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Equilibrium Concepts in the Large Household Model

by

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Abstract

This paper formulates two alternative equilibrium concepts in the large household model: one which allows individual household agents to make choices in their separate meetings, and the other which commits individual household agents to contingent actions prior to their meetings. In the first formulation, *large* converts a model with non-linear preferences for the household into one with quasi-linear preferences for the individual household's agents, which is critical to make degeneracy—all households experience the same distribution of meeting outcomes—as an equilibrium; in the second formulation, commitment instead of *large* is the critical factor.

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1 Introduction

Search models now play a dominant role in labor economics and a prominent role in monetary economics. In such models, meeting-specific shocks are obvious sources of heterogeneity. For example, in a money model with complete specialization in consumption and production and random pairwise meetings (e.g. Kyotaki and Wright [1]), two people who start with the same wealth end up with different wealth if one becomes a buyer and the other

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becomes a seller in the relevant meetings; or, in a labor model with random job destruction (e.g. Pissarides [7, Ch 1, 2]), two workers who start with the same wealth end up with different wealth if the relevant worker-firm pairs experience different separation or productivity shocks. Because heterogeneity precludes closed-form solutions, efforts have been made to create models in which equilibria have degenerate distributions of wealth.

One such model is the so-called large household model, initiated by Merz [3] in labor economics and by Shi [5] in monetary economics. In this model, each household is large—it consists of a non atomic measure of agents, and each agent from a household meets someone from outside the household, a firm in [3], or an agent from another household in [5]. If all households start with the same wealth, then it is *feasible* that all households experience the same distribution of meeting outcomes, and, by a law of large numbers argument, end up with the same wealth. Of course, whether such degeneracy in wealth happens depends not only on feasibility, but on whether the same distribution of meeting outcomes is an equilibrium. Whether it is is unclear because the literature contains neither clear definitions of equilibrium or existence proofs. Rauch [4] points out a defect in the formulation of [5] (also a defect of [3]), but Rauch’s suggested alternative is itself not correct. More recent literature, initiated by Shi [6], avoids the problem pointed out by Rauch, but suffers from other deficiencies.

Here, in the context of a money model, I formulate two alternative concepts of search equilibrium. One completes and corrects Rauch’s formulation. That formulation allows individual household agents to make choices in their separate meetings and, therefore, is called the *no-commitment* approach. The other completes the more recent approach of Shi [6]. It commits individual household agents to contingent actions prior to their meetings and, therefore, is called the *commitment* approach. In order to study the role of *large* in determining whether degeneracy is an equilibrium, I use a model in which degeneracy is feasible whether or not the household is large. In particular, I study a model in which each meeting is a single-coincidence meeting, and I study a finite household version—a household that consists of n buyers and n sellers, and a large household version—a household that consists of equal non atomic measures of buyers and sellers.

In the no commitment approach, *large* converts a model with nonlinear preferences for the household into one with quasi-linear preferences for the individual household agents. The quasi-linearity is critical for degeneracy as an equilibrium. In the commitment approach, the surplus split in each

pairwise meeting is essentially conducted by a Nash demand game, which permits a variety of linear pricing, and, consequently, a continuum of degenerate equilibria. That is, commitment itself is critical for degeneracy as an equilibrium, while it results in a weak equilibrium concept. Seemingly, the no-commitment approach is a better approach, and the present formulation of this approach and the analysis provide a basis for applications of the large household model.

2 The environment

Time is discrete. There is a non atomic measure of each of $K \geq 3$ types of infinitely lived households. Each household consists of a set of buyers indexed by I , and a set of sellers indexed by I . The set I is either a finite set $\{1, \dots, n\}$, or a non atomic measure space with mass n . I refer the model as the finite household model when I is finite, and as the large household model when I is infinite. (In the large household model, the mass n can always be normalized as unity. This general notation helps simplify exposition below.)

There are $K \geq 3$ types of produced and perishable goods. For a type- k household, its buyers only consume type- k good; its sellers only produce type- $k + 1$ good; and the household period utility is

$$\int_{i \in I} u(q_{ib}) di - \int_{i \in I} c(q_{is}) di,$$

where q_{ib} is the consumption of its buyer i , and q_{is} is the production of its seller i .¹ The household maximizes expected discounted utility with discount factor $\beta \in (0, 1)$. As is standard, u is bounded, $u' > 0$, $u'' < 0$, $u(0) = 0$, and $u'(0) = \infty$; and it is without loss of generality to set $c(q) = q$.

