that since $A^o(\underline{x},\underline{x}) = \overline{A}$, it is clear that once the planner has accumulated a level of bonds equal to $-\underline{x}$, he will maintain it. On the other hand, when the planner has bond holdings less than $-\underline{x}$ then he must anticipate using distortionary taxation in the future. To smooth taxes he has an incentive to acquire additional bonds when the value of the public good is low in the current period. This leads to an upward drift in goverment bond holdings over time.

Pulling all this together, we have the following proposition.

Proposition 1. *The social planner's solution converges to a steady state in which the debt level is* $\underline{x}$*, the tax rate is* 0*, the public good level is* $g_S(A)$*, and citizens receive* $\rho(-\underline{x})-pg_S(A)$ *in transfers.*

This result illustrates in the context of our model the problem with the tax smoothing approach identified by Aiyagari et al. Though the planner can not issue state contingent bonds, he can smooth taxation across states by accumulating assets. As shown by Aiyagari et al. by numerical methods, this phenomenon is general and can characterize the planner's solution under less restrictive assumptions on the functional forms of the citizens' utilities and the stochastic process of government spending.

One way to avoid the absorbing state in which $x = \underline{x}$ is to assume that the social planner faces what Aiyagari et al. call "ad hoc" constraints on asset accumulation. If the planner is not allowed to accumulate more bonds than, say, $-z$ where $z \in (\underline{x}, 0)$, then even in the long run the optimal debt level will fluctuate and taxes will be positive at least some of the time.[10] This is because, by definition of $\underline{x}$, even when the planner has accumulated $-z$ in bonds he can not finance the Samuelson level of public goods from the interest earnings when A is very high. In these high realizations, it will be optimal to finance additional public good provision by a combination of levying taxes and reducing bond holdings. Reducing bond holdings temporarily allows the planner to smooth taxes. The dynamic pattern of debt suggested by Barro is created by the rebuilding of bond holdings in future periods when A is low. However, the difficulty with this resolution of the problem is obvious: why should the planner be so constrained and, if he is, what should determine the level z?

[10] In order for taxes to be *always* positive it must be the case that $\rho(-z) < pg_S(\underline{A})$.

4 The political equilibrium

We look for a symmetric Markov-perfect equilibrium in which any representative selected to propose at round $\tau \in \{1,...,T\}$ of the meeting at some time t makes the same proposal and this depends only on the current level of public debt (b) and the value of the public good (A).[11] As standard in the theory of legislative voting, we assume that legislators vote for a proposal if they prefer it (weakly) to continuing on to the next proposal round.[12] We focus, without loss of generality, on equilibria in which at each round τ, proposals are immediately accepted by at least q legislators, so that on the equilibrium path, no meeting lasts more than one proposal round. Accordingly, the policies that are actually implemented in equilibrium are those proposed in the first round.

Let $\{r(b, A),\ g(b, A),\ x(b, A)\}$ denote the tax rate, public good and public debt policies that are implemented in equilibrium and let $B(b, A)$ be the total amount of revenues devoted to transfers (i.e., $B(b, A) = B(r(b, A), g(b, A), x(b, A); b)$). In addition, let $v(b, A)$ denote the legislators' common (net of initial bond holdings) value function. Reflecting the fact that legislators are ex ante equally likely to receive transfers, this is defined recursively by:

$$v(b, A) = u(w(1 - r(b, A)), g(b, A); b) + \frac{B(b, A)}{n} + \delta E v(x(b, A), A'). \tag{17}$$

This is also the (net of initial bond holdings) value function for each citizen, since as noted earlier, representatives have the same policy preferences as their constituents.[13]

We restrict attention to a particular type of equilibrium, which we refer to as a "well-behaved

[11] A Markov-perfect equilibrium is a particular type of subgame perfect equilibrium in which strategies do not depend on payoff irrelevant past events. By focusing on Markov-perfect equilibria we rule out, for example, equilibria in which proposers punish earlier proposers for not providing their districts with transfers. Markov games (such as the game studied here) generally have a large set of subgame perfect equilibria and the Markov-perfect requirement allows us to dramatically shrink this set. Non Markov equilibria are often supported by complex strategies, or by strategies that (even when they are simple) require unrealistic degrees of coordination from the players. Markov equilibria do not require coordination and are very simple. The idea of simplicity has been formalized by Baron and Kalai (1992) for the static Baron and Ferejohn game. Given a standard definition of simplicity (Kalai and Stanford (1988)), they have shown that the unique simplest equilibrium of this game is stationary (i.e., Markov). Stationarity is also supported in a recent laboratory experiment. Frechette, Kagel and Morelli (2005) have shown that there is no evidence of of non stationary behavior in the data of their experimental study of the Baron and Ferejohn game.

[12] As in all voting games, it is possible to construct equilibria in which legislators vote against a proposal even if they strictly prefer it to continuing on to the next proposal round. If all voters always vote no to a proposal and there are three or more voters, then no voter will be pivotal and voting no will be weakly optimal no matter what preferences are. These equilibria are implausible and uninteresting. This assumption on legislators' voting behavior rules them out.

[13] The expected lifetime payoff of a citizen with bond holdings y at the beginning of a period in which the state is (b, A) will be $y(1 + \rho) + v(b, A)$.

equilibrium". To define what this is, call the interval of debt levels $[\inf_{(b,A)} x(b,A), \overline{x}]$ the *policy domain*. Levels of debt outside this range will never be observed except when exogenously assumed at date zero. An equilibrium is said to be *well-behaved* if the associated legislators' value function satisfies the following three properties: (i) v is continuous on the state space, (ii) for all A, $v(\cdot, A)$ is concave on $[\underline{x}, \overline{x}]$ and $Ev(\cdot, A)$ is strictly concave on the policy domain, and (iii) for all b, $v(\cdot, A)$ is differentiable at b for almost all A. In the Appendix, we demonstrate:

Proposition 2. *There exists a unique well-behaved equilibrium.*

This is the equilibrium that we characterize in the sequel.

4.1 The equilibrium policies

The basic structure of the equilibrium policies is easily understood. To get support for his proposal, the proposer must obtain the votes of $q - 1$ other representatives. Accordingly, given that utility is transferable, he is effectively making decisions to maximize the utility of q legislators.[14] It is therefore *as if* a randomly chosen coalition of q representatives is selected in each period and this coalition chooses a policy choice to maximize its aggregate utility.

The proposer's policy will depend upon the state (b, A). As in the social planner's solution, there are two possibilities: either the proposer will propose transfers for his coalition or he will not. Because the proposer is only taking into account the welfare of q legislators and transfers are financed collectively, his incentive to choose transfers is obviously greater than the planner's. Nonetheless, transfers require reducing public good spending or increasing taxation in the present or the future (if financed by issuing additional debt). When b and/or A are sufficiently high, the marginal benefit of spending on the public good and the marginal cost of increasing taxation may be too high to make this attractive. In this case, the proposer will not propose transfers and the outcome will be *as if* the proposer is maximizing the utility of the legislature as a whole.

In equilibrium, therefore, there will exist a cut-off value of the public good, inversely related to the level of public debt, that divides the state space into two ranges. Above the cut-off, the proposer will propose a no-transfer policy package that maximizes aggregate legislator utility. This proposal will be supported by the entire legislature. Below the cut-off, the proposer chooses a policy package that provides pork for his own district and those of a minimum winning coalition

[14] This is demonstrated formally in the appendix.

of representatives. The transfer paid out to coalition members will be just sufficient to make them favor accepting the proposal. Thus, only those legislators whose districts receive pork vote for the proposal. We will refer to the first regime as *responsible-policy-making* (RPM) and the second as *business-as-usual* (BAU).

To develop this more precisely, consider the problem of choosing the tax rate-public good-public debt triple that maximizes the collective utility of q representatives under the assumption that they divide the net of transfer surplus among their districts and that the constraint that this surplus be non-negative is non-binding. Formally, the problem is:

$$\begin{array}{c} \max_{(r,g,x)} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{q} + \delta E v(x, A') \\ s.t. \quad x \in [\underline{x}, \overline{x}]\,. \end{array} \tag{18}$$

Using the first-order conditions for this problem, the solution is $(r^*, g^*(A), x^*)$ where the tax rate r^* satisfies the condition that

$$\frac{1}{q} = \frac{[\frac{1-r^*}{1-r^*(1+\varepsilon)}]}{n}, \tag{19}$$

the public good level $g^*(A)$ satisfies the condition that

$$\alpha A g^*(A)^{\alpha-1} = \frac{p}{q}, \tag{20}$$

and the public debt level x^* satisfies

$$\frac{1}{q} \geq -\delta E[\frac{\partial v(x^*, A')}{\partial x}] \;\; (= \text{ if } x^* < \overline{x}). \tag{21}$$

Condition (19) says that the benefit of raising taxes in terms of increasing the per-legislator transfer $(1/q)$ must equal the per-capita cost of the increase in the tax rate. Condition (20) says that the per-capita benefit of increasing the public good must equal the per-legislator reduction in transfers that providing the additional unit necessitates. Condition (21) tells us that the benefit of increasing debt in terms of increasing the per-legislator transfer must equal the per-capita cost of an increase in the debt level.

Now define $A^*(b, x)$ to be the largest value of A consistent with the triple $(r^*, g^*(A), x)$ satisfying the constraint that $B(r^*, g^*(A), x; b) \geq 0$.[15] Then, if the state (b, A) is such that $A < A^*(b, x^*)$, the proposer proposes the triple $(r^*, g^*(A), x^*)$ together with a transfer just sufficient to induce members of the coalition to accept the proposal and the legislature is in the BAU

[15] If $B(r^*, 0, x; b) < 0$, let $A^*(b, x) = 0$.

regime. If $A > A^*(b, x^*)$, then the constraint that $B(r, g, x; b) \geq 0$ must bind and the solution equals that which maximizes aggregate legislator utility. The legislature is therefore in the RPM regime. Thus, we have:

Lemma 1. *There exists some debt level x^* such that if $A \geq A^*(b, x^*)$*

$$(r(b,A), g(b,A), x(b,A)) = \arg\max \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta E v(x, A') \\ B(r,g,x;b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}] \end{array} \right\}$$

and $B(b, A) = 0$, while if $A < A^*(b, x^*)$

$$(r(b,A), g(b,A), x(b,A)) = (r^*, g^*(A), x^*)$$

and $B(b, A) > 0$.

In the RPM regime (i.e., when $A \geq A^*(b, x^*)$), just as in the social planner's solution, the equilibrium tax rate-public good-public debt triple is implicitly defined by conditions (10), (11) and (12) (obviously, with the equilibrium value function). Thus, as in the planner's problem, the tax rate and debt level are increasing in b and A, while the public good level is decreasing in b and increasing in A.

Note that at $A = A^*(b, x^*)$ the triple that maximizes aggregate legislator utility equals $(r^*, g^*(A), x^*)$. To see this, note first that $(r^*, g^*(A), x^*)$ satisfies the budget balance condition (12) at $A = A^*(b, x^*)$. In addition, using the definition of r^* in (19) we may write the first order conditions (20) and (21) in the same form as (10) and (11). Thus, the equilibrium policy proposal is a continuous function of the state (b, A). Moreover, given the monotonicity properties of the solution in the RPM regime, it follows that when $A > A^*(b, x^*)$, the equilibrium policy proposal involves a tax rate higher than r^*, the provision of a public good level below $g^*(A)$, and a level of debt that exceeds x^*. Thus, in the political equilibrium, the government's debt level is always at least x^*, the tax rate is always at least r^*, and the public good level is always no greater than $g^*(A)$.

4.2 The debt level x^*

The next step is to characterize the debt level x^* that the proposer chooses when providing pork to his coalition. We use a similar strategy to that used to characterize x^o in the planner's problem.

From Lemma 1 we know that, in equilibrium,

$$v(x,A)=\begin{cases}\max_{\{r,g,z\}}\left\{\begin{array}{c}u(w(1-r),g;A)+\frac{B(r,g,z;x)}{n}+\delta Ev(z,A)\\ B(r,g,z;x)\geq 0\ \ \&\ \ z\in[\underline{x},\overline{x}]\end{array}\right\} & \text{if } A\geq A^*(x,x^*)\\ u(w(1-r^*),g^*(A);A)+\frac{B(r^*,g^*(A),x^*;x)}{n}+\delta Ev(x^*,A') & \text{if } A<A^*(x,x^*)\end{cases}. \tag{22}$$

Thus, by the *Envelope Theorem*:

$$\frac{\partial v(x,A)}{\partial x}=\begin{cases}-(\frac{1-r(x,A)}{1-r(x,A)(1+\varepsilon)})(\frac{1+\rho}{n}) & \text{if } \ A\geq A^*(x,x^*)\\ -(\frac{1+\rho}{n}) & \text{if } \ A<A^*(x,x^*)\end{cases}. \tag{23}$$

In contrast to the planner's solution, there is a discontinuity in the derivative of the value function when $A=A^*(x,x^*)$. This reflects the fact that the tax rate $r(x,A^*)$ equals r^* and hence the marginal cost of taxation strictly exceeds 1. Intuitively, a higher future level of debt reduces pork if the legislature is in the BAU regime and increases taxes if the legislature is in the RPM regime. Increasing taxes is more costly than reducing pork because taxes are positive in RPM and thus the marginal cost of public funds exceeds 1.

Using this expression to compute the expected marginal cost of borrowing and using our first order condition (21), we find that x^* must satisfy

$$\frac{n}{q}\geq G(A^*(x^*,x^*))+\int_{A^*(x^*,x^*)}^{\overline{A}}(\frac{1-r(x^*,A)}{1-r(x^*,A)(1+\varepsilon)})dG(A)\ \ (=\ \text{if } x^*<\overline{x}). \tag{24}$$

Our assumption concerning the maximum debt level $\overline{x}$ implies that $A^*(\overline{x},\overline{x})<\underline{A}$. Thus, since taxes exceed r^* in the RPM regime, the expected marginal social cost of debt must exceed n/q when $x^*=\overline{x}$. It follows that x^* is strictly less than $\overline{x}$ and condition (24) must hold as an equality.

Notice that for condition (24) to be satisfied, $A^*(x^*,x^*)$ must lie strictly between $\underline{A}$ and $\overline{A}$. Intuitively, this means that the debt level x^* must be such that the legislature will transition out of BAU with positive probability and stay in it with positive probability. This has important implications for the magnitude of x^* which we will draw out below.

4.3 Dynamics

The equilibrium policies determine a distribution of public debt levels in each period. In the Appendix, we show that this sequence of debt distributions converges to a unique invariant distribution. Thus, no matter what the economy's initial debt level, the same distribution of debt emerges in the long run. The lower bound of the support of this distribution is x^* - the level of public debt chosen in the BAU regime. There is a mass point at this debt level, since the probability of remaining at x^* having reached it is $G(A^*(x^*, x^*))$ - which is positive. However, the distribution of debt is non-degenerate because, as just noted, x^* must be such that there is a positive probability of leaving the BAU regime.

Combining this with our earlier discussion, yields the following proposition:

Proposition 3. *The equilibrium debt distribution converges to a unique invariant distribution whose support is a subset of* $[x^*, \overline{x}]$. *This distribution has a mass point at* x^* *but is non-degenerate. When the debt level is* x^*, *the tax rate is* r^*, *the public good level is* $g^*(A)$, *and a minimum winning coalition of districts receive transfers. When the debt level exceeds* x^*, *the tax rate exceeds* r^*, *the public good level is less than* $g^*(A)$, *and no districts receive transfers.*

The dynamics of the equilibrium are such that, in the long-run, legislative policy-making oscillates between BAU and RPM. Periods of BAU are brought to an end by high realizations of the value of public goods. These trigger an increase in debt and taxes to finance higher public good spending and a cessation of pork-barrel spending. Once in the RPM regime, further high realizations of the value of the public good trigger further increases in debt and higher taxes. Policy-making returns to BAU only after a suitable sequence of low realizations of the value of the public good. The larger the amount of debt that has been built up, the greater the expected time before returning to BAU.

To get a graphical feel for the long run dynamics of the system, let A_L be less than $A^*(x^*, x^*)$ and A_H be larger than $A^*(x^*, x^*)$. Suppose that the legislature is in BAU in period $t-1$ so that the level of debt is x^* at the beginning of period t. Further suppose that in periods t through t_L the value of the public good is A_L; in periods $t_L + 1$ through t_H the value of the public good is A_H; and in periods $t_H + 1$ and beyond the value of the public good returns to A_L. Then, the dynamic pattern of debt, tax rates and public good provision is as represented in Figure 1. At date $t_L + 1$ debt, taxes and public good levels jump up in response to the increase in A. During periods $t_L + 1$ through t_H, debt and taxes continue to rise, while public good provision falls. In period $t_H + 1$, public good provision drops in response to the fall in A, overshooting its natural level $g^*(A_L)$. After period $t_H + 1$, debt and taxes start to fall and public good provision increases. Eventually, the legislature returns to BAU.

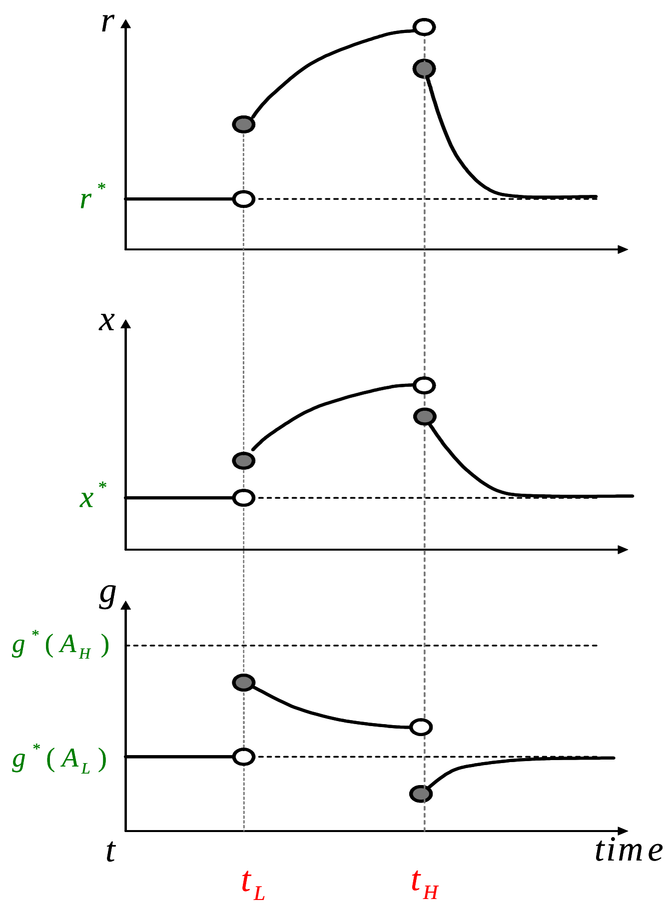

Figure 1: The dynamics of the political equilibrium.

If the debt level x^* is positive, the economy is in perpetual debt, with the extent of debt spiking up after a sequence of high values of the public good. When x^* is negative, the government will have positive asset holdings at least some of the time. A key question is therefore what determines the magnitude of x^*. As noted earlier, x^* must be such that the legislature will transition out of BAU with positive probability and stay in it with positive probability. We now use this observation to shed light on the determinants of x^*.

Given the definition of the function A^*, the critical value $A^*(x^*, x^*)$ satisfies the equation $R(r^*) - \rho x^* = pg^*(A)$. Thus, since $A^*(x^*, x^*)$ must lie between $\underline{A}$ and $\overline{A}$, if $R(r^*)$ exceeds $pg^*(\overline{A})$ then x^* must be positive. Intuitively, if x^* were equal to zero, the legislature would be in BAU with probability one so interest payments must be positive to soak up some of the surplus revenues.

On the other hand, if $R(r^*)$ is less than $pg^*(\underline{A})$ then x^* must be negative. If x^* were equal to zero, the legislature would be in RPM with probability one so interest earnings must be positive to supplement scarce tax revenues. The key determinant of the magnitude of x^* is therefore the size of the tax base as measured by $R(r^*)$ relative to the economy's desired public good spending as measured by $pg^*(A)$. The greater the relative size of the tax base, the larger is the debt level chosen in the BAU regime. Paradoxically, therefore, it is economies with relatively larger tax bases that are more likely to be in perpetual debt.

5 Political distortions

A central mission of the contemporary literature on political economy is to understand how political decision-making distorts policy choices. Propositions 1 and 3 tell us precisely how the equilibrium policy sequence differs from the planner's policy sequence. Specifically, in the long run, the level of debt held by the government is too high relative to the optimal level, tax rates are too high, and public good levels are too low. Moreover, tax rates are too volatile. Since in both the planner's solution and the equilibrium all citizens receive the same expected utility (modulo any differences in initial bond holdings), these distortions mean that the political equilibrium is Pareto dominated by the planner's solution. Thus, the equilibrium exhibits "political failure" in the sense defined by Besley and Coate (1998).

This inefficiency not withstanding, in the RPM regime, legislators behave exactly as the social planner would want them given the constraint that future policy choices are politically determined. This suggests the intriguing idea that the equilibrium policies might solve an appropriately constrained planning problem. The following proposition makes this idea precise.

Proposition 4. *The equilibrium value function $v(b, A)$ solves the functional equation*

$$v(b, A) = \max_{(r,g,x)}\{u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta E v(x, A') :$$
$$B(r, g, x; b) \geq 0,\ r \geq r^*,\ g \leq g^*(A),\ \&\ x \in [x^*, \overline{x}]\},$$

and the equilibrium policies $\{r(b, A),\ g(b, A),\ x(b, A)\}$ are the optimal policy functions for this program.

Comparing the program described in the proposition with that in (8), we see that political determination imposes three constraints on the planning problem. First, the tax rate cannot be below r^*. Were it to be so, the proposer could benefit his coalition by raising the income tax rate and dividing the proceeds among coalition members. Second, the public good level cannot be above $g^*(A)$. If it were, the proposer could benefit his coalition by reducing public good spending and dividing the proceeds among coalition members. Third, the debt level cannot be below x^*. If it were, the proposer could benefit his coalition by borrowing more and dividing the proceeds among coalition members.

The proposition is interesting because it tells us that political determination can be interpreted as imposing additional constraints on the planner's problem, rather than fundamentally changing the nature of fiscal policy. The result affirms that the principles of tax smoothing operate even when the assumption of a benevolent government is relaxed. Indeed, since the constraints serve to break the unpalatable implication of zero taxation in the long run, adding political determination increases the relevance of the basic tax smoothing idea. The result also helps develop intuition on the nature of the political equilibrium: the planner's problem in the tax smoothing context is already reasonably well understood and it is relatively straightforward to think through the consequences of the additional constraints.

The proposition makes clear that political economy considerations give rise to distortions in taxes and public good spending as well as in debt. Importantly, the equilibrium policy choices are not the same as those obtained from the tax smoothing problem with a constraint that the government's debt level must exceed x^* (which is the problem studied by Aiyagari et al). Constraints on the tax rate and public good level must also be imposed for the solution to be declared a positive prediction. The distortions in the tax rate and the public good level are *static distortions* in the sense that within any period in which the constraints are binding aggregate citizen welfare would be higher if the tax rate were reduced and the public good level increased. The distortion in the debt level is a *dynamic distortion* in the sense that the future benefits to citizens from lower debt offset the costs of lower revenues in the present.

There are two basic reasons for the distortions arising from political decision-making. First, the fact that q is less than n means that the decisive coalition does not fully internalize the costs of raising taxes or reducing public good spending. If the legislature operated by unanimity rule (i.e., $q = n$) then legislative decision-making would reproduce the planner's solution. This follows immediately from Proposition 4 once it is noted that with $q = n$, the tax rate r^* is 0, the public good level $g^*(A)$ is just the Samuelson level $g_S(A)$, and the debt level x^* is $\underline{x}$. More generally, moving from majority to super-majority rule will improve welfare since raising q reduces r^* and x^* and raises $g^*(A)$, thereby relaxing the constraints on the planning problem.

Second, the random allocation of proposal power in the legislature creates uncertainty about the identity of the minimum winning coalition. If the proposer is making transfers to his coalition and anticipates that the legislature will be in BAU the next period, this uncertainty means that he will always want to issue more debt. Issuing an additional dollar of debt would gain $1/q$ units for each legislator in the minimum winning coalition and would lead to a one unit reduction in pork in the next period. This has an expected cost of only $1/n$ because members of the current minimum winning coalition are not sure they will be included in the next period. The critical role of uncertainty can be appreciated by noting that if the identity of the minimum winning coalition were constant through time the resulting equilibrium policy sequence would be Pareto efficient.[16]

6 Empirical implications of the theory

Our theory has two types of empirical implications. First, for a given economy, it has implications for the pattern of debt, taxes, public spending and legislative voting behavior.[17] Second, comparing across economies, it provides predictions on how the distributions of debt, taxes and public spending should vary with the underlying fundamentals.

For a given economy, the most obvious implication of the theory concerns the impact of an increase in the value of public goods as a result, say, of a war or natural disaster. Recall that the equilibrium public good level is strictly increasing in A, while the equilibrium tax and debt levels are constant in A in the BAU regime and and increasing in the RPM regime. Moreover, a sufficiently high realization of A must shift the equilibrium from BAU to RPM. Thus, the theory predicts that we should observe increases in debt, taxes and public good spending following a significant increase in the value of public goods. This prediction is consistent with the fact that historically the debt/GDP ratio in the U.S. and the U.K. tends to have increased in periods of high government spending needs and decreased in periods of low needs (Barro (1979), (1986), and

[16] It is relatively straightforward to find the equilibrium in this "oligarchic" case in which a constant coalition of q representatives choose policy. The economy would converge to a deterministic steady state in which the tax rate is r^*, the public good level is $g^*(A)$, and the debt level is such that $\rho(-x) + R(r^*) \geq pg^*(\overline{A})$. Excess revenues would be shared by the q representatives.

[17] In light of Proposition 4, these implications are obviously going to be similar to those of a tax smoothing model with an "ad hoc" limit on bond accumulation.

(1987)). The theory also suggests that we should see a reduction in pork-barrel spending and an increase in the size of majorities passing budget bills. While there are issues here concerning the empirical measurement of pork, it would be well worth trying to investigate these auxiliary predictions.

The theory also has implications for how the equilibrium policies should depend on the current stock of debt. Recall that the equilibrium public good level is constant in b in the BAU regime and decreasing in the RPM regime, while the equilibrium tax and debt levels are constant in b in BAU and increasing in RPM. Moreover, the economy is more likely to be in RPM the higher is b. Combining these observations, it is possible to derive some interesting implications for the relationship between what is known as the "primary surplus" and the level of debt. The primary surplus is the difference between tax revenues and public spending other than interest payments. In our model, it is the difference between tax revenues and spending on the public good and pork. Using the budget constraint, we may write this as $PS(b, A) = (1 + \rho)b - x(b, A)$. To understand what happens to the primary surplus when debt increases, consider a small increase Δb in b while holding A constant. If the legislature is in BAU, then because $x(b, A)$ is constant, $\Delta PS(b, A)/\Delta b = (1 + \rho)$. On the other hand, if the legislature is in RPM, then $\Delta PS(b, A)/\Delta b = (1 + \rho) - \Delta x(b, A)/\Delta b$ which is positive but less than $1 + \rho$ since $x(b, A)$ is increasing in b but at a rate smaller than $1 + \rho$.[18] In both cases, therefore, the relationship between the primary surplus and debt is positive, but the effect is smaller in the RPM regime. The first implication is consistent with the work of Bohn (1998) who finds that for the U.S. federal government the relationship between the primary surplus and debt is positive. However, since the economy is more likely to be in the RPM regime when b is high, the second implication is inconsistent with Bohn's finding that the relationship is convex. It seems plausible that $x(b, A)$ might be concave in the RPM regime which would yield Bohn's finding for sufficiently high levels of debt but, unfortunately, this is not something that can be established analytically.

With respect to the impact of the current debt level, the theory also has an interesting implication for winning margins in the legislature. The expected size of the coalition voting in favor of the winning proposal with debt level b is $G(A^*(b, x^*))q + (1 - G(A^*(b, x^*))n$, which is increasing in b. Thus, the winning margin on budget bills should be increasing in the current debt level. This

[18] This follows from the facts that in the RPM regime $x(b, A) + R(r(b, A))$ equals $(1 + \rho)b + pg(b, A)$, $r(b, A)$ is increasing in b, and $g(b, A)$ is decreasing in b.

is a novel prediction which is well worth investigating.

Comparing across economies, the model has three groups of underlying parameters: *preference parameters* which include the labor supply elasticity ε, the discount rate δ, and the distribution of public good values $G(A)$; *technological parameters* which consist of labor productivity w and the price of public goods p; and *institutional parameters* which consist of the number of districts n and the size of the majorities required to pass legislation q. For any given set of parameters, the model predicts a unique long run distribution of debt and associated distributions of tax rates, public good spending, and voting coalitions. In principle, therefore, it is possible to explore how changes in each of these parameters impact these distributions. For example, one could ask how moving from a high to a low productivity economy would impact the distribution of debt. This type of comparative static exercise could be quite valuable as it would allow the development of predictions concerning cross country (or state) debt distributions which is something the tax smoothing model has trouble explaining (see Alesina and Perotti (1995) and Roubini and Sachs (1989)). Unfortunately, however, it is something that appears difficult to do analytically. Thus, it requires computing a calibrated version of the equilibrium which, while feasible, would take us well beyond the scope of this paper.[19]

Not withstanding the difficulties in characterizing the comparative statics, it is possible to make some general informed speculations based on what we know about the underlying logic of the model. The key equilibrium variable is x^* - the level of debt that is chosen in the BAU regime. As we have explained, x^* adjusts to ensure that in BAU the economy transitions to RPM with a probability which is positive but less than one. The equilibrium debt distribution has a mass point at x^* and it seems likely that if x^* increases, the debt distribution will shift rightward. We have identified the relative size of the tax base as being the key factor in the determination of x^*. Thus, parameters that raise $R(r^*)$ relative to $Epg^*(A)$ would seem likely to raise the average level of debt. This suggests that the average level of debt will be decreasing in ε, q/n, and rightward shifts in $G(A)$, and increasing in w.

While we are not aware of empirical evidence on the relationship between debt levels and our specific parameters, there is a literature investigating the relationship between fiscal policy and political variables in the OECD countries.[20] A central theme of this literature is that a

[19] We are currently working on computing the model with co-author Marina Azzimonti.

[20] Our theory is designed to apply to political systems (like the U.S.) in which political parties are relatively

"fragmented" policy-making process leads to present-biased fiscal outcomes. Influenced by the "common pool" view of fiscal policy (see section 8 below for discussion), Perotti and Kontopoulos (2002) define fragmentation "as the degree to which individual fiscal policy-makers internalize the costs of one dollar of aggregate expenditure". Using a number of empirical measures of this variable, they have shown that it is positively correlated with higher deficits.[21] In our model, as we argued in section 4.1, fiscal policy is effectively chosen collectively in each period by a coalition of q randomly chosen representatives. Thus, the sole "fiscal policy-maker" is the group of q representatives and this group internalizes q/n of the cost of any dollar of spending. Accordingly, a prediction that average debt levels are increasing in q/n would appear consonant with this literature.