There is another durable and intrinsically useless object called money. The per household money holding is fixed at unity. Each household evenly distributes money among its sellers.²

¹An alternative assumption is that buyers pool goods together after search, and the household's utility from consuming q is $u(q)$. It is straightforward to adapt the formulations given below for this assumption. Also, results in Propositions 1-3 below hold for adapted formulations.

²An alternative assumption is that the household can distribute money arbitrarily among its agents. As long as the household's choice of money distribution is common knowledge in meetings, it is straightforward to adapt the formulations given below for this assumption. Also, results in Propositions 1-3 below hold for adapted formulations.

At each date, agents from households are randomly matched in pairs, but in a way that makes each meeting a meeting between a seller who produces type- k good and a buyer who consumes type- k good. That is, each meeting is a single-coincidence meeting.

In each meeting, each agent's money holding and his household's start-of-date money holding are common knowledge. These common knowledge assumptions permit me to avoid dealing with asymmetric information.

Throughout, any candidate equilibrium is *symmetric*: symmetry is across specialization types, and across buyers of a given household and sellers of a given household. By symmetry and by the initial distribution of money, all households hold one unit of money at the start of each date, so the equilibrium is *degenerate*. However, in order to analyze the consequences for a household of a deviation, it is necessary to describe the value to a household of starting with an arbitrary money holding. I do that using recursive techniques. In what follows, I refer to a household with 1 as a *regular* household, an agent from a regular household as a regular agent, and a meeting between two regular agents as a regular meeting. Furthermore, any candidate equilibrium is *stationary*, *monetary*, and *binding*, and with a continuous, weakly increasing, and weakly concave value function v . Stationary is self explanatory; monetary means the regular-meeting output is positive; and binding means the regular-meeting transfer of money equals the buyer's money holding.

3 The no-commitment approach

In this section, I assume that households are not committed to pre-search plans; instead, each agent makes his own decision in a meeting. Following Shi [5] and Rauch [4], I assume generalized Nash bargaining in meetings.³

3.1 Equilibrium definition

As indicated above, I shall describe the expected lifetime discounted utility of an arbitrary household with an arbitrary money holding $x—v(x)$. Since each agent makes his own decision in meeting, he ought to evaluate each feasible trade. Here, I define the payoff of a trade to an agent as the *additional* or *marginal* contribution of the trade to the household's lifetime expected

³Although generalized Nash bargaining does not explicitly describe the agent's decision making, as is well known, its solution can be interpreted as the limit of equilibrium outcomes of some game which explicitly describes the agent's decision making.

utility, taking as given other trade outcomes obtained by other agents from the same household when meeting regular agents.

Throughout, I denote a trade by (q, l) , where q is the transfer of the good and l is the transfer of money. In this section, I denote the trade in a meeting between a buyer from the household with x and a regular seller by $(q_b(x), l_b(x))$, and the trade in a meeting between a seller from the same household and a regular seller by $(q_s(x), l_s(x))$.

In the finite household model, when a buyer from the household with x acquires l in a meeting, his household ends up with $x + nl_s(x) - (n-1)l_b(x) - l$ with probability one, for each of other $n-1$ buyers from the household transfers $l_b(x)$ and each of n sellers from the household acquires $l_s(x)$ in meeting regular agents; therefore, the payoff of trade (q, l) to the buyer is

$$\Pi_b(q, l, x) = u(q) + \beta v[x + nl_s(x) - (n-1)l_b(x) - l]. \quad (1)$$

Similarly, the payoff of trade (q, l) to a seller from the household with x is

$$\Pi_s(q, l, x) = -q + \beta v[x + (n-1)l_s(x) - nl_b(x) + l]. \quad (2)$$

In the large household model, analogously, the payoff of trade (q, l) to a buyer from the household with x is

$$\Pi_b(q, l, x) = u(q) + (x-l)\beta v'[x + nl_s(x) - nl_b(x)], \quad (3)$$

and the payoff of trade (q, l) to a seller from the household with x is

$$\Pi_s(q, l, x) = -q + l\beta v'[x + nl_s(x) - nl_b(x)]. \quad (4)$$

Here, because of *large*, the marginal payoff of money to an agent is $\beta v'(x_+)$ when the household's end-of-match money holding is x_+ ; as a critical implication, the payoff functions $\Pi_b(\cdot, \cdot, x)$ and $\Pi_s(\cdot, \cdot, x)$ are quasi linear.