7 An application of the theory

To illustrate the potential usefulness of the theory for policy analysis, we briefly explore its implications for the desirability of *balanced budget requirements*. There has been considerable debate in academic and policy circles concerning this issue.[22] Many of the U.S. states have some form of balanced budget requirement and there is evidence that they do have an effect.[23] Proponents argue that they dampen politicians' ability to borrow to spend inappropriately. Opponents point out that they restrict the state's ability to adjust to revenue and spending shocks without having to raise taxes. Both positions seem reasonable, but to provide sharper policy guidance it is necessary to understand the features of the environment that determine when the benefits outweigh the potential costs.[24]

We consider a fiscal restraint that requires the legislature to ensure that tax revenues equal

weak and legislators care a great deal about bringing resources back to their districts. Since the strength of parties and the importance of pork barrel spending in motivating legislators seems to vary significantly across the OECD countries, the U.S. states may be the most natural place to look to test the cross economy implications of the theory. That said, in the spirit of Alesina and Tabellini (1990), it is possible to interpret the n legislators as distinct political parties and the pork-barrel spending as transfers to party constituents. The coalition of q legislators choosing policy in each period could then be interpreted as that election cycle's governing coalition.

[21] See also Roubini and Sachs (1989) and Volkerink and de Haan (2001) for related findings.

[22] For relevant discussion see Bohn and Inman (1996), Brennan and Buchanan (1980), Niskanen (1992), Poterba (1994), (1995), Poterba and von Hagen (1999) and Primo (2007).

[23] For example, Poterba (1994) shows that states with restraints were quicker to reduce spending and/or raise taxes in response to negative revenue shocks than those without.

[24] There appears to be surprisingly little welfare analysis of fiscal restraints beyond the original work of Brennan and Buchanan (1980). Besley and Smart (2007) provide a general treatment of restraints in the context of a two period political agency model. Bassetto and Sargent (2006) study the welfare case for separating capital and ordinary government budgets and allowing the government to issue debt only to finance capital items.

public spending in every period. We assume that in the first period the government begins with no debt (i.e., $b_0 = 0$), so that spending is just on public goods and transfers. We seek to understand when citizens' welfare will be enhanced by the constraint that public spending be financed solely by tax revenues.

Let $(r_c(A), g_c(A))$ denote the equilibrium tax rate and public good level when the value of the public good is A under the balanced budget requirement. Then, following the logic of Lemma 1, we have that

$$(r_c(A), g_c(A)) = \begin{cases} \arg\max\{u(w(1-r), g; A) + \frac{B(r,g,0;0)}{n} : B(r, g, 0; 0) \geq 0\} & \text{if } A \geq A^*(0,0) \\ (r^*, g^*(A)) \quad \text{if } A < A^*(0,0) \end{cases} . \tag{25}$$

Thus, if $A < A^*(0,0)$, the legislature is in the BAU regime and districts receive pork, while if $A \geq A^*(0,0)$, the legislature is in the RPM regime. The solution is stationary because government cannot issue debt or acquire bonds. If $v_c(A)$ denotes citizen expected utility under the balanced budget requirement given that the current value of the public good is A, then

$$v_c(A) = u(w(1 - r_c(A)), g_c(A); A) + \frac{B(r_c(A), g_c(A), 0; 0)}{n} + \delta E v_c(A'). \tag{26}$$

Expected citizen welfare under the constraint is $Ev_c(A)$ and equation (26) implies that

$$Ev_c(A) = \int_{\underline{A}}^{\overline{A}} [u(w(1 - r_c(A)), g_c(A); A) + \frac{B(r_c(A), g_c(A), 0; 0)}{n}] dG(A)/(1 - \delta). \tag{27}$$

As in section 4, let $\{r(b, A),\ g(b, A),\ x(b, A)\}$ denote the tax rate, public good and public debt policies that are implemented in the unconstrained equilibrium and let $v(b, A)$ denote the legislator's value function. Starting from a situation in which the government has no debt, citizen expected utility in the unconstrained equilibrium is $Ev(0, A)$. Thus, a balanced budget requirement will be desirable if and only if $Ev_c(A) > Ev(0, A)$.

Our first result is that when the revenues raised by the tax rate r^* are *never* sufficient to cover the cost of the optimal level of public goods, a balanced budget requirement is not desirable.

Proposition 5. *If $R(r^*) \leq pg^*(\underline{A})$, a balanced budget requirement is not desirable.*

To see this, recall from section 4.3 that the condition of the proposition implies that x^* must be non-positive, so that in the BAU regime, the winning proposals involve the purchase of bonds. These bond holdings allow the legislature to lower taxes and provide higher levels of public goods in the long run. Moreover, the legislature only issues debt in the RPM regime which means that borrowing will be used only when it will raise aggregate utility. Such borrowing must therefore be socially beneficial.

An interesting feature of this case, is that under a balanced budget requirement, the legislature never engages in pork barrel spending. (This follows from the fact that the condition implies that $A^*(0,0) \leq \underline{A}$.) By contrast, in the unconstrained equilibrium, the legislature does provide pork in the BAU regime. Thus, the balanced budget requirement is undesirable despite eliminating transfers. This underscores the lesson that there is nothing necessarily undesirable about transfers - indeed, in the planner's solution the government redistributes excess revenues from its interest earnings back to the citizens in each period.

Our second result is the mirror image of the first: when the revenues raised by the tax rate r^* are *always* sufficient to cover the cost of the optimal level of public goods when the tax rate is r^*, a balanced budget requirement is desirable.

Proposition 6. *If $R(r^*) \geq pg^*(\overline{A})$, a balanced budget requirement is desirable.*

To see this, note that with a balanced budget restraint, the equilibrium will involve the tax rate r^* and the public good level $g^*(A)$ in every period. Without the restraint, the equilibrium will involve the legislature immediately borrowing x^* and using the revenues to finance extra pork. The amount x^* must be sufficiently large that in future periods there is positive probablity that the tax rate will exceed r^* and the public good level will be less than $g^*(A)$. There is no offsetting benefit, and hence eliminating the government's ability to borrow, increases citizen welfare.

If $R(r^*)$ is between $pg^*(\underline{A})$ and $pg^*(\overline{A})$ but x^* is nonpositive, then the argument underlying Proposition 4 remains and imposing a balanced budget requirement will be harmful. However, if x^* is positive the picture is murkier because there are offsetting effects from imposing the requirement. On the one hand, the government does not need to service the debt and hence long run taxes and public good levels must be lower on average with the requirement. On the other, the government's ability to smooth tax rates and public good levels by varying the debt level is lost.

Intuitively, it seems natural to suppose that the larger the size of the tax base as measured by $R(r^*)$ the more likely is a balanced budget requirement to be desirable. After all, the larger the tax base, the less the need to borrow to meet desired public good spending and the greater the debt level that will need to be financed when there is no restraint. This idea can be investigated formally by noting that the size of $R(r^*)$ is determined by the magnitude of the private sector wage w. From (4), we see that $R(r^*)$ equals $pg^*(A)$ if and only if $w = [pg^*(A)/nr^*\varepsilon^\varepsilon(1-r^*)^\varepsilon]^{\frac{1}{1+\varepsilon}}$. Thus, as we increase w between $[pg^*(\underline{A})/nr^*\varepsilon^\varepsilon(1-r^*)^\varepsilon]^{\frac{1}{1+\varepsilon}}$ and $[pg^*(\overline{A})/nr^*\varepsilon^\varepsilon(1-r^*)^\varepsilon]^{\frac{1}{1+\varepsilon}}$, we move the size of the tax base through the interval $(pg^*(\underline{A}), pg^*(\overline{A}))$. Our conjecture is that there must exist a critical wage w^* greater than $[pg^*(\underline{A})/nr^*\varepsilon^\varepsilon(1-r^*)^\varepsilon]^{\frac{1}{1+\varepsilon}}$ but less than $[pg^*(\overline{A})/nr^*\varepsilon^\varepsilon(1-r^*)^\varepsilon]^{\frac{1}{1+\varepsilon}}$ such that a balanced budget requirement is desirable if and only w exceeds w^*. Unfortunately, however, the argument that this is indeed the case has so far proven elusive.

To summarize: the theory suggests the key determinant of the desirability of a balanced budget requirement is the size of the tax base relative to the economy's desired public good spending. When this size is large, a balanced budget requirement is a good idea and when it is small, the opposite conclusion holds. The relative size of the tax base will be reflected in the magnitude of the debt level that is chosen in the BAU regime. Thus, the theory supports the common sense conclusion that economies with large and perpetual deficits should introduce balanced budget requirements.

8 Related political economy literature

This paper is related to two established branches of the political economy literature.[25] The first concerns the implications of pork-barrel spending for the efficiency of legislative policy-making. In a well-known paper, Weingast, Shepsle and Johnsen (1981) argue that pork-barrel spending will lead to a government that is too large. They do not model the process of passing legislation, assuming instead that legislative policy-making is governed by a "norm of universalism". Under this norm, each legislator unilaterally decides on the level of spending he would like on projects in his own district and the aggregate level of taxation is determined by the need to balance the budget. Policy-making then becomes a pure common pool problem.

More recently, the bargaining approach of Baron and Ferejohn (1989) has emerged as the standard way to model legislative decision making and a number of papers have employed it to study the efficiency implications of pork-barrel spending.[26] In a one period model, Baron

[25] For excellent reviews of this literature see Drazen (2000) and Persson and Tabellini (2000).

[26] The main criticism of the common pool perspective is that it does not model the voting process. For game theoretic approaches that are alternatives to Baron-Ferejohn, see Chari and Cole (1995) and Morelli (1999). For

(1991) shows that legislators may propose projects whose aggregate benefits are less than their costs, when these benefits can be targeted to particular districts.[27] This is because the decisive coalition does not fully internalize the costs of financing projects. In a similar static framework, Volden and Wiseman (2007) study the allocation of a fixed budget between public goods and pork barrel spending and show that public goods will be under-provided in some circumstances. In the context of a finite horizon dynamic model, LeBlanc, Snyder and Tripathi (2000) argue that legislatures will under-invest in public capital. In their model, in each period a legislature allocates a fixed amount of revenue between targeted transfers and a public investment that serves to increase the amount of revenue available in the next period.[28]

The second related branch of political economy literature is that discussing public debt. This literature offers two main explanations for why governments may run deficits even when there is no social role for so doing. Following Weingast, Shepsle and Johnsen (1981), the first explanation is based on viewing the accumulation of public debt as a dynamic common pool problem (see, for example, Inman (1990) and von Hagen and Harden (1995)). A formal model of this type is developed by Velasco (2000). While there is no economic role for debt in his model, he demonstrates the existence of an equilibrium in which deficits and debt accumulation continue unabated until the government's debt ceiling is reached. Chari and Cole (1993) present a related two period model, where legislators facing a common pool problem that drives spending too high try to restrain second period spending by issuing excessive levels of debt in the first period.

The second explanation is based on political instability. The basic idea is that when politicians choose current policy, they realize that with some probability they will not be choosing policy in the future. This may induce too much borrowing because the costs in terms of future spending cuts are not fully internalized. This idea was introduced independently by Alesina and Tabellini (1990) and Persson and Svensson (1989). Alesina and Tabellini consider a model in which in each period, two political parties hold office with exogenous probability. There are two goods that

experimental studies of the Baron and Ferejohn model, see Frechette, Kagel and Morelli (2005) who study the classic static divide-the-dollar version, and Battaglini and Palfrey (2007) who study a dynamic version.

[27] Related models are elaborated by Austen-Smith and Banks (2005) and Persson and Tabellini (2000).

[28] As noted in the introduction, our earlier paper (Battaglini and Coate (2007)) also explores how pork barrel spending distorts investment in public capital goods. Our model differs from LeBlanc et al in that it is an infinite horizon model in which the government can levy taxes as well as allocate public revenues and public investment yields benefits for more than one period. We find conditions under which the equilibrium size of government is too large and the level of public goods too low but also show that there are conditions under which legislative decisions are efficient and/or government is too small.

may be publicly-provided, but each party's constituency values only one. The government may finance public provision with debt and/or distortionary taxation. In each period, the winning party chooses taxes, debt and how much to spend on the good its constituency cares about. Alesina and Tabellini present conditions under which the steady state debt level is positive even though the optimal debt level is zero. Persson and Svensson consider a two period model featuring two political parties who differ in their preferences on the level of spending. Spending is financed by debt and distortionary taxes. They show that the party who prefers less spending may run a deficit in the first period to constrain the spending choices of the second period government.[29]

While both these arguments require policy-motivated political parties, Lizzeri (1999) shows that the same logic works even in a world with Downsian parties. He considers a two period economy in which elections are held at the beginning of each period. Parties can make binding promises before the elections as to how they will redistribute the available resources across voters and over time. Rational voters reward myopic behavior, however, favoring a party promising to distribute all resources today, because resources left for the future may be spent on others by the opposing party if the first period incumbent is not re-elected. Lizzeri refers to the force generating the present bias as "redistributive uncertainty".

It is evident that the arguments underlying the political distortions in our model are similar to those made in these two branches of the political economy literature. The static distortions in taxes and public goods parallel those that emerge in other legislative bargaining models with pork barrel spending and the logic of the dynamic distortion is similar to that underlying the various models of political instability. As noted in the introduction, what is novel about our analysis is that we incorporate these types of forces into a tax smoothing model in which there is a social role for debt.[30] Our analysis thus provides insights into how economic and political considerations interact to determine the dynamic pattern of fiscal policy. Rather than seeing political economy models as alternatives to the tax smoothing model, our work shows how the two approaches can be profitably integrated to reduce the shortcomings of each approach individually taken.

The objective of incorporating political decision making into a tax smoothing model is also

[29] An incumbent government may also want to choose debt issue strategically for another reason; namely, to influence its likelihood of re-election. This argument is developed by Aghion and Bolton (1990).

[30] Corsetti and Roubini (1997) introduce a motive for tax smoothing into a two period version of the Alesina and Tabellini (1990) model. They show in numerical simulations that the effect of this motive on the first period deficit is largely independent of the political incentives to run deficits as measured by the probability that the first period incumbent is re-elected.

shared by the recent work of Yared (2007). However, he analyzes the complete market version of the tax smoothing model due to Lucas and Stokey (1983) in which state contingent debt is available. He also adopts a more stylized model of the political process than us, assuming that citizens choose the income tax rate but that a self-interested, infinitely-lived ruler chooses borrowing and allocates the budget between public goods, his own consumption, and debt repayment. He looks at non Markov equilibria, in which citizens punish the ruler by levying no taxes if he "abuses" his executive power and the ruler punishes the citizens by underfunding public goods if they provide too little revenue. To solve the problem of multiple subgame perfect equilibria, he assumes that players can coordinate on equilibria on the Pareto frontier and the bulk of his analysis focuses on the equilibrium which maximizes the citizens' welfare - the "efficient sustainable equilibrium". In the spirit of our Proposition 4, he shows that it is possible to represent this equilibrium as the solution of a particular constrained planner's problem.[31] His main finding is that efficient sustainable taxes and debt respond more persistently to shocks than do taxes and debt in Lucas and Stokey's planning problem. Thus, political constraints may explain why an economy with complete markets may behave as observed in the empirical literature (and predicted by Barro's incomplete markets tax smoothing model).

9 Conclusion

This paper has presented a dynamic theory of public spending, taxation and debt. The theory brings together ideas from the optimal taxation and political economy literatures. From the former, the theory adopts the basic framework underlying the tax smoothing approach to fiscal policy. From the latter, the theory incorporates pork barrel spending and employs the legislative bargaining approach to modelling policy-making. The result is a tractable dynamic general equilibrium model that yields a rich set of predictions concerning the dynamics of fiscal policy and permits a rigorous analysis of the normative properties of equilibrium policies.

There are numerous ways the theory might usefully be extended. A particularly interesting extension would be to introduce cyclical fluctuations in tax revenues due to the business cycle.

[31] Yared's work draws on Acemoglu, Golosov and Tsyvinski (2006a) who characterize efficient sustainable equilibria in a dynamic economy with a self-interested ruler. However, their model is a dynamic Mirrlees model rather than a tax smoothing model. Like Yared, they demonstrate that it is possible to represent the efficient sustainable equilibrium as the solution to a planner's problem with constraints. Acemoglu, Golosov and Tysvinski (2006b) present a similar result for a simpler economic model with a richer political process in which political power fluctuates between distinct social groups.

This could be achieved by specifying a stochastic process (with persistence) for the private sector wage. Such a model would deliver predictions concerning the cyclical behavior of fiscal policy. While the tax smoothing paradigm suggests that deficits might be observed in recessions and surpluses in booms, observed fiscal policy is often procyclical (Alesina and Tabellini (2005)). It would be interesting to know what the type of theory developed here predicts.

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10 Appendix

10.1 Proof of Proposition 1

From the analysis in the text we know that if $A \leq A^o(b, \underline{x})$ the planner would select a tax rate of 0, a public good level of $g_S(A)$, a debt level of $\underline{x}$, and transfer $B(0, g_S(A), \underline{x}; b)$ to the citizens. Since $A^o(\underline{x}, \underline{x}) = \overline{A}$, it follows that once the planner has selected the debt level $\underline{x}$ the economy enters a deterministic steady state in which the debt level is $\underline{x}$, the tax rate is 0, the public good level is $g_S(A)$, and citizens receive $\rho(-\underline{x}) - pg_S(A)$ in transfers. Thus, it only remains to show that whatever the initial debt level, the planner will eventually select the debt level $\underline{x}$ with probability one. We will establish this following the proof of Proposition 4 below. ■

10.2 Definition of political equilibrium

As background for the analysis to follow, we need to provide a more precise definition of political equilibrium. An equilibrium is described by a collection of proposal functions: $\{r_\tau(b, A),\ g_\tau(b, A),\ x_\tau(b, A),\ s_\tau(b, A)\}_{\tau=1}^{T}$. Here $r_\tau(b, A)$ is the income tax rate that is proposed at round τ when the state is (b, A); $g_\tau(b, A)$ is the level of the public good; $x_\tau(b, A)$ is the new level of public debt, and $s_\tau(b, A)$ is a transfer the proposer offers to the districts of $q-1$ randomly selected representatives.[32]

Any remaining surplus revenues are used to finance a transfer for the proposer's own district. Following the notation used in Section 4, we will sometimes drop the subscript and refer to the first round policy proposal as $\{r(b, A),\ g(b, A),\ x(b, A),\ s(b, A)\}$.

The collection of proposal functions $\{r_\tau(b, A),\ g_\tau(b, A),\ x_\tau(b, A),\ s_\tau(b, A)\}_{\tau=1}^{T}$ is an *equilibrium* if at each proposal round τ and all states (b, A), the prescribed proposal maximizes the proposer's payoff subject to the incentive constraint of getting the required number of affirmative votes and the appropriate feasibility constraints. To state this formally, let $v_\tau(b, A)$ denote the legislators' value function at round τ which describes the expected future payoff of a legislator at the beginning of a period in which the state is (b, A). Again, following the notation of Section 4, we refer to the value function at round 1 as $v(b, A)$. Then, for each proposal round τ and all states (b, A), the

[32] It should be clear that there is no loss of generality in assuming that the proposer only offers transfers to $q-1$ representatives.

proposal $(r_\tau(b,A),\ g_\tau(b,A),\ x_\tau(b,A),\ s_\tau(b,A))$ must solve the problem

$$\max_{(r,g,x,s)} u(w(1-r),g;A) + B(r,g,x;b) - (q-1)\,s + \delta Ev(x,A')$$

$$s.t. \quad u(w(1-r),g;A) + s + \delta Ev(x,A') \geq v_{\tau+1}(b,A),$$

$$B(r,g,x;b) \geq (q-1)s, \quad s \geq 0 \;\&\; x \in [\underline{x},\overline{x}].$$

The first constraint is the incentive constraint and the remainder are feasibility constraints. The formulation reflects the assumption that on the equilibrium path, the proposal made in the first proposal round is accepted.

As noted in the text, the legislators' round one value function is defined recursively by (17). To understand this recall that a legislator is chosen to propose in round one with probability $1/n$. If chosen to propose, he obtains a payoff in that period of

$$u(w(1-r(b,A)),g(b,A);A) + B(r(b,A),g(b,A),x(b,A);b) - (q-1)s(b,A).$$

If he is not chosen to propose, but is included in the minimum winning coalition, he obtains $u(w(1-r(b,A)),g(b,A);A)+s(b,A)$ and if he is not included he obtains just $u(w(1-r(b,A)),g(b,A);A)$. The probability that he will be included in the minimum winning coalition, conditional on not being chosen to propose, is $(q-1)/(n-1)$. Taking expectations, the pork barrel transfers $s(b,A)$ cancel and the period payoff is as described in (17).

Once we have the round one value function, the other value functions can be readily derived. For all proposal rounds $\tau = 1,..,T-1$ the expected future payoff of a legislator if the round τ proposal is rejected is

$$v_{\tau+1}(b,A) = u\left(w(1-r_{\tau+1}(b,A)),g_{\tau+1}(b,A);A\right) + \tfrac{B(r_{\tau+1}(b,A),g_{\tau+1}(b,A),x_{\tau+1}(b,A);b)}{n}$$
$$+\delta Ev(x_{\tau+1}(b,A),A').$$

This reflects the assumption that the round $\tau+1$ proposal will be accepted. Recall that if the round T proposal is rejected, the assumption is that a legislator is appointed to choose a default tax rate, public goods level, level of debt and a uniform transfer. Thus,

$$v_{T+1}(b,A) = \max_{(r,g,x)} \left\{ u(w(1-r),g;A) + \frac{B(r,g,x;b)}{n} + \delta Ev(x,A') : B(r,g,x;b) \geq 0 \;\&\; x \in [\underline{x},\overline{x}] \right\}.$$

10.3 Proof of Lemma 1

We begin by establishing the claim made in the text that, given that utility is transferable, the proposer is effectively making decisions to maximize the collective utility of q legislators under the assumption that they get to divide any surplus revenues among their districts.

Lemma A.1: *Let* $\{r_\tau(b,A), g_\tau(b,A), x_\tau(b,A), s_\tau(b,A)\}_{\tau=1}^{T}$ *be an equilibrium with associated value function* $v(b,A)$. *Then, for all states* (b,A), *the tax rate-public good-public debt triple* $(r_\tau(b,A),$ $g_\tau(b,A), x_\tau(b,A))$ *proposed in any round* τ *solves the problem*

$$\max_{(r,g,x)} u(w(1-r),g;A) + \frac{B(r,g,x;b)}{q} + \delta Ev(x,A')$$
$$s.t. \quad B(r,g,x;b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}] \,.$$

Moreover, the transfer to coalition members is given by

$$s_\tau(b,A) = v_{\tau+1}(b,A) - u(w(1-r_\tau(b,A), g_\tau(b,A);A) - \delta Ev(x_\tau(b,A), A').$$

Proof: We begin with proposal round T. Let (b,A) be given. Multiplying the objective function through by q, we need to show that if (r_T, s_T, g_T, x_T) solves the round T proposer's problem when the state is (b,A), (r_T, g_T, x_T) solves the problem

$$\begin{array}{c} \max_{(r,g,x)} q[u(w(1-r),g;A) + \delta Ev(x,A')] + B(r,g,x;b) \\ s.t. \quad B(r,g,x;b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}] \end{array} , \tag{A.1}$$

and $s_T = v_{T+1}(b,A) - u(w(1-r_T), g_T;A) - \delta Ev(x_T, A')$. Recall that the round T proposer's problem is:

$$\max_{(r,g,x,s)} u(w(1-r),g;A) + B(r,g,x;b) - (q-1)\,s + \delta Ev(x,A)$$
$$s.t. \quad u(w(1-r),g;A) + s + \delta Ev(x,A) \geq v_{T+1}(b,A),$$
$$B(r,g,x;b) \geq (q-1)s, \quad s \geq 0 \ \& \ x \in [\underline{x}, \overline{x}].$$

It is easy to see that $s_T = v_{T+1}(b,A) - \delta Ev(x_T, A') - u(w(1-r_T), g_T;A)$, for if this were not the case it would follow from the definition of $v_{T+1}(b,A)$ that $s_T > 0$ and we could create a preferred proposal by just reducing s_T. It follows that we can write the proposer's payoff as

$$q\,[u(w(1-r_T), g_T;A) + \delta Ev(x_T, A')] + B(r_T, g_T, x_T;b).$$

Now suppose that (r_T, g_T, x_T) does not solve problem (A.1). Let (r', g', x') solve problem (A.1) and $s' = v_{T+1}(b,A) - u(w(1-r'), g';A) - \delta Ev(x', A')$. Then, the proposer's payoff under the proposal (r', g', x', s') is $q\,[u(w(1-r'), g'; A) + \delta Ev(x', A')] + B(r', g', x'; b)$. By construction, the incentive constraint is satisfied and, by definition of $v_{T+1}(b, A)$, $s' \geq 0$. Moreover, $x' \in [\underline{x}, \overline{x}]$. Finally, note that

$$\begin{aligned} B(r', g', x'; b) - (q-1)s' &= (q-1)[u(w(1-r'), g'; A) + \delta Ev(x', A')] + B(r', g', x'; b) \\ &\quad -(q-1)v_{T+1}(b, A) \geq 0 \end{aligned}$$

where the last inequality follows from the fact that (r', g', x') solves problem (A.1) and the definition of $v_{T+1}(b, A)$. It follows that (r', g', x', s') is feasible for the proposer's problem and yields a higher payoff than (r_T, g_T, x_T, s_T) - a contradiction.

Now consider the round $T-1$ proposer's problem

$$\begin{aligned} &\max_{(r,g,x,s)} u(w(1-r), g; A) + B(r, g, x; b) - (q-1)\,s + \delta Ev(x, A') \\ &s.t. \quad u(w(1-r), g; A) + s + \delta Ev(x, A') \geq v_T(b, A), \\ &B(r, g, x; b) \geq (q-1)s, \quad s \geq 0 \;\&\; x \in [\underline{x}, \overline{x}]. \end{aligned} \tag{A.2}$$

From what we know about the round T proposer's problem,

$$v_T(b, A) = u(w(1-r_T), g_T; A) + \frac{B(r_T, g_T, x_T; b)}{n} + \delta Ev(x_T, A'),$$

where (r_T, g_T, x_T) solves problem (A.1).

We need to show that if $(r_{T-1}, s_{T-1}, g_{T-1}, x_{T-1})$ is the solution to the round $T-1$ proposer's problem, $(r_{T-1}, g_{T-1}, x_{T-1})$ solves problem (A.1) and

$$s_{T-1} = v_T(b, A) - u(w(1-r_{T-1}), g_{T-1}; A) - \delta Ev(x_{T-1}, A').$$

The result would follow from our earlier argument if we could show that

$$s_{T-1} = v_T(b, A) - u(w(1-r_{T-1}), g_{T-1}; A) - \delta Ev(x_{T-1}, A'),$$

so suppose that $s_{T-1} > v_T(b, A) - u(w(1-r_{T-1}), g_{T-1}; A) - \delta Ev(x_{T-1}, A')$. Then it must be the case that $s_{T-1} = 0$, or we could obtain a preferred proposal by simply reducing s_{T-1}. It follows that

$$v_T(b, A) < u(w(1-r_{T-1}), g_{T-1}; A) + \delta Ev(x_{T-1}, A'). \tag{A.3}$$

This implies that $(r_{T-1}, g_{T-1}, x_{T-1})$ solves

$$\max_{(r,g,x)} u(w(1-r), g; A) + B(r, g, x; b) + \delta Ev(x, A)$$
$$s.t. \;\; B(r, g, x; b) \geq 0 \; \& \; x \in [\underline{x}, \overline{x}].$$

Now consider the proposal $(r_T, g_T, x_T, \frac{B(r_T, g_T, x_T; b)}{n})$. Clearly, this proposal satisfies all the constraints of the proposer's problem. The payoff to the proposer under this policy is

$$q[u(w(1-r_T), g_T; A) + \delta Ev(x_T, A')] + B(r_T, g_T, x_T; b) - (q-1)v_T(b, A).$$

From (A.3), this payoff is strictly larger than

$$\begin{aligned} & q[u(w(1-r_T), g_T; A) + \delta Ev(x_T, A)] + B(r_T, x_T, g_T; b) \\ & -(q-1)[u(w(1-r_{T-1}), g_{T-1}; A) + \delta Ev(x_{T-1}, A)]. \end{aligned}$$

The payoff to the proposer under the optimal policy $(r_{T-1}, g_{T-1}, x_{T-1})$ is

$$u(w(1-r_{T-1}), g_{T-1}; A) + B(r_{T-1}, x_{T-1}, g_{T-1}; b) + \delta Ev(x_{T-1}, A).$$

Thus, it must be the case that

$$\begin{aligned} & u(w(1-r_{T-1}), g_{T-1}; A) + B(r_{T-1}, x_{T-1}, g_{T-1}; b) + \delta Ev(x_{T-1}, A') \\ > \; & q[u(w(1-r_T), g_T; A) + \delta Ev(x_T, A')] + B(r_T, x_T, g_T; b) \\ & -(q-1)[u(w(1-r_{T-1}), g_{T-1}; A) + \delta Ev(x_{T-1}, A')], \end{aligned}$$

implying that

$$\begin{aligned} & q[u(w(1-r_{T-1}), g_{T-1}; A) + \delta Ev(x_{T-1}, A')] + B(r_{T-1}, x_{T-1}, g_{T-1}; b) \\ > \; & q[u(w(1-r_T), g_T; A) + \delta Ev(x_T, A')] + B(r_T, x_T, g_T; b). \end{aligned}$$

This contradicts the fact that (r_T, g_T, x_T) solves problem (A.1).

Application of the same logic to proposal rounds $\tau = T-2, ..., 1$ implies the lemma. ■

Using this result, we can prove:

Lemma A.2: *Let* $\{r_\tau(b, A),\; g_\tau(b, A),\; x_\tau(b, A),\; s_\tau(b, A)\}_{\tau=1}^{T}$ *be an equilibrium with associated value function* $v(b, A)$*. Then, there exists some debt level* x^* *such that for any proposal round* τ *if*

$A \geq A^*(b, x^*)$

$$(r_\tau(b,A), g_\tau(b,A), x_\tau(b,A)) = \arg\max \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta Ev(x; A') \\ \\ B(r,g,x;b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}] \end{array} \right\}$$

and $s_\tau(b, A) = 0$*, while if* $A < A^*(b, x^*)$

$$(r_\tau(b,A), g_\tau(b,A), x_\tau(b,A)) = (r^*, g^*(A), x^*)$$

and

$$s_\tau(b,A) = \left\{ \begin{array}{cc} \frac{B(r^*,g^*(A),x^*;b)}{n} & \text{if } \tau = 1, ..., T-1 \\ \\ v_{T+1}(b,A) - u(w(1-r^*, g^*(A); A) - \delta Ev(x^*, A') & \text{if } \tau = T \end{array} \right. .$$

Proof: The argument in Section 4.1 of the paper together with Lemma A.1 implies that for any proposal round τ if $A \geq A^*(b, x^*)$

$$(r_\tau(b,A), g_\tau(b,A), x_\tau(b,A)) = \arg\max \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta Ev(x, A') \\ \\ B(r,g,x;b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}] \end{array} \right\},$$

while if $A < A^*(b, x^*)$

$$(r_\tau(b,A), g_\tau(b,A), x_\tau(b,A)) = (r^*, g^*(A), x^*).$$

Turning to the equilibrium transfers, it is clear that, since there are no surplus revenues when $A \geq A^*(b, x^*)$, transfers are zero. If $A < A^*(b, x^*)$ it follows that for all proposal rounds $\tau = 1, ..., T-1$ we have that

$$v_{\tau+1}(b,A) = u(w(1-r^*), g^*(A); A) + \frac{B(r^*, g^*(A), x^*; b)}{n} + \delta Ev(x^*, A').$$

Thus, Lemma A.1 implies that the transfers to coalition members are given by:

$$s_\tau(b,A) = \left\{ \begin{array}{cc} B(r^*, g^*(A), x^*; b)/n & \tau = 1, ..., T-1 \\ \\ v_{T+1}(b,A) - u(w(1-r^*, g^*(A); A) - \delta Ev(x^*, A') & \tau = T \end{array} \right. .$$

■

Lemma 1 now follows immediately from Lemma A.2.