In both models, letting the buyer's bargaining power be denoted by θ , then $(q_b(x), l_b(x))$ must satisfy

$$\begin{aligned} & (q_b(x), l_b(x)) \\ \in & \arg \max_{q \geq 0, 0 \leq l \leq x/n} [\Pi_b(q, l, x) - \Pi_b(0, 0, x)]^\theta [\Pi_s(q, l, 1) - \Pi_s(0, 0, 1)]^{1-\theta}; \end{aligned} \quad (5)$$

this is because when (q, l) is the trade between a buyer from the household with x and a regular seller, $\Pi_b(q, l, x) - \Pi_b(0, 0, x)$ is the buyer's surplus, and

$\Pi_s(q, l, 1) - \Pi_s(0, 0, 1)$ is the seller's surplus. Similarly, $(q_s(x), l_s(x))$ must satisfy

$$(q_s(x), l_s(x)) \in \arg \max_{q \geq 0, 0 \leq l \leq 1/n} [\Pi_b(q, l, 1) - \Pi_b(0, 0, 1)]^\theta [\Pi_s(q, l, x) - \Pi_s(0, 0, x)]^{1-\theta}. \quad (6)$$

In turn, the value function v must satisfy

$$v(x) = nu[q_b(x)] - nq_s(x) + \beta v[x + nl_s(x) - nl_b(x)]. \quad (7)$$

Finally, bindingness requires

$$nl_b(1) = nl_s(1) = 1. \quad (8)$$

Therefore, I have the following definitions.

Definition 1 *In the finite household model, a no-commitment equilibrium is a value function v (continuous, weakly increasing, weakly concave) on \mathbb{R}_+ , and a collection of functions (q_b, l_b, q_s, l_s) on \mathbb{R}_+ , that satisfy (1), (2), and (5)-(8). In the large household model, a no-commitment equilibrium is a value function v (continuous, weakly increasing, weakly concave) on \mathbb{R}_+ , and a collection of functions (q_b, l_b, q_s, l_s) on \mathbb{R}_+ , that satisfy (3), (4), and (5)-(8).*

3.2 Comparison to the literature

Aside from details, Shi [5] and Rauch [4] share all the important assumptions of the environment, including the common knowledge assumptions. Shi [5], who initiated the use of the large household model for money applications, describes the household's problem in terms of sequences of the household's choices. In his formulation, each household takes as given that the regular-meeting trade is the trade that its buyers and sellers will make—independent of the household's start-of-date money holding. However, such trade is not feasible for a household with $x < 1$, which leaves $v(x)$ for $x < 1$ undefined. It also implies that $v(x) = v(1)$ for $x \geq 1$. As Rauch [4] points out in a comment on [5], neither is satisfactory. He proposes an alternative formulation.

In Rauch's formulation, each agent's action is a function of his household's end-of-match money holdings. Because of *large*, this gets around the issue that the payoff of a trade to one agent depends on the trade outcomes of other agents from the same household. Such treatment looks awkward, though;

moreover, it does not work in the finite household model, for the household's end-of-match money holding depends on the agent's action.

There is another problem in Rauch's formulation. Rauch describes the household problem in terms of sequences of the household's choices. But to define the payoff of an arbitrary sequence, Rauch uses the Lagrangian multipliers that are associated with an optimal sequence, which makes the household's problem ill-defined. This problem can be avoided by using a recursive approach and introducing a value function.

3.3 Main results

I present two results here: non existence of a no-commitment equilibrium in the finite household model with $n = 1$, and existence in the large household model.

First, I give the non-existence result.

Proposition 1 *In the finite household model with $n = 1$, there does not exist a no-commitment equilibrium*

Proof. Suppose (v, q_b, l_b, q_s, l_s) is a no-commitment equilibrium. Let $\hat{q} \equiv q_b(1)$. Setting $x = 1$ in (5), and by (8) and $\hat{q} > 0$ (the equilibrium is monetary), we have $v(1) > v(0)$. So v is strictly increasing over $[0, \bar{x}]$, where $\bar{x} \equiv \min\{x \leq 1 : v(x) = v(1)\} > 0$. Then, by this strict monotonicity, $u'(0) = \infty$, (5), and $l_b(1) = 1$ (see (8)), we have

$$\begin{aligned} & (1 - \theta)\{u[q_b(x)] + \beta v[x + l_s(x) - l_b(x)] - \beta v[x + l_s(x)]\} \\ & = \theta u'[q_b(x)]\{-q_b(x) + \beta v[l_b(x)] - \beta v(0)\}, \quad \forall x > 0. \end{aligned} \quad (9)$$