10.4 Properties of the equilibrium policy functions

In this section we establish some properties of the equilibrium (and optimal) policy functions that are mentioned in the text and that will be used in the following proofs.

We first show that when $A \geq A^*(b, x^*)$, the tax rate, public debt level and the level of the public good depend positively on the value of the public good (A), the tax rate and level of public debt depend positively on the current level of debt (b) and the level of the public good depends negatively on b. From Lemma A.2 we know that when $A \geq A^*(b, x^*)$, the equilibrium tax rate-public good-public debt triple $(r_\tau(b, A), g_\tau(b, A), x_\tau(b, A))$ solve

$$\begin{array}{c} \max_{(r,g,x)} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta E v(x, A') \\ s.t. \quad B(r, g, x; b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}] \end{array}.$$

Moreover, from the discussion in the text, $(r_\tau(b, A), g_\tau(b, A), x_\tau(b, A))$ is implicitly defined by equations (10), (11) and (12) in Section 3.1 (with the appropriate equilibrium value function).

Lemma A.3: *Let $b \in [\underline{x}, \overline{x}]$ and let $A_0, A_1 \in [\underline{A}, \overline{A}]$ be such that $A^*(b, x^*) < A_0 < A_1$. Then, it is the case that $g_\tau(b, A_0) < g_\tau(b, A_1)$ and $r_\tau(b, A_0) < r_\tau(b, A_1)$. Moreover, it is also the case that $x_\tau(b, A_0) \leq x_\tau(b, A_1)$ with strict inequality if $x_\tau(b, A_0) < \overline{x}$.*

Proof of Lemma A.3: We begin by showing that $g_\tau(b, A_0) < g_\tau(b, A_1)$. Let $\varphi(A'; b, A)$ be the value of the objective function for the problem when the state is A' and the policies are those that are optimal given state (b, A); that is,

$$\varphi(A'; b, A) = u(w(1-r_\tau(b, A), g_\tau(b, A); A') + \frac{B(r_\tau(b, A), g_\tau(b, A), x_\tau(b, A); b)}{n} + \delta E v(x_\tau(b, A), A').$$

Then, we have that $\varphi(A_0; b, A_0) > \varphi(A_0; b, A_1)$ and $\varphi(A_1; b, A_1) > \varphi(A_1; b, A_0)$ (the strict inequality follows from the fact that the problem has a unique solution).

Moreover, using the definition of the indirect utility function $u(w(1-r), g; A)$ (see equation (3) in Section 2) and letting $\Delta A = A_1 - A_0$, we can write $\varphi(A_0; b, A_0) = \varphi(A_1; b, A_0) - \Delta A g_\tau(b, A_0)$ and $\varphi(A_0; b, A_1) = \varphi(A_1; b, A_1) - \Delta A g_\tau(b, A_1)$. Since $\varphi(A_0; b, A_0) > \varphi(A_0; b, A_1)$, this means that $\varphi(A_1; b, A_0) - \Delta A g_\tau(b, A_0) > \varphi(A_1; b, A_1) - \Delta A g_\tau(b, A_1)$, and hence

$$\Delta A[g_\tau(b, A_0) - g_\tau(b, A_1)] < \varphi(A_1; b, A_0) - \varphi(A_1; b, A_1) < 0.$$

Since $\Delta A > 0$, this implies that $g_\tau(b, A_0) < g_\tau(b, A_1)$ as required.

We next show that $r_\tau(b, A_0) < r_\tau(b, A_1)$. Suppose to the contrary that $r_\tau(b, A_0) \geq r_\tau(b, A_1)$. Then the first order condition for x (i.e., (11)) and the concavity of $Ev(\cdot, A)$ imply that $x_\tau(b, A_0) \geq x_\tau(b, A_1)$. But then it follows that

$$B(r_\tau(b, A_0), g_\tau(b, A_0), x_\tau(b, A_0); b) > B(r_\tau(b, A_1), g_\tau(b, A_1), x_\tau(b, A_1); b) = 0$$

which is a contradiction.

Finally, we show that $x_\tau(b, A_0) \leq x_\tau(b, A_1)$ with strict inequality if $x_\tau(b, A_0) < \overline{x}$. This follows immediately from the first order condition for x and the concavity of v given that $r_\tau(b, A_0) < r_\tau(b, A_1)$. ■

Lemma A.4: *Let $b_0, b_1 \in [\underline{x}, \overline{x}]$ be such that $b_0 < b_1$ and let $A \in [\underline{A}, \overline{A}]$ be such that $A^*(b_0, x^*) < A$. Then, it is the case that $r_\tau(b_0, A) < r_\tau(b_1, A)$ and $g_\tau(b_0, A) > g_\tau(b_1, A)$. Moreover, it is also the case that $x_\tau(b_0, A) \leq x_\tau(b_1, A)$ with strict inequality if $x_\tau(b_0, A) < \overline{x}$.*

Proof of Lemma A.4: We first show that $r_\tau(b_0, A) < r_\tau(b_1, A)$. Suppose to the contrary that $r_\tau(b_0, A) \geq r_\tau(b_1, A)$. Then the first order conditions for g and x (i.e., (10) and (11)) and the concavity of $Ev(\cdot, A)$ imply that $g_\tau(b_0, A) \leq g_\tau(b_1, A)$ and $x_\tau(b_0, A) \geq x_\tau(b_1, A)$. But then it follows that

$$B(r_\tau(b_0, A), g_\tau(b_0, A), x_\tau(b_0, A); b_0) > B(r_\tau(b_1, A), g_\tau(b_1, A), x_\tau(b_1, A); b_1) = 0$$

which is a contradiction.

The fact that $g_\tau(b_0, A) > g_\tau(b_1, A)$ follows immediately from the first order condition for g and the fact that $r_\tau(b_0, A) < r_\tau(b_1, A)$. In addition, the fact that $x_\tau(b_0, A) \leq x_\tau(b_1, A)$ with strict inequality if $x_\tau(b_0, A) < \overline{x}$ follows immediately from the first order condition for x (i.e., (11)), the concavity of $Ev(\cdot, A)$ and the fact that $r_\tau(b_0, A) < r_\tau(b_1, A)$. ■

10.5 Proof of Proposition 2

We begin by proving the existence of an equilibrium. The proof is divided into seven steps.

Step 1: Let F denote the set of all real valued functions $v(\cdot, \cdot)$ defined over the compact set $[\underline{x}, \overline{x}] \times [\underline{A}, \overline{A}]$. Let F^* be the subset of these functions that are continuous and concave in x for all A. For any $z \in [\frac{R(r^*) - pg^*(\overline{A})}{\rho}, \overline{x}]$ and $v \in F^*$ consider the maximization problem

$$\begin{array}{l} \max_{(r,g,x)} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta Ev(x, A') \\ s.t. \quad B(r, g, x; b) \geq 0,\ r \geq r^*,\ g \leq g^*(A)\ \&\ x \in [z, \overline{x}] \end{array}.$$

For all $\mu > 0$, let

$$X_z^\mu(v) = \arg\max_x\{\frac{x}{\mu} + \delta Ev(x, A') : x \in [z, \overline{x}]\}$$

and let $x_z^\mu(v)$ be the largest element of the compact set $X_z^\mu(v)$. Notice that $x_z^\mu(v)$ is non-increasing in μ.

Suppose that (r, g, x) is a solution to the maximization problem. It is straightforward to show that (i) if $A \leq A^*(b, x_z^n(v))$ then $(r, g) = (r^*, g^*(A))$ and $x \in X_z^n(v) \cap \{x : B(r^*, g^*(A), x; b) \geq 0\}$; (ii) if $A \in (A^*(b, x_z^n(v)), A^*(b, x_z^q(v))]$ then $(r, g) = (r^*, g^*(A))$ and $B(r^*, g^*(A), x; b) = 0$; and (iii) if $A > A^*(b, x_z^q(v))$ (r, g, x) is uniquely defined and the budget constraint is binding. Moreover, $r > r^*$ and $g < g^*(A)$. Note that in all cases the tax rate and public good level are uniquely defined.

Step 2: For any $z \in [\frac{R(r^*)-pg^*(\overline{A})}{\rho}, \overline{x}]$, define the operator $T_z : F^* \to F$ as follows:

$$T_z(v)(b, A) = \max_{(r,g,x)} \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta Ev(x, A') \\ B(r, g, x; b) \geq 0,\ r \geq r^*,\ g \leq g^*(A)\ \&\ x \in [z, \overline{x}] \end{array} \right\}.$$

It can be verified that $T_z(v) \in F^*$ and that T_z is a contraction. Thus, there exists a unique fixed point $v_z(b, A)$ which is continuous and concave in b for all A. This fixed point satisfies the functional equation

$$v_z(b, A) = \max_{(r,g,x)} \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta Ev_z(x, A') \\ B(r, g, x; b) \geq 0,\ r \geq r^*,\ g \leq g^*(A)\ \&\ x \in [z, \overline{x}] \end{array} \right\}.$$

Let (b, A) be given and let (r, g, x) denote an optimal policy. By Step 1, we have that (i) if $A \leq A^*(b, x_z^n(v_z))$ then $(r, g) = (r^*, g^*(A))$ and $x \in X_z^n(v_z) \cap \{x : B(r^*, g^*(A), x; b) \geq 0\}$; (ii) if $A \in (A^*(b, x_z^n(v_z)), A^*(b, x_z^q(v_z))]$ then $(r, g) = (r^*, g^*(A))$ and $B(r^*, g^*(A), x; b) = 0$; and (iii) if $A > A^*(b, x_z^q(v_z))$ (r, g, x) is uniquely defined and the budget constraint is binding. Moreover, $r > r^*$ and $g < g^*(A)$. Again, in all cases the tax rate and public good level is uniquely defined. Let these be given by $(r_z(b, A), g_z(b, A))$ - these are also continuous functions on the state space.

Step 3: For any $z \in [\frac{R(r^*)-pg^*(\overline{A})}{\rho}, \overline{x}]$, the expected value function $Ev_z(\cdot, A)$ is strictly concave on the set $\{b \in [\underline{x}, \overline{x}] : A^*(b, x_z^q(v_z)) < \overline{A}\}$.

Proof: It suffices to show that for any $v \in F^*$, the function $ET_z(v)(\cdot, A)$ is strictly concave on the set $\{b \in [\underline{x}, \overline{x}] : A^*(b, x_z^q(v)) < \overline{A}\}$. Since $T_z(v) \in F^*$, we know already that the function

$T_z(v)(\cdot, A)$ is concave for all A. We now show that for all A, the function $T_z(v)(\cdot, A)$ is strictly concave on $\{b \in [\underline{x}, \overline{x}] : A^*(b, x_z^q(v)) < A\}$. In this case, the budget constraint is strictly binding and $g_z(b, A) < g^*(A)$, $r_z(b, A) > r^*$. We can therefore write:

$$T_z(v)(b, A) = \max_{(r,g,x)} \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta Ev\left(x, A'\right) \\ B(r, g, x; b) \geq 0 \ \& \ x \in [z, \overline{x}] \end{array} \right\}$$

Take two points b_1 and b_2 in the set $\{b \in [\underline{x}, \overline{x}] : A^*(b, x_z^q(v)) < \overline{A}\}$ and assume that $b_1 < b_2$. Let λ be a point in the interval $[0, 1]$. Define (r_i, g_i, x_i) to be the optimal policies associated with (b_i, A) for $i = 1, 2$ (as noted above these are unique). Let $b_\lambda = \lambda b_1 + (1-\lambda) b_2$, $r_\lambda = \lambda r_1 + (1-\lambda) r_2$, $g_\lambda = \lambda g_1 + (1-\lambda) g_2$ and $x_\lambda = \lambda x_1 + (1-\lambda) x_2$. Since $v(x, A') + x/n$ is weakly concave in x, $u(w(1-r), g; A) + [R(r) - pg]/n$ is strictly concave in (r, g), and $(r_1, g_1, x_1) \neq (r_2, g_2, x_2)$, we have that:

$$\begin{aligned} \lambda T_z(v)(b_1, A) + (1-\lambda) T_z(v)(b_2, A) &= \lambda \left[\begin{array}{c} u(w(1-r_1), g_1; A) \\ + \frac{B(r_1,g_1,x_1;b_1)}{n} + \delta Ev\left(x_1, A'\right) \end{array} \right] \\ &\quad + (1-\lambda) \left[\begin{array}{c} u(w(1-r_2), g_2; A) \\ + \frac{B(r_2,g_2,x_2;b_2)}{n} + \delta Ev\left(x_2, A'\right) \end{array} \right] \\ &< u(w(1-r_\lambda), g_\lambda; A) + \frac{B(r_\lambda, g_\lambda, x_\lambda; b_\lambda)}{n} + \delta Ev\left(x_\lambda, A'\right) \end{aligned}$$

Since $R(r)$ is concave in r, we have that $B(r_\lambda, g_\lambda, x_\lambda; b_\lambda) > 0$ and, in addition, $x_\lambda \in [z, \overline{x}]$. Therefore:

$$\begin{aligned} & u(w(1-r_\lambda), g_\lambda; A) + \frac{B(r_\lambda, g_\lambda, x_\lambda; b_\lambda)}{n} + \delta Ev\left(x_\lambda, A'\right) \\ \leq & \max_{(r,g,x)} \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,x;b_\lambda)}{n} + \delta Ev\left(x, A'\right) \\ B(r, g, x; b_\lambda) \geq 0 \ \& \ x \in [z, \overline{x}] \end{array} \right\} = T_z(v)(b_\lambda, A). \end{aligned}$$

We conclude that $\lambda T_z(v)(b_1, A) + (1-\lambda) T_z(v)(b_2, A) < T_z(v)(b_\lambda, A)$ as required.

Now take any two points b_1 and b_2 in the set $\{b \in [\underline{x}, \overline{x}] : A^*(b, x_z^q(v)) < \overline{A}\}$ and assume that $b_1 < b_2$. Then, we have that

$$\lambda ET_z(v)(b_1, A) + (1-\lambda) ET_z(v)(b_2, A)$$

$$
\begin{aligned}
&= \lambda\{\int_{\underline{A}}^{A^*(b_1,x_z^q(v))} T_z(v)(b_1,A)dG(A) + \int_{A^*(b_1,x_z^q(v))}^{\overline{A}} T_z(v)(b_1,A)dG(A)\} \\
&\quad +(1-\lambda)\{\int_{\underline{A}}^{A^*(b_2,x_z^q(v))} T_z(v)(b_2,A)dG(A) + \int_{A^*(b_2,x_z^q(v))}^{\overline{A}} T_z(v)(b_2,A)dG(A)\} \\
&= \int_{\underline{A}}^{A^*(b_1,x_z^q(v))} [\lambda T_z(v)(b_1,A) + (1-\lambda)T_z(v)(b_2,A)]dG(A) \\
&\quad + \int_{A^*(b_1,x_z^q(v))}^{\overline{A}} [\lambda T_z(v)(b_1,A) + (1-\lambda)T_z(v)(b_2,A)]dG(A) \\
&< \int_{\underline{A}}^{A^*(b_1,x_z^q(v))} T_z(v)(b_\lambda,A)dG(A) + \int_{A^*(b_1,x_z^q(v))}^{\overline{A}} T_z(v)(b_\lambda,A)dG(A) = ET_z(v)(b_\lambda,A)
\end{aligned}
$$

Step 4: For any $z \in [\frac{R(r^*)-pg^*(\overline{A})}{\rho}, \overline{x}]$, let

$$M(z) = \arg\max\{\frac{x}{q} + \delta E v_z(x,A) : x \in [\frac{R(r^*)-pg^*(\overline{A})}{\rho}, \overline{x}]\}.$$

Then there exists $z^* \in [\frac{R(r^*)-pg^*(\overline{A})}{\rho}, \overline{x}]$ such that $z^* \in M(z^*)$.

Proof: The result follows from *Kakutani's Fixed Point Theorem* if $M(z)$ is non-empty, upper hemi-continuous, and convex and compact-valued. We have:

Claim: $M(z)$ is non-empty, upper hemi-continuous, and convex and compact-valued.

Proof: Let F_z denote the set of all bounded and continuous real valued functions $\varphi(\cdot,\cdot,\cdot)$ defined over the compact set $[\frac{R(r^*)-pg^*(\overline{A})}{\rho}, \overline{x}] \times [\underline{x}, \overline{x}] \times [\underline{A}, \overline{A}]$. Define the operator:

$$\Psi(\varphi)(z,b,A) = \max_{(r,g,x)} \left\{ \begin{array}{c} u(w(1-r),g;A) + \frac{B(r,g,x;b)}{n} + \delta E\varphi(z,x,A') \\ B(r,g,x;b) \geq 0,\ g \leq g^*(A),\ r \geq r^* \ \&\ x \in [z,\overline{x}] \end{array} \right\}$$

It is easy to verify that Ψ maps F_z into itself and is a contraction. Thus, it has a unique fixpoint $\varphi^* = \Psi(\varphi^*)$ which belongs to F_z. Now note that for any $z \in [\frac{R(r^*)-pg^*(\overline{A})}{\rho}, \overline{x}]$, $v_z(b,A) = \varphi^*(z,b,A)$. To see this, note that for any given z, $\varphi^*(z,b,A) \in F^*$, so we can define $T_z(\varphi^*(z,b,A))$. The definition of φ^*, however, implies $T_z(\varphi^*(z,b,A)) = \varphi^*(z,b,A)$. Since T_z has a unique fixpoint, it must be that $v_z(b,A) = \varphi^*(z,b,A)$.

Given this, we conclude that $v_z(b,A)$ is continuous in z and the *Theorem of the Maximum* then implies that $M(z)$ is non-empty, upper hemi-continuous, and compact-valued. Convexity of $M(z)$ follows from the fact that $Ev_z(x,A)$ is weakly concave. ■

Step 5: Let z^* be such that $z^* \in M(z^*)$. Then, $x_{z^*}^q(v_{z^*}) = z^*$.

Proof: By definition, $x^q_{z^*}(v_{z^*})$ is the largest element in the set $X^q_{z^*}(v_{z^*})$. By construction, z^* belongs to the set

$$M(z^*) = \arg\max\{\frac{x}{q} + \delta E v_{z^*}(x, A) : x \in [\frac{R(r^*) - pg^*(\overline{A})}{\rho}, \overline{x}]\}.$$

Since z^* obviously satisfies the constraint that $x \geq z^*$, it must be the case that $z^* \in X^q_{z^*}(v_{z^*})$. If $z^* \neq x^q_{z^*}(v_{z^*})$, then it must be the case that $z^* < x^q_{z^*}(v_{z^*})$ and that

$$\frac{x^q_{z^*}(v_{z^*})}{q} + \delta E v_{z^*}(x^q_{z^*}(v_{z^*}), A) = \frac{z^*}{q} + \delta E v_{z^*}(z^*, A).$$

This implies that the expected value function $Ev_{z^*}(\cdot, A)$ is linear on the interval $[z^*, x^q_{z^*}(v_{z^*})]$. However, we know that

$$x^q_{z^*}(v_{z^*}) > z^* \geq \frac{R(r^*) - pg^*(\overline{A})}{\rho}$$

which implies that $pg^*(\overline{A}) + \rho x^q_{z^*}(v_{z^*}) > R(r^*)$, and hence that $A^*(x^q_{z^*}(v_{z^*}), x^q_{z^*}(v_{z^*})) < \overline{A}$. By continuity, therefore, there must exist an interval $[x', x^q_{z^*}(v_{z^*})] \subset [z^*, x^q_{z^*}(v_{z^*})]$ such that for all x in this interval $A^*(x, x^q_{z^*}(v_{z^*})) < \overline{A}$. But by Step 3, the expected value function $Ev_{z^*}(\cdot, A)$ is strictly concave on the interval $[x', x^q_{z^*}(v_{z^*})]$ - a contradiction.

Step 6: Let z^* be such that $z^* \in M(z^*)$. Then, the function $v_{z^*}(\cdot, A)$ is differentiable for all b such that $A \neq A^*(b, z^*)$. Moreover:

$$\frac{\partial v_{z^*}(b, A)}{\partial x} = \begin{cases} -(\frac{1 - r_{z^*}(b,A)}{1 - r_{z^*}(b,A)(1+\varepsilon)})(\frac{1+\rho}{n}) & \text{if } A > A^*(b, z^*) \\ -(\frac{1+\rho}{n}) & \text{if } A < A^*(b, z^*) \end{cases}.$$

Proof: Let $A \in [\underline{A}, \overline{A}]$ and let x_o be given. By Step 5, we know that $x^q_{z^*}(v_{z^*}) = z^*$ which immediately implies that $x^n_{z^*}(v_{z^*}) = z^*$. Suppose first that $A < A^*(x_o, z^*)$. Then, we have that in a neighborhood of x_o that

$$v_{z^*}(x, A) = u(w(1 - r^*), g^*(A); A) + \frac{B(r^*, g^*(A), z^*; x)}{n} + \delta E v_{z^*}(z^*, A').$$

Thus, it is immediate that the value function $v_{z^*}(x, A)$ is differentiable at x_o and that

$$\frac{\partial v_{z^*}(x_o, A)}{\partial x} = -(\frac{1+\rho}{n}).$$

Now suppose that $A > A^*(x_o, z^*)$. Then, we know that the budget constraint is binding, and that the constraints $r \geq r^*$ and $g \leq g^*(A)$ are not binding. Thus, we have that in a neighborhood of x_o that

$$v_{z^*}(x,A) = \max_{(r,g,y)} \left\{ \begin{array}{c} u(w(1-r),g;A) + \frac{B(r,g,y;x)}{n} + \delta E v_{z^*}(y,A') \\ B(r,g,y;x) \geq 0 \ \& \ x \in [z^*, \overline{x}] \end{array} \right\}.$$

Define the function

$$g(x) = \frac{R(r_{z^*}(x_o,A)) + x_{z^*}(x_o,A) - (1+\rho)x}{p}$$

and let

$$\eta(x) = u(w(1-r_{z^*}(x_o,A)),g(x);A) + \frac{B(r_{z^*}(x_o,A),g(x),x_{z^*}(x_o,A);x)}{n} + \delta E v_{z^*}(x_{z^*}(x_o,A),A').$$

Notice that $(r_{z^*}(x_o,A),g(x),x_{z^*}(x_o,A))$ is a feasible policy when the initial debt level is x so that in a neighborhood of x_o we have that $v_{z^*}(x,A) \geq \eta(x)$. Moreover, $\eta(x)$ is twice continuously differentiable with derivatives

$$\eta'(x) = -\alpha A g(x)^{\alpha-1}(\tfrac{1+\rho}{p})$$

$$\eta''(x) = -(1-\alpha)\alpha A g(x)^{\alpha-2}(\tfrac{1+\rho}{p})^2 < 0$$

The second derivative property implies that $\eta(x)$ is strictly concave. It follows from Theorem 4.10 of Stokey and Lucas (1989) that $v_{z^*}(x,A)$ is differentiable at x_o with derivative $\frac{\partial v_{z^*}(x_o,A)}{\partial x} = \eta'(x_o) = -\alpha A g_{z^*}(x_o,A)^{\alpha-1}(\frac{1+\rho}{p})$. To complete the proof note that $(r_{z^*}(x_o,A),g_{z^*}(x_o,A))$ must solve the problem:

$$\max_{(r,g)} \left\{ \begin{array}{c} u(w(1-r),g;A) + \frac{B(r,g,x_{z^*}(x_o,A);x_o)}{n} \\ B(r,g,x_{z^*}(x_o,A);x_o) \geq 0 \end{array} \right\},$$

which implies that $\alpha n A g_{z^*}(x_o,A)^{\alpha-1} = p[\frac{1-r_{z^*}(x_o,A)}{1-r_{z^*}(x_o,A)(1+\varepsilon)}]$. Thus, we have that

$$\frac{\partial v_{z^*}(x_o,A)}{\partial x} = -[\frac{1-r_{z^*}(x_o,A)}{1-r_{z^*}(x_o,A)(1+\varepsilon)}](\frac{1+\rho}{n}).$$

Step 7: Let z^* be such that $z^* \in M(z^*)$. Then, the following constitutes an equilibrium. For each proposal round τ

$$(r_\tau(b,A),g_\tau(b,A),x_\tau(b,A)) = (r_{z^*}(b,A),g_{z^*}(b,A),x_{z^*}(b,A));$$

for proposal rounds $\tau = 1,...,T-1$

$$s_\tau(b,A) = B(r_{z^*}(b,A),g_{z^*}(b,A),x_{z^*}(b,A);b)/n;$$

and for proposal round T

$$s_T(b;A) = v_{T+1}(b,A) - u(w(1-r_{z^*}(b,A)), g_{z^*}(b,A);A) - \delta E v_{z^*}(x_{z^*}(b,A),A');$$

where

$$v_{T+1}(b,A) = \max_{(r,g,x)} \left\{ \begin{array}{c} u(w(1-r),g;A) + \frac{B(r,x,g;b)}{n} + \delta E v_{z^*}(x,A') \\ \text{s.t. } B(r,x,g;b) \geq 0 \ \& \ x \in [\underline{x},\overline{x}] \end{array} \right\}.$$

Proof: Given these proposals, the legislators' round one value function is given by $v_{z^*}(b,A)$. This follows from the fact that

$$\begin{aligned} v(b,A) &= u(w(1-r_{z^*}(b,A)), g_{z^*}(b,A);A) + \frac{B(r_{z^*}(b,A), g_{z^*}(b,A), x_{z^*}(b,A);b)}{n} \\ &\quad + \delta E v_{z^*}(x_{z^*}(b,A),A') = v_{z^*}(b,A). \end{aligned}$$

Similarly, the round $\tau = 2,...,T$ legislators' value function $v_\tau(b,A)$ is given by $v_{z^*}(b,A)$. It follows from Steps 3 and 4 that the value function $v_{z^*}(b,A)$ has the properties required for an equilibrium to be well-behaved. Thus, we need only show: (i) that for proposal rounds $\tau = 1,..,T-1$ the proposal

$$(r_{z^*}(b,A), g_{z^*}(b,A), x_{z^*}(b,A), \frac{B(r_{z^*}(b,A), g_{z^*}(b,A), x_{z^*}(b,A);b)}{n})$$

solves the problem

$$\begin{aligned} &\max_{(r,g,x,s)} u(w(1-r),g;A) + B(r,g,x;b) - (q-1)\,s + \delta E v_{z^*}(x;A') \\ &s.t. \quad u(w(1-r),g;A) + s + \delta E v_{z^*}(x;A') \geq v_{z^*}(b;A), \\ &\qquad B(r,g,x;b) \geq (q-1)s, \quad s \geq 0 \ \& \ x \in [\underline{x},\overline{x}], \end{aligned}$$

and (ii) that for proposal round T the proposal

$$(r_{z^*}(b,A), g_{z^*}(b,A), x_{z^*}(b,A), v_{T+1}(b,A) - u(w(1-r_{z^*}(b,A)), g_{z^*}(b,A);A) - \delta E v_{z^*}(x_{z^*}(b,A),A'))$$

solves the problem

$$\begin{aligned} &\max_{(r,g,x,s)} u(w(1-r),g;A) + B(r,g,x;b) - (q-1)\,s + \delta E v_{z^*}(x;A') \\ &s.t. \quad u(w(1-r),g;A) + s + \delta E v_{z^*}(x;A') \geq v_{T+1}(b,A), \\ &\qquad B(r,g,x;b) \geq (q-1)s, \quad s \geq 0 \ \& \ x \in [\underline{x},\overline{x}], \end{aligned}$$

We show only (i) - the argument for (ii) being analogous.

Consider some proposal round $\tau = 1, ..., T-1$. Let (b, A) be given. To simplify notation, let

$$(\widehat{r}, \widehat{g}, \widehat{x}, \widehat{s}) = (r_{z^*}(b, A), g_{z^*}(b, A), x_{z^*}(b, A), \frac{B(r_{z^*}(b, A), g_{z^*}(b, A), x_{z^*}(b, A); b)}{n}).$$

It is clear from the argument in the text that $(\widehat{r}, \widehat{g}, \widehat{x})$ solves the problem

$$\max_{(r,g,x)} u(w(1-r), g; A) + \tfrac{B(r,g,x;b)}{q} + \delta E v_{z^*}(x; A')$$

$$s.t. \quad B(r, g, x; b) \geq 0 \ \& \ x \in [\underline{x}, \overline{x}],$$

and that $\widehat{s} = v_{z^*}(b, A) - u(w(1-\widehat{r}), \widehat{g}; A) - \delta E v_{z^*}(\widehat{x}; A')$. Suppose that $(\widehat{r}, \widehat{g}, \widehat{x}, \widehat{s})$ does not solve the round τ proposer's problem. Then there exist some (r', g', x', s') which achieves a higher value of the proposer's objective function. We know that $s' \geq v_{z^*}(b; A) - u(w(1-r'), g'; A) - \delta E v_{z^*}(x'; A')$. Thus, we have that the value of the proposer's objective function satisfies

$$\begin{aligned} & u(w(1-r'), g'; A) + B(r', g', x'; b) - (q-1)\, s' + \delta E v_{z^*}(x'; A') \\ \leq \quad & q[u(w(1-r'), g'; A) + \delta E v_{z^*}(x'; A')] + B(r', g', x'; b). \end{aligned}$$

But since $B(r', g', x'; b) \geq 0$, we know that

$$\begin{aligned} & q[u(w(1-r'), g'; A) + \delta E v_{z^*}(x'; A')] + B(r', g', x'; b) \\ \leq \quad & q[u(w(1-\widehat{r}), \widehat{g}; A) + \delta E v_{z^*}(\widehat{x}; A')] + B(\widehat{r}, \widehat{g}, \widehat{x}; b). \end{aligned}$$

But the right hand side of the inequality is the value of the proposer's objective function under the proposal $(\widehat{r}, \widehat{g}, \widehat{x}, \widehat{s})$. This therefore contradicts the assumption that (r', g', x', s') achieves a higher value for the proposer's problem. ■

We now turn to proving that the equilibrium is unique. Let $\{r^0_\tau(b, A),\ g^0_\tau(b, A),\ x^0_\tau(b, A),\ s^0_\tau(b, A)\}_{\tau=1}^T$ and $\{r^1_\tau(b, A),\ g^1_\tau(b, A),\ x^1_\tau(b, A),\ s^1_\tau(b, A)\}_{\tau=1}^T$ be two equilibria with associated round one value functions $v^0(b, A)$ and $v^1(b, A)$. Let x^*_0 and x^*_1 be the debt levels chosen in the BAU regimes of the two equilibria and suppose that $x^*_0 \leq x^*_1$. We will demonstrate that it must be the case that $x^*_0 = x^*_1$. To do this, we will show that the assumption that $x^*_0 < x^*_1$ results in a contradiction.

As in the proof of existence, define the operator $T_z : F^* \to F$ as follows:

$$T_z(v)(b, A) = \max_{(r,g,x)} \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta E v(x, A') \\ \\ B(r, g, x; b) \geq 0,\ g \leq g^*(A),\ r \geq r^* \ \&\ x \in [z, \overline{x}] \end{array} \right\}.$$

We know that $T_z(v) \in F^*$ and that T_z is a contraction. Moreover, for $i \in \{0,1\}$, we have that $T_{x_i^*}(v^i) = v^i$.