Next, we **claim** $\exists z \in (0, \bar{x}]$ s.t. $\forall x \leq z$, $l_b(x) = x$, $l_s(x) \in [\bar{x}, 1]$, and

$$v(x) = u[q_b(x)] - \hat{q} + \beta v(1). \quad (10)$$

The proof of the claim is by the standard argument that exploits concavity and monotonicity of u and v , and it is delegated to the appendix.

Now fix $x \in (0, z]$. By (10), $q_b(0) = 0$ (see (5)), and $u(0) = 0$, we have

$$v(x) - v(0) = u[q_b(x)]. \quad (11)$$

By (9), monotonicity of v , $l_b(x) = x$, and (11), we have

$$(1 - \theta)u[q_b(x)] \geq \theta u'[q_b(x)]\{-q_b(x) + \beta u[q_b(x)]\}. \quad (12)$$

By (5) and continuity of v , as $x \rightarrow 0$, $q_b(x) \rightarrow 0$, and therefore, $u'[q_b(x)] \rightarrow \infty$ and $q_b(x)/u[q_b(x)] \rightarrow 0$. But then (12) cannot hold as $x \rightarrow 0$. ■

Next, I turn to existence in the large household model. I shall construct an equilibrium in which $l_b(x) = x$ and $l_s(x) = 1$ as x is not too large. Some components of this equilibrium are in the next lemma.

Lemma 1 *If \tilde{q} satisfies $u(\tilde{q}) > \tilde{q}$ and*

$$u'(\tilde{q})[\theta u'(\tilde{q}) + (1 - \theta)][1 + \beta\theta - \beta\theta u'(\tilde{q}) - \beta] = (1 - \theta)\theta u''(\tilde{q})[u(\tilde{q}) - \tilde{q}], \quad (13)$$

then let

$$\omega = \frac{\theta u'(\tilde{q})\tilde{q} + (1 - \theta)u(\tilde{q})}{\theta u'(\tilde{q}) + (1 - \theta)}, \quad (14)$$

and then $\forall x \geq 0$, let

$$q(x) = \arg \max_{q \geq 0} [u(q) - \omega x]^\theta [-q(x) + \omega x]^{1-\theta}. \quad (15)$$

(i) $u'(\tilde{q}) > 1$. (ii) $x \mapsto q(x)$ is strictly increasing. (iii) $x \mapsto q(x)$ is differentiable at $x > 0$. (iv) $q(1) = \tilde{q}$. (v) $\omega = \beta u'[q(1)]q'(1)$.

Proof. Parts (i) and (ii) are obvious. As $x > 0$, $q(x)$ satisfies

$$\theta u'[q(x)]\{-q(x) + \omega x\} = (1 - \theta)\{u[q(x)] - \omega x\}. \quad (16)$$

Now the implicit function theorem implies part (iii). Setting $x = 1$ in (16) and comparing it with (14) gives part (iv). Then differentiating (16) gives

$$[-\theta u''(\tilde{q})(-\tilde{q} + \omega) + u'(\tilde{q})]q'(1) = \omega[\theta u'(\tilde{q}) + (1 - \theta)].$$

This, (13), and (14) imply part (v). ■

For the reason to be clear soon, in my proof, I need a *nonbinding upper bound* on the household's money holdings. The equilibrium with such a bound in the large household model is a number $Z > 1$, a value function v (continuous, weakly increasing, weakly concave) on $[0, Z]$, and a collection of functions (q_b, l_b, q_s, l_s) on $[0, Z]$, that satisfy (3), (4), and (5)-(8).

Also, I need the following assumptions about u ,

(A1) $\exists \tilde{q}$ satisfying $u(\tilde{q}) > \tilde{q}$ and (13);

(A2) $\exists X > 1$ s.t. $u[q(\cdot)] : [0, X] \rightarrow \mathbb{R}_+$ is strictly concave.

As $\theta = 1$, (A1) and (A2) are guaranteed by assumptions about u given in Section 2. As $\theta < 1$, (A1) and (A2) can be satisfied by some familiar functions, e.g., $u(q) = q^\delta \forall q \in [0, 1]$ and with $0 < \delta \leq 0.5$; also, (A1) and (A2) are satisfied if $u''u'' > u'u'''$.⁴

Now I can show the existence result.