Now let $v \in F^*$ be such that for all b, $v(\cdot, A)$ is differentiable at b for almost all A and for each $i \in \{0,1\}$ consider the sequence of functions $\langle v_k^i \rangle_{k=1}^{\infty}$ defined inductively as follows: $v = T_{x_i^*}(v)$, and $v_{k+1}^i = T_{x_i^*}(v_k^i)$. Notice that since T_{z_i} is a contraction, $\langle v_k^i \rangle_{k=1}^{\infty}$ converges to v^i. We now establish the following result:

Claim: Let $\rho' \in (0, \rho)$. Then, for all k and for any $x \in [x_1^*, \overline{x}]$ we have that

$$-E(\frac{\partial v_k^1(x,A)}{\partial b}) > -E(\frac{\partial v_k^0(x - \frac{x_1^* - x_0^*}{1+\rho'}, A)}{\partial b}).$$

Proof: The proof proceeds via induction. Consider first the claim for $k = 1$. Recall from Step 1 of the existence part of the proof of Proposition 2 that if (r, g, x) is a solution to the problem

$$\begin{array}{c} \max u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta E v(x, A') \\ \text{s.t. } B(r,g,x;b) \geq 0,\ g \leq g^*(A),\ r \geq r^*,\ x \in [z, \overline{x}] \end{array},$$

then: (i) if $A \leq A^*(b, x_z^n(v))$ then $(r, g) = (r^*, g^*(A))$ and $x \in X_z^n(v) \cap \{x : B(r^*, g^*(A), x; b) \geq 0\}$; (ii) if $A \in (A^*(b, x_z^n(v)), A^*(b, x_z^q(v))]$ then $(r, g) = (r^*, g^*(A))$ and $B(r^*, g^*(A), x; b) = 0$; and (iii) if $A > A^*(b, x_z^q(v))$ (r, g, x) is uniquely defined, the budget constraint is binding, $r > r^*$ and $g < g^*(A)$. Denote the solution in case (iii) as $(r_z(b, A; v), g_z(b, A; v), x_z(b, A; v))$.

It follows from this that, if $A \leq A^*(b, x_z^n(v))$

$$T_z(v)(b,A) = u(w(1-r^*), g^*(A); A) + \frac{B(r^*), g^*(A), x_z^n(v); b)}{n} + \delta E v(x_z^n(v), A')$$

and

$$-\frac{\partial T_z(v)(b,A)}{\partial b} = \frac{1+\rho}{n}.$$

If $A \in (A^*(b, x_z^n(v)), A^*(b, x_z^q(v))]$, then

$$T_z(v)(b,A) = u(w(1-r^*), g^*(A); A) + \delta E v(pg^*(A) + (1+\rho)b - R(r^*), A')$$

and, since v is differentiable,

$$-\frac{\partial T_z(v)(b,A)}{\partial b} = -\delta E v'(pg^*(A) + (1+\rho)b - R(r^*), A')(1+\rho).$$

Notice for future reference that in this range, $x \in (x_z^n(v), x_z^q(v)]$ and hence

$$-\delta E v'(pg^*(A) + (1+\rho)b - R(r^*), A')(1+\rho) \in (\frac{1+\rho}{n}, \frac{1+\rho}{q}].$$

If $A > A^*(b, x_z^q(v))$ then

$$T_z(v)(b,A) = \max_{(r,g,x)} \left\{ \begin{array}{c} u(w(1-r), g; A) + \frac{B(r,g,x;b)}{n} + \delta E v\left(x, A'\right) \\ \\ B(r,g,x;b) \geq 0 \ \& \ x \in [z, \overline{x}] \end{array} \right\} \tag{A.4}$$

and

$$-\frac{\partial T_z(v)(b,A)}{\partial b} = \frac{1 - r_z(b,A;v)}{n(1 - r_z(b,A;v)(1+\varepsilon))}(1+\rho).$$

Since $r_z(b, A; v) > r^*$, in this range we have that

$$-\frac{\partial T_z(v)(b,A)}{\partial b} > \frac{(1+\rho)}{q}.$$

Combining all this and using the fact that $T_z(v)(b, \cdot)$ is continuous, we have that:

$$\begin{aligned} -n\delta E(\frac{\partial T_z(v)(b,A)}{\partial b}) \quad &= \quad G(A^*(b, x_z^n(v))) \\ &\quad -n\int_{A^*(b,x_z^n(v))}^{A^*(b,x_z^q(v))} E[\frac{\partial v(pg^*(A) + (1+\rho)b - R(r^*), A')}{\partial b}]dG(A) \\ &\quad + \int_{A^*(b,x_z^q(v))}^{\overline{A}} [\frac{1 - r_z(b,A;v)}{1 - r_z(b,A;v)(1+\varepsilon)}]dG(A). \end{aligned}$$

Applying this to the problem at hand, let $x \in [x_1^*, \overline{x}]$ and $f(x) = x - \frac{x_1^* - x_0^*}{1+\rho'}$. Then, to prove the claim for $k = 1$, we need to show that

$$\begin{aligned} G(A^*(x, x_{x_1^*}^n(v))) - n\int_{A^*(x,x_{x_1^*}^n(v))}^{A^*(x,x_{x_1^*}^q(v))} E[\tfrac{\partial v(pg^*(A)+(1+\rho)x-R(r^*),A')}{\partial b}]dG(A) \\ + \int_{A^*(x,x_{x_1^*}^q(v))}^{\overline{A}} [\tfrac{1-r_{x_1^*}(x,A;v)}{1-r_{x_1^*}(x,A;v)(1+\varepsilon)}]dG(A) \\ > G(A^*(f(x), x_{x_0^*}^n(v))) - n\int_{A^*(f(x),x_{x_0^*}^n(v))}^{A^*(f(x),x_{x_0^*}^q(v))} E[\tfrac{\partial v(pg^*(A)+(1+\rho)f(x)-R(r^*),A')}{\partial b}]dG(A) \\ + \int_{A^*(f(x),x_{x_0^*}^q(v))}^{\overline{A}} [\tfrac{1-r_{x_0^*}(f(x),A;v)}{1-r_{x_0^*}(f(x),A;v)(1+\varepsilon)}]dG(A). \end{aligned}$$

It is straightforward to verify that the following four conditions are sufficient for this inequality to hold: (i) $A^*(x, x_{x_1^*}^n(v)) < A^*(f(x), x_{x_0^*}^n(v))$, (ii) $A^*(x, x_{x_1^*}^q(v)) < A^*(f(x), x_{x_0^*}^q(v))$, (iii) for all $A \in (A^*(f(x), x_{x_0^*}^n(v)), A^*(x, x_{x_1^*}^q(v))]$

$$-E\frac{\partial v(pg^*(A) + (1+\rho)x - R(r^*), A')}{\partial b} \geq -E\frac{\partial v(pg^*(A) + (1+\rho)f(x) - R(r^*), A')}{\partial b},$$

and (iv) for all $A \in (A^*(f(x), x_{x_0^*}^q(v)), \overline{A}]$

$$\frac{1 - r_{x_0^*}(f(x), A; v)}{1 - r_{x_0^*}(f(x), A; v)(1+\varepsilon)} \leq \frac{1 - r_{x_1^*}(x, A; v)}{1 - r_{x_1^*}(x, A; v)(1+\varepsilon)}.$$

We will now show that these four conditions are satisfied. We begin with condition (i). If it were not satisfied, then $A^*(x, x^n_{x^*_1}(v)) \geq A^*(f(x), x^n_{x^*_0}(v))$ which is equivalent to $x^n_{x^*_1}(v) - (1+\rho)x \geq x^n_{x^*_0}(v) - (1+\rho)f(x)$. Thus,

$$\begin{aligned} x^n_{x^*_0}(v) &\leq x^n_{x^*_1}(v) - (1+\rho)[x - f(x)] \\ &= x^n_{x^*_1}(v) - \frac{1+\rho}{1+\rho'}(x^*_1 - x^*_0) < x^n_{x^*_1}(v) - (x^*_1 - x^*_0) \end{aligned}$$

which implies that $x^n_{x^*_1}(v) - x^n_{x^*_0}(v) > x^*_1 - x^*_0$. Given the definition of $x^n_{x^*_1}(v)$, this requires that $x^n_{x^*_1}(v) > x^*_1$. This in turn implies that

$$-\delta E \frac{\partial v(x^n_{x^*_1}(v), A')}{\partial b} = \frac{1}{n}$$

and hence that $x^n_{x^*_0}(v) = x^n_{x^*_1}(v)$ - a contradiction.

Condition (ii) can be established in the same way and condition (iii) follows directly from the assumption that $v(\cdot, A)$ is concave. This leaves condition (iv). From the first order conditions associated with problem (A.4) (see (10), (11) and (12)), we have that

$$\frac{1 - r_{x^*_1}(x, A; v)}{1 - r_{x^*_1}(x, A; v)(1+\varepsilon)} \geq -\delta n E \frac{\partial v(x_{x^*_1}(x, A; v), A')}{\partial b} \quad (= \text{ if } x_{x^*_1}(x, A; v) < \overline{x})$$

and that

$$\frac{1 - r_{x^*_0}(f(x), A; v)}{1 - r_{x^*_0}(f(x), A; v)(1+\varepsilon)} \geq -\delta n E \frac{\partial v(x_{x^*_0}(f(x), A; v), A')}{\partial b} \quad (= \text{ if } x_{x^*_0}(f(x), A; v) < \overline{x}).$$

Suppose that

$$\frac{1 - r_{x^*_0}(f(x), A; v)}{1 - r_{x^*_0}(f(x), A; v)(1+\varepsilon)} > \frac{1 - r_{x^*_1}(x, A; v)}{1 - r_{x^*_1}(x, A; v)(1+\varepsilon)}.$$

Then this implies that $r_{x^*_0}(f(x), A; v) > r_{x^*_1}(x, A; v)$, $g_{x^*_0}(f(x), A; v) < g_{x^*_1}(x, A; v)$, and $x_{x^*_0}(f(x), A; v) \geq x_{x^*_1}(x, A; v)$. Thus, we have that

$$\begin{aligned} pg_{x^*_0}(f(x), A; v) + (1+\rho)f(x) &= R(r_{x^*_0}(f(x), A; v)) + x_{x^*_0}(f(x), A; v) \\ &> R(r_{x^*_1}(x, A; v)) + x_{x^*_1}(x, A; v) = pg_{x^*_1}(x, A; v) + (1+\rho)x \end{aligned}$$

This means that $(1+\rho)[x - f(x)] = \frac{1+\rho}{1+\rho'}(x^*_1 - x^*_0) < 0$, which is a contradiction.

Now assume the claim is true for $1, ..., k$ and consider it for $k+1$. We have that

$$-n\delta E(\frac{\partial T_{x^*_i}(v^i_k)(b, A)}{\partial b}) \quad = \quad G(A^*(b, x^n_{x^*_i}(v^i_k)))$$

$$-n\int_{A^*(b,x^n_{x_i^*}(v_k^i))}^{A^*(b,x^q_{x_i^*}(v_k^i))} E[\frac{\partial v_k^i(pg^*(A)+(1+\rho)b-R(r^*),A')}{\partial b}]dG(A)$$
$$+\int_{A^*(b,x^q_{x_i^*}(v_k^i))}^{\overline{A}} [\frac{1-r_{x_i^*}(b,A;v_k^i)}{1-r_{x_i^*}(b,A;v_k^i)(1+\varepsilon)}]dG(A).$$

Thus, we need to show that

$$G(A^*(x,x^n_{x_1^*}(v_k^1))) - n\int_{A^*(x,x^n_{x_1^*}(v_k^1))}^{A^*(x,x^q_{x_1^*}(v_k^1))} E[\tfrac{\partial v_k^1(pg^*(A)+(1+\rho)x-R(r^*),A')}{\partial b}]dG(A)$$
$$+\int_{A^*(x,x^q_{x_1^*}(v_k^1))}^{\overline{A}} [\tfrac{1-r_{x_1^*}(x,A;v_k^1)}{1-r_{x_1^*}(x,A;v_k^1)(1+\varepsilon)}]dG(A)$$
$$> G(A^*(f(x),x^n_{x_0^*}(v_k^0))) - n\int_{A^*(f(x),x^n_{x_0^*}(v_k^0))}^{A^*(f(x),x^q_{x_0^*}(v_k^0))} E[\tfrac{\partial v_k^0(pg^*(A)+(1+\rho)f(x)-R(r^*),A')}{\partial b}]dG(A)$$
$$+\int_{A^*(f(x),x^q_{x_0^*}(v_k^0))}^{\overline{A}} [\tfrac{1-r_{x_0^*}(f(x),A;v_k^0)}{1-r_{x_0^*}(f(x),A;v_k^0)(1+\varepsilon)}]dG(A)$$

Following the same approach as above, for this inequality to hold, the following four conditions are sufficient: (i) $A^*(x,x^n_{x_1^*}(v_k^1)) < A^*(f(x),x^n_{x_0^*}(v_k^0))$, (ii) $A^*(x,x^q_{x_1^*}(v_k^1)) < A^*(f(x),x^q_{x_0^*}(v_k^0))$, (iii) for all $A \in (A^*(f(x),x^n_{x_0^*}(v_k^0)), A^*(x,x^q_{x_1^*}(v_k^1))]$

$$-E\frac{\partial v_k^1(pg^*(A)+(1+\rho)x-R(r^*),A')}{\partial b} \geq -E\frac{\partial v_k^0(pg^*(A)+(1+\rho)f(x)-R(r^*),A')}{\partial b},$$

and (iv) for all $A \in (A^*(f(x),x^q_{x_0^*}(v_k^0)),\overline{A}]$

$$\frac{1-r_{x_0^*}(f(x),A;v_k^0)}{1-r_{x_0^*}(f(x),A;v_k^0)(1+\varepsilon)} \leq \frac{1-r_{x_1^*}(x,A;v_k^1)}{1-r_{x_1^*}(x,A;v_k^1)(1+\varepsilon)}.$$

We will again show that these four conditions are satisfied. We begin with condition (i). If it were not satisfied, then it must be the case that $A^*(x,x^n_{x_1^*}(v_k^1)) \geq A^*(f(x),x^n_{x_0^*}(v_k^0))$ which is equivalent to $x^n_{x_1^*}(v_k^1)-(1+\rho)x \geq x^n_{x_0^*}(v_k^0)-(1+\rho)f(x)$. This implies that

$$x^n_{x_0^*}(v_k^0) \leq x^n_{x_1^*}(v_k^1) - \frac{1+\rho}{1+\rho'}[x_1^*-x_0^*] < x^n_{x_1^*}(v_k^1) - [x_1^*-x_0^*]. \quad \text{(A.5)}$$

Thus, we know by the induction step that

$$-\delta E\frac{\partial v_k^1(x^n_{x_1^*}(v_k^1),A')}{\partial b} > -\delta E\frac{\partial v_k^0(x^n_{x_0^*}(v_k^0),A')}{\partial b} \geq \frac{1}{n}.$$

This implies that $x^n_{x_1^*}(v_k^1) = x_1^*$ which in turn, together with (A.5), implies that $x^n_{x_0^*}(v_k^0) < x_0^*$ which is a contradiction.

We can use similar logic to conclude that condition (ii) is satisfied. Condition (iii) follows immediately from the induction step since we have that

$$pg^*(A)+(1+\rho)x-R(r^*)-[x_1^*-x_0^*] > pg^*(A)+(1+\rho)f(x)-R(r^*).$$

This leaves condition (iv). Suppose to the contrary that for some A

$$\frac{1-r_{x_0^*}(f(x),A;v_k^0)}{1-r_{x_0^*}(f(x),A;v_k^0)(1+\varepsilon)} > \frac{1-r_{x_1^*}(x,A;v_k^1)}{1-r_{x_1^*}(x,A;v_k^1)(1+\varepsilon)}$$

Again, from the first order conditions associated with problem (A.4) we have that

$$\frac{1-r_{x_1^*}(x,A;v_k^1)}{1-r_{x_1^*}(x,A;v_k^1)(1+\varepsilon)} \geq -\delta nE\frac{\partial v_k^1(x_{x_1^*}(x,A;v_k^1),A')}{\partial b} \quad (= \text{ if } x_{x_1^*}(x,A;v_k^1) < \overline{x})$$

and that

$$\frac{1-r_{x_0^*}(f(x),A;v_k^0)}{1-r_{x_0^*}(f(x),A;v_k^0)(1+\varepsilon)} \geq -\delta nE\frac{\partial v_k^0(x_{x_0^*}(f(x),A;v_k^0),A')}{\partial b} \quad (= \text{ if } x_{x_0^*}(f(x),A;v_k^0) < \overline{x}).$$

If $x_{x_0^*}(f(x),A;v_k^0) < \overline{x}$ these first order conditions imply that

$$-\delta nE\frac{\partial v_k^1(x_{x_1^*}(x,A;v_k^1),A')}{\partial b} < -\delta nE\frac{\partial v_k^0(x_{x_0^*}(f(x),A;v_k^0),A')}{\partial b}.$$

We know by the induction step that for any $x \geq x_1^*$, $-\delta nE(\frac{\partial v_k^1(x,A)}{\partial x}) > -\delta nE(\frac{\partial v_k^0(f(x),A)}{\partial x})$. Thus, it must be the case that

$$x_{x_0^*}(f(x),A;v_k^0) > f(x_{x_1^*}(x,A;v_k^1)) = x_{x_1^*}(x,A;v_k^1) - \frac{x_1^*-x_0^*}{1+\rho'}$$

In addition, we know that $r_{x_0^*}(f(x),A;v_k^0) > r_{x_1^*}(x,A;v_k^1)$ and that $g_{x_0^*}(f(x),A;v_k^0) < g_{x_1^*}(x,A;v_k^1)$. But this means that

$$\begin{aligned} pg_{x_0^*}(f(x),A;v_k^0)+(1+\rho)f(x) &= x_{x_0^*}(f(x),A;v_k^0)+R(r_{x_0^*}(f(x),A;v_k^0)) \\ &> x_{x_1^*}(x,A;v_k^1)+R(r_{x_1^*}(x,A;v_k^1)) - \frac{x_1^*-x_0^*}{1+\rho'} \\ &= pg_{x_1^*}(x,A;v_k^1)+(1+\rho)x - \frac{x_1^*-x_0^*}{1+\rho'}. \end{aligned}$$

This in turn implies that $(1+\rho)[x-f(x)] < \frac{x_1^*-x_0^*}{1+\rho'}$ which is a contradiction. If $x_{x_0^*}(f(x),A;v_k^0) = \overline{x}$, then it must be the case that $x_{x_1^*}(x,A;v_k^1) \leq x_{x_0^*}(f(x),A;v_k^0)$ and the same contradiction arises. ■

To complete the uniqueness proof, observe that for $i \in \{0,1\}$ the function $E(v^i(\cdot,A))$ is concave and differentiable. In addition, $\langle E(v_k^i(\cdot,A))\rangle$ is a sequence of concave and differentiable functions such that for all x $\lim_{k\to\infty} E(v_k^i(x,A)) = E(v^i(x,A))$. Thus, by Theorem 25.7 of Rockafellar (1970), we know that $\lim_{k\to\infty}\frac{dE(v_k^i(x,A))}{dx} = \frac{dE(v^i(x,A))}{dx}$. It follows that for any $x \in [x_1^*,\overline{x}]$

$$\begin{aligned} -\delta E(\frac{\partial v^1(x,A)}{\partial b}) &= \lim_{k\to\infty} -\delta E(\frac{\partial v_k^1(x,A)}{\partial b}) \\ &\geq \lim_{k\to\infty} -\delta E(\frac{\partial v_k^0(x-\frac{(x_1^*-x_0^*)}{1+\rho'},A)}{\partial b}) = -\delta E(\frac{\partial v^0(x-\frac{(x_1^*-x_0^*)}{1+\rho'},A)}{\partial b}). \end{aligned}$$

From equation (21) in the text, we know that

$$-\delta E(\frac{\partial v^1(x_1^*, A)}{\partial b}) = -\delta E(\frac{\partial v^0(x_0^*, A)}{\partial x}) = \frac{1}{q}.$$

Thus, it follows that

$$-\delta E(\frac{\partial v^0(x_1^* - \frac{(x_1^* - x_0^*)}{1+\rho'}, A)}{\partial b}) \leq -\delta E(\frac{\partial v^1(x_1^*, A)}{\partial b}) = -\delta E(\frac{\partial v^0(x_0^*, A)}{\partial b}).$$

But this implies that $x_1^* - \frac{x_1^* - x_0^*}{1+\rho'} \leq x_0^*$, which contradicts the fact that $x_1^* > x_0^*$.

It follows that $x_0^* = x_1^*$. This, in turn, implies that $v^0 = v^1$ and hence that $\{r_\tau^0(b, A), g_\tau^0(b, A), x_\tau^0(b, A), s_\tau^0(b, A)\}_{\tau=1}^T$ equals $\{r_\tau^1(b, A), g_\tau^1(b, A), x_\tau^1(b, A), s_\tau^1(b, A)\}_{\tau=1}^T$. ■

10.6 Proof of Proposition 3

Given the discussion in the text, the only thing we need to show is that the equilibrium debt distribution converges to a unique invariant distribution. Let $\psi_t(x)$ denote the distribution function of the current level of debt at the beginning of period t. The distribution function $\psi_1(x)$ is exogenous and is determined by the economy's initial level of debt b_0. To describe the distribution of debt in periods $t \geq 2$, we must first describe the *transition function* implied by the equilibrium. First, define the function $\widehat{A} : [\underline{x}, \overline{x}] \times (x^*, \overline{x}] \rightarrow [\underline{A}, \overline{A}]$ as follows:

$$\widehat{A}(b, x) = \begin{cases} \underline{A} & \text{if } x < x(b, \underline{A}) \\ \min\{A \in [\underline{A}, \overline{A}] : x(b, A) = x\} & \text{if } x \in [x(b, \underline{A}), x(b, \overline{A})] \\ \overline{A} & \text{if } x > x(b, \overline{A}) \end{cases}.$$

Intuitively, $\widehat{A}(b, x)$ is the smallest value of public goods under which the equilibrium debt level would be x given an initial level of debt b. Then, the transition function is given by

$$H(b, x) = \begin{cases} G(\widehat{A}(b, x)) & \text{if } x \in (x^*, \overline{x}] \\ G(A^*(b, x^*)) & \text{if } x = x^* \end{cases}.$$

Intuitively, $H(b, x)$ is the probability that in the next period the initial level of debt will be less than or equal to $x \in [x^*, \overline{x}]$ if the current level of debt is b. Using this notation, the distribution of debt at the beginning of any period $t \geq 2$ is defined inductively by

$$\psi_t(x) = \int_b H(b, x) d\psi_{t-1}(b).$$

The sequence of distributions $\langle \psi_t(x) \rangle$ *converges* to the distribution $\psi(x)$ if for all $x \in [x^*, \overline{x}]$, we have that $\lim_{t\to\infty} \psi_t(x) = \psi(x)$.[33] Moreover, $\psi^*(x)$ is an *invariant distribution* if

$$\psi^*(x) = \int_b H(b, x) d\psi^*(b).$$

We can now establish that the sequence of debt distributions $\langle \psi_t(x) \rangle$ converges to a unique invariant distribution $\psi^*(x)$.

It is easy to prove that the transition function $H(b, x)$ has the Feller Property and that it is monotonic in b (see Ch. 12.4 in Stokey, Lucas and Prescott (1989) for definitions). By Theorem 12.12 in Stokey, Lucas and Prescott (1989), therefore, the result follows if the following "mixing condition" is satisfied:

Mixing Condition: *There exists an $\epsilon > 0$ and $m \geq 1$, such that $H^m(\overline{x}, x^*) \geq \epsilon$ and $1 - H^m(\underline{x}, x^*) \geq \epsilon$ where the function $H^m(b, x)$ is defined inductively by $H^1(b, x) = H(b, x)$ and $H^m(b, x) = \int_z H(z, x) dH^{m-1}(b, z)$.*

Intuitively, this condition requires that if we start out with the highest level of debt $\overline{x}$, then we will end up at x^* with probability greater than ϵ after m periods, while if we start out with the lowest level of debt $\underline{x}$, we will end up above x^* with probability greater than ϵ in m periods. For any $b \in [\underline{x}, \overline{x}]$ and $A \in [\underline{A}, \overline{A}]$ define the sequence $\langle \phi_m(b, A) \rangle$ as follows: $\phi_0(b, A) = b$, $\phi_{m+1}(b, A) = x(\phi_m(b, A), A)$. Thus, $\phi_m(b, A)$ is the level of debt if the debt level were b at time 0 and the shock was A in periods 1 through m. Recall that, by assumption, there exists some positive constant $\xi > 0$, such that for any pair of realizations satisfying $A < A'$, the difference $G(A') - G(A)$ is at least as big as $\xi(A' - A)$. This implies that for any $b \in [\underline{x}, \overline{x}]$, $H^m(b, \phi_m(b, \underline{A} + \lambda)) - H^m(b, \phi_m(b, \underline{A})) \geq \xi^m \lambda^m$ for all λ such that $0 < \lambda < \overline{A} - \underline{A}$. Using this observation, we can prove:

Claim 1: For m sufficiently large, $H^m(\overline{x}, x^*) > 0$.

Proof: It suffices to show that for m sufficiently large $A^*(\phi_m(\overline{x}, \underline{A}), x^*) > \underline{A}$. Then, for any such m, by continuity there is a λ_m small enough such that $A^*(\phi_m(\overline{x}, \underline{A} + \lambda_m), x^*) > \underline{A}$. It then follows that

$$H^m(\overline{x}, x^*) \quad = \quad \int_z H(z, x^*) dH^{m-1}(\overline{x}, z) = \int_z G(A^*(z, x^*)) dH^{m-1}(\overline{x}, z)$$

[33] In the present environment, this definition is equivalent to the requirement that the sequence of probability measures associated with $\langle \psi_t(x) \rangle$ converges weakly to the probability measure associated with $\psi(x)$ (see Stokey, Lucas and Prescott (1989) Theorem 12.8).

$$\begin{aligned}
&\geq \int_{\phi_m(\overline{x},\underline{A})}^{\phi_m(\overline{x},\underline{A}+\lambda_m)} G(A^*(z,x^*))dH^{m-1}(\overline{x},z) \\
&\geq G(A^*(\phi_m(\overline{x},\underline{A}+\lambda_m),x^*)) \left[H^{m-1}(\overline{x},\phi_{m-1}(\overline{x},\underline{A}+\lambda_m)) - H^{m-1}(\overline{x},\phi_{m-1}(\overline{x},\underline{A}))\right] \\
&\geq G(A^*(\phi_m(\overline{x},\underline{A}+\lambda_m),x^*)) \left(\xi\lambda_m\right)^{m-1} > 0.
\end{aligned}$$

Suppose, to the contrary, that for all m we have that $A^*(\phi_m(\overline{x},\underline{A}),x^*) \leq \underline{A}$. Then, it must be the case that the sequence $\langle\phi_m(\overline{x},\underline{A})\rangle$ is decreasing. To see this note that since $r(b,A)$ is increasing in A we have that

$$\frac{1-r(\phi_k(\overline{x},\underline{A}),\underline{A})}{1-r(\phi_k(\overline{x},\underline{A}),\underline{A})(1+\varepsilon)} < \int_{\underline{A}}^{\overline{A}} (\frac{1-r(\phi_k(\overline{x},\underline{A}),A)}{1-r(\phi_k(\overline{x},\underline{A}),A)(1+\varepsilon)})dG(A).$$

But the first order condition for $x(b,A)$ (see (11) with appropriate value function) and the derivative of the value function (23) imply that:

$$\frac{1-r(\phi_{k-1}(\overline{x},\underline{A}),\underline{A})}{1-r(\phi_{k-1}(\overline{x},\underline{A}),\underline{A})(1+\varepsilon)} = \int_{\underline{A}}^{\overline{A}} (\frac{1-r(\phi_k(\overline{x},\underline{A}),A)}{1-r(\phi_k(\overline{x},\underline{A}),A)(1+\varepsilon)})dG(A). \qquad \text{(A.6)}$$

Since $r(b,\underline{A})$ is increasing in b and A, this implies $\phi_{k-1}(\overline{x},\underline{A}) > \phi_k(\overline{x},\underline{A})$.

We can therefore assume without loss of generality that $\phi_m(\overline{x},\underline{A})$ converges to some finite $\beta \geq \underline{x}$. We now prove that this yields a contradiction. Taking the limit as $m \to \infty$, continuity of $r(\cdot,\underline{A})$ would imply $\lim_{k\to\infty} r(\phi_k(\overline{x},\underline{A}),A) = r(\phi_\infty(\overline{x},\underline{A}),A)$ for all A. Using condition (A.4):

$$\frac{1-r(\phi_\infty(\overline{x},\underline{A}),\underline{A})}{1-r(\phi_\infty(\overline{x},\underline{A}),\underline{A})(1+\varepsilon)} = \int_{\underline{A}}^{\overline{A}} (\frac{1-r(\phi_\infty(\overline{x},\underline{A}),A)}{1-r(\phi_\infty(\overline{x},\underline{A}),A)(1+\varepsilon)})dG(A)$$

which is impossible since $r(\phi_\infty(\overline{x},\underline{A}),A)$ is strictly increasing in A. We conclude therefore that for m sufficiently large $A^*(\phi_m(\overline{x},\underline{A}),x^*) > \underline{A}$, which yields the result. ■

Next, we can establish:

Claim 2: For all $m \geq 2$, $1 - H^m(\underline{x},x^*) \geq G(A^*(\underline{x},x^*))G(A^*(x^*,x^*))^{m-2}\left[1-G(A^*(x^*,x^*))\right]$.

Proof: With probability $G(A^*(\underline{x},x^*))$ the level of debt chosen in period 1 is x^* when the initial level of debt is $\underline{x}$; so with probability $G(A^*(\underline{x},x^*))G(A^*(x^*,x^*))^{m-2}$ the level of debt is x^* for the first $m-1$ periods. Given this, the probability that the level of debt is larger than x^* in period m is at least $G(A^*(\underline{x},x^*))G(A^*(x^*,x^*))^{m-2}\left[1-G(A^*(x^*,x^*))\right]$. ■

These two Claims imply that the Mixing Condition is satisfied if $q < n$. To see this, choose m sufficiently large so that $H^m(\overline{x},x^*) > 0$. This is always possible by Claim 1. Now let

$$\epsilon = \min\left\{G(A^*(\underline{x},x^*))G(A^*(x^*,x^*))^{m-2}\left[1-G(A^*(x^*,x^*))\right]; H^m(\overline{x},x^*)\right\}$$

Assuming that $q < n$, we know from the definition of x^* that $A^*(x^*, x^*) \in \left(\underline{A}, \overline{A}\right)$ (see equation (24)) and $A^*(\underline{x}, x^*) > A^*(x^*, x^*) > \underline{A}$. Thus, $\epsilon > 0$ and the condition is satisfied. ■

10.7 Proof of Proposition 4

This result follows from Step 7 of the existence part of the proof of Proposition 2. ■

10.8 Completion of the Proof of Proposition 1

As discussed in the text, Proposition 4 implies that legislative decision-making delivers the planner's solution when $q = n$. Thus, we just need to show that when $q = n$, the equilibrium debt level will reach $\underline{x}$ with probability one. Since $x^* = \underline{x}$ when $q = n$, Claim 1 of Proposition 3 implies that there exists a $\epsilon > 0$ and a m such that for any initial b, the probability that $x = \underline{x}$ in the next m periods is at least ϵ. Thus, the probability that x is never equal to $\underline{x}$ in the next $j \cdot m$ periods is not larger than $(1-\epsilon)^j$. Since $\lim_{j\to\infty} (1-\epsilon)^j = 0$, we conclude that the probability that x is never equal to $\underline{x}$ is zero. ■