Proposition 2 *Suppose (A1) and (A2) hold. In the large household model, there exists a no-commitment equilibrium with a nonbinding upper bound on the household's money holdings .*

Proof. Without loss of generality, let the mass n be normalized as 1. Fix \tilde{q} satisfying $u(\tilde{q}) > \tilde{q}$ and (13), and let ω and $q(x)$ be defined as in Lemma 1. By Lemma 1 (i)-(iv), $\exists Z \in (1, X)$ s.t. $u'[q(x)] \geq 1 \forall x \leq Z$. Fix such Z , and let (v, q_b, l_b, q_s, l_s) on $[0, Z]$ be defined by

$$v(x) = u[q(x)] - q(1) + \beta v(1), \quad (17)$$

$$(q_b(x), l_b(x)) = (q(x), x), \quad (18)$$

$$(q_s(x), l_s(x)) = (q(1), 1). \quad (19)$$

Now we verify Z and functions (v, q_b, l_b, q_s, l_s) in (17)-(19) constitute a no-commitment equilibrium. First, those functions satisfy (7) and (8). Next, by Lemma 1 (ii) and (iii) and by (A2), v is strictly increasing, strictly concave and differentiable. Next, by Lemma 1 (v), $\beta v'(1) = \omega$. Given this and functions (q_b, l_b, q_s, l_s) , by (15) and $u'[q(x)] \geq 1$, $(q(x), x)$ solves the problem in (5) with $x \leq Z$, and by (15) and $u'[q(1)] > 1$, $(q(1), 1)$ solves the problem in (6) with $x \leq Z$. ■

The role of *large* can be seen from proofs of Propositions 1 and 2. In the finite household model, (8) implies $l_b(x) = x$ and $v[g(x)] = v(1)$ for x in a closed interval, where $g(x) \equiv x + l_s(x) - l_b(x)$. This, in turn, implies that $q_b(x)$ depends on $v(x) - v(0)$ (see (10)), and, in particular, the period return for the household with x depends on $v(x)$, which leads to the contradiction. In the large household model, the agent's payoff functions in (3)-(4) are quasi linear, so $q_b(x)$ (also $q_s(x)$) depends on $v'[g(x)]$ but not on any other term related to v ; in particular, when $l_b(x) = x$ and $g(x) = 1$, the period return for the household with x depends on $v'(1)$ but not on $v(x)$.

⁴As $\theta < 1$, Lagos and Wright [2] also appeal to $u''u'' > u'u'''$ in their existence proof. See the last section for the connection between the present model and the Lagos-Wright model.

The role of the bound Z should also be clear. In general, $g(x)$ depends on x , and if it does, then little can be said about the relationship between $q_b(x)$ (also $q_s(x)$) and $v'[g(x)]$. The bound Z is constructed so that $g(x)$ does not depend on x ; in particular, it is constructed so that $g(x) = 1 \forall x \leq Z$.

Two remarks on Proposition 1 are in order. First, it is straightforward to show Proposition 1 holds if there is a nonbinding upper bound on the household's money holdings; notice that, to be nonbinding, now any such upper bound should be no less than 2. Second, Proposition 1 resembles a result in Wallace and Zhu [8, section 2]. They study a model in which each household consists of one agent, each meeting is a single-coincidence meeting, and $\theta = 1$. They show that if β is close to unity, then there does not exist any stationary, degenerate, and monetary equilibrium. Proposition 1 is weaker than that result for it only rules out any such equilibrium with a continuous and concave value function (but it is stronger for it deals with general θ and β); in the proof of Proposition 1, continuity and concavity over $[0, 2]$ are used to establish the claim, and continuity is used to draw the final contradiction.

4 The commitment approach

In this section, I assume that households are committed to pre-search plans; that is, each household chooses a binding contingent plan for all its buyers and sellers prior to meetings. Following Shi [6], I assume that buyers make take-it-or-leave-it offers in meetings.

4.1 Equilibrium

I start by describing an arbitrary plan made by an arbitrary household with arbitrary money holding x . Such a plan, denoted $p_x \equiv (\sigma_x, \lambda_x)$, is a contingent plan. When a buyer from the household with x meets a seller whose household money holding is y , the buyer's contingency is y ; by the plan, the buyer makes an offer $\sigma_x(y)$. When a seller from the household with x meets a buyer whose household money holding is y and the buyer makes an offer σ , the seller's contingency is (y, σ) ; by the plan, the seller makes an acceptance-rejection action $\lambda_x(y, \sigma)$.

To avoid non convexity, I allow stochastic offers and responses. So an offer is a probability measure over the set of feasible trades, where feasibility means the transfer of money does not exceed the buyer's money holding. Also, an acceptance-rejection action is a probability to accept a relevant offer, and, in

particular, $\lambda_x(y, \sigma)$ is the probability to accept the offer σ made by a buyer whose household's money holding is y .

Given the plan made by regular households, denoted $p_1^* \equiv (\sigma_1^*, \lambda_1^*)$, p_x induces a distribution of realizations of $(q_{ib}, l_{ib}, q_{is}, l_{is})_{i \in I}$, where (q_{ib}, l_{ib}) is the trade between buyer i from the household with x and a regular seller, and (q_{is}, l_{is}) is the trade between seller i from the household with x and a regular buyer. Letting $\pi(p_x, p_1^*)$ denote this distribution, and letting $E_{\pi(p_x, p_1^*)}$ stand for the expectation over the distribution $\pi(p_x, p_1^*)$, the payoff of p_x to the household with x is

$$f(p_x, p_1^*) = E_{\pi(p_x, p_1^*)}[\int u(q_{ib})di - \int q_{is}di + \beta v(x - \int l_{ib}di + \int l_{is}di)]. \quad (20)$$

It follows that

$$v(x) = \max_{p_x} f(p_x, p_1^*). \quad (21)$$

Bindingness requires

$$\pi(p_1^*, p_1^*)\{(q_{ib}, l_{ib}, q_{is}, l_{is})_{i \in I} : \int l_{ib}di = \int l_{is}di = 1\} = 1. \quad (22)$$

As is obvious, (21) imposes dependence of $p_x^* \equiv (\sigma_x^*, \lambda_x^*)$ on p_1^* , $\forall x \geq 0$. But (21) does not impose any dependence of p_1^* on $p_x^* \forall x \neq 1$. In fact, because the payoff of a plan is computed before matching and equilibrium is degenerate, any p_1 is a best response to $p_x^* \forall x \neq 1$. To strength the equilibrium concept, I introduce two constraints on p_1^* .

In the finite household model, the first constraint is

$$\sigma_1^*(x) \in \arg \max_{\sigma} \lambda_x^*(1, \sigma) E_{\sigma}[u(q) + \beta v(1 + 1/n - l)]; \quad (23)$$

that is, for a regular household, when one of its buyers meets a seller from a household with x , the buyer's offer $\sigma_1^*(x)$ is a best response to the seller's acceptance-rejection action dictated by $\lambda_x^*(1, \cdot)$, taking as given that other agents from the regular household are in regular meetings. The second constraint is

$$\lambda_1^*[x, \sigma_x^*(1)] \in \arg \max_{\lambda} \lambda E_{\sigma_x^*(1)}[-q + \beta v(1 - 1/n + l)]; \quad (24)$$

that is, for the regular household, when one of its sellers meets a buyer from a household with x , the seller's acceptance-rejection action $\lambda_1^*[x, \sigma_x^*(1)]$ is a best response to the buyer's offer $\sigma_x^*(1)$, taking as given that other agents from the regular household are in regular meetings.

In the large household model, the analogous constraints are

$$\sigma_1^*(x) \in \arg \max_{\sigma} \lambda_x^*(1, \sigma) E_{\sigma} [u(q) - l\beta v'(1)], \quad (25)$$

and

$$\lambda_1^*[x, \sigma_x^*(1)] \in \arg \max_{\lambda} \lambda E_{\sigma_x^*(1)} [-q + l\beta v'(1)]. \quad (26)$$

Therefore, I have the following definitions.

Definition 2 *In the finite household model, a commitment equilibrium is a value function v (continuous, weakly increasing, weakly concave) on \mathbb{R}_+ , and plans $p_x^* = (\sigma_x^*, \lambda_x^*) \forall x \geq 0$, that satisfy (20)-(22), (23), and (24). In the large household model, a commitment equilibrium is a value function v (continuous, weakly increasing, weakly concave) on \mathbb{R}_+ , and plans $p_x^* = (\sigma_x^*, \lambda_x^*) \forall x \geq 0$, that satisfy (20)-(22), (25), and (26).*

4.2 Comparison to the literature

Shi [6] initiates the commitment approach. In [6], there is no explicit description of contingencies and no analogue of (25), but there is a special version of (26). In that version, in a meeting between a regular seller and a buyer from a household with x , the seller accepts any offer whose payoff to the regular household is no worse than no trade, taking as given that other agents from the regular household are in regular meetings. Therefore, the regular-meeting output in any equilibrium must be equal to $\beta v'(1)$ (compare this with Proposition 4 below).

Aside from details, when the mass n is normalized as 1, the purported equilibrium in [6] has the value function

$$v(x) = \max_{0 \leq l \leq x, 0 \leq \rho \leq 1} u(l\bar{q}) - \rho\bar{q} + \beta v(x + \rho - l), \quad (27)$$

where $\beta u'(\bar{q}) = 1$. It follows that $\beta v'(1) = \bar{q}$. Letting $(l(x), \rho(x))$ be the optimal solution to the maximization problem in (27), the plans in the purported equilibrium are: (a) a regular seller accepts an offer (q, l) if and only if $q \leq l\bar{q}$, and a buyer from the household with x offers $(l(x)\bar{q}, l(x))$ to a regular seller; and (b) a regular buyer offers $(\bar{q}, 1)$ to a seller from the household with x , and a seller from the household with x accepts $(\bar{q}, 1)$ from a regular buyer with probability one.

This purported equilibrium has a defect, though. Given the offer and acceptance-rejection action of regular agents in (a) and (b), the household with x should let each of its sellers accept $(\bar{q}, 1)$ with probability $\rho(x)$. This is also consistent with the way that the value function is described. It can be shown that $\rho(x) < 1$ if x is sufficiently large. Therefore, it is not optimal for the household with x to choose for its sellers the acceptance-rejection action in (b).

4.3 Main results

I present two results here: existence of a continuum of commitment equilibria in the finite household model with $n = 1$, and existence of a continuum of commitment equilibria in the large household model.

Proposition 3 *Let $\beta u'(\bar{q}) = 1$ and let $\hat{q} \in (0, \bar{q}]$. In the finite household model with $n = 1$, there exists a commitment equilibrium in which \hat{q} is the regular-meeting output and $\hat{q} < \beta v(1) - \beta v(0)$.*

Proof. See the appendix. ■

Proposition 4 *Let $\beta u'(\bar{q}) = 1$ and let $\hat{q} \in (0, \bar{q}]$. In the large household model, there exists a commitment equilibrium in which \hat{q} is the regular-meeting output and $\hat{q} < \beta v'(1)$ as $\hat{q} < \bar{q}$.*

Proof. Without loss of generality, let the mass n be normalized as 1. Fix \hat{q} and we first construct a candidate equilibrium. Let v be defined by

$$v(x) = \max_{0 \leq l \leq x, 0 \leq \rho \leq 1} u(l\hat{q}) - \rho\hat{q} + \beta v(x + \rho - l). \quad (28)$$

It is standard to show there exists a unique v satisfying (28) and v is strictly increasing, strict concave, and differentiable. Also, there is a unique solution, denoted $(l(x), \rho(x))$, to the maximization problem in (28). By $\beta u'(\hat{q}) \geq 1$, we have $\rho(1) = l(1) = 1$ and

$$v'(1) = u'(\hat{q})\hat{q}. \quad (29)$$

Regarding p_1^* , let the offer made by a regular buyer to a seller from a household with x be

$$\sigma_1^*(x)\{(q, l) : q = \rho(x)\hat{q}, l = \rho(x)\} = 1 \quad \forall x; \quad (30)$$

that is, the buyer asks $\rho(x)\hat{q}$ units of good for $\rho(x)$ units of money (or, the support of $\sigma_1^*(x)$ is the singleton set $\{(\rho(x)\hat{q}, \rho(x))\}$). For a feasible offer σ made by a buyer from a household with x (the support of σ is a subset of $\{(q, l) : l \leq x\}$), let the action by a regular seller be

$$\lambda_1^*(x, \sigma) = 1 \text{ if } \sigma\{(q, l) : q \leq l\hat{q}, l = l(x)\} = 1, \lambda_1^*(x, \sigma) = 0 \text{ otherwise; } \quad (31)$$

that is, the seller accepts σ if and only if the implied price of money is no greater than \hat{q} and the transfer of money is $l(x)$.

Regarding p_x^* for general x , we only describe $\sigma_x^*(1)$ and $\lambda_x^*(1, \sigma)$. Let the offer made by a buyer from a household with x to a regular seller be

$$\sigma_x^*(1)\{(q, l) : q = l(x)\hat{q}, l = l(x)\} = 1; \quad (32)$$

that is, the buyer asks $l(x)\hat{q}$ units of good for $l(x)$ units of money. For a feasible offer σ made by a regular buyer (the support of σ is a subset of $\{(q, l) : l \leq 1\}$), let the action by a seller from a household with x be

$$\lambda_x^*(1, \sigma) = 1 \text{ if } \sigma\{(q, l) : q \leq l\hat{q}, l = \rho(x)\} = 1, \lambda_x^*(1, \sigma) = 0 \text{ otherwise; } \quad (33)$$

that is, the seller accepts σ if and only if the implied price of money is no greater than \hat{q} and the transfer of money is $\rho(x)$.

Now we verify that v in (28) and p_x^* in (30)-(33) constitute a commitment equilibrium. First, by $\rho(1) = 1$, p_1^* satisfies (22). Next, given λ_x^* and (29), $\sigma_1^*(x)$ solves the problem in (25). Next, given σ_x^* and (29), by $\beta u'(\hat{q}) \geq 1$, $\lambda_1^*[x, \sigma_x^*(1)]$ solves the problem in (26). Next, as is clear, $v(x) = f(p_x^*, p_1^*)$. Finally, given p_1^* and v , by the fact that $(l(x), \rho(x))$ solves the problem in (28), p_x^* solves the problem in (21). ■

The role of commitment can be seen from proofs of Propositions 3 and 4. With commitment, the surplus split in each pairwise meeting is essentially conducted by a Nash demand game (note, in particular, the buyer need not have all the bargaining power in the meeting, though he makes a take-it-or-leave-it offer). This permits a variety of linear pricing—see (30)-(33) for the large household model, and (43)-(46) in the appendix for the finite household model; any such linear pricing makes degeneracy an equilibrium.

It shall not be surprising that in models where a household consists of one agent, with commitment, degeneracy can be equilibrium as long as it is feasible; in fact, it is straightforward to establish a version of Proposition 3 in the model studied by Wallace and Zhu [8, section 2].

5 Concluding remarks

In the no-commitment formulation, the roles of quasi-linearity of the individual agent's payoff functions and the bound Z are similar to their roles in the Lagos-Wright model [2]. In [2], agents trade in a centralized market after random matching, and preferences over centralized-trade goods are quasi-linear. For an internal solution in the centralized market, the agent must enter the centralized market with money holdings that are not too large. In that case, the assumed quasi-linear preferences imply that the value function for the agent's end-of-match money holdings is affine, and that, in turn, implies that in a pairwise meeting, the buyer and seller payoff functions are quasi linear, linear in end-of-match money holdings. Moreover, those functions have the same linear coefficient, provided that the sum of the buyer and seller money holdings is consistent with an internal solution in the centralized market.

My description of the household's problem in the no-commitment approach can be applied to the labor search model of Merz [3]. It can also be adapted to describe the large firm's decision problem in the labor search literature. The large firm has many job positions, and the wage in each position is determined by bargaining with a worker. But in the literature (see Pissarides [7, Ch 3.1]), the firm takes the prevailing wage as given. This seems problematic. Instead, following the approach used above, it could be assumed that the wage in each position is determined by bargaining between a firm's agent and a worker, while taking as given the bargaining outcome between other agents of the firm and workers.

Finally, a natural way to refine the commitment equilibrium—a seemingly weak equilibrium concept—is to require that each household's plan is optimal in each contingency (or, each equilibrium strategy is subgame perfect). After all, this is consistent with the formulation adopted by most search models in which a household consists of one agent. Of course, doing so leads to the no-commitment equilibrium.

Appendix

Completion of the proof of Proposition 1

Proof. Here we prove the claim in the main text. First, by the strict monotonicity of v over $[0, \bar{x}]$, $u'(0) = \infty$, (6), and $l_s(1) = 1$ (see (8)), we have

$$\begin{aligned} & (1 - \theta)\{u[q_s(x)] + \beta v[2 - l_s(x)] - \beta v(2)\} \\ &= \theta u'[q_s(x)]\{-q_s(x) + \beta v[x + l_s(x) - l_b(x)] - \beta v[x - l_b(x)]\}. \end{aligned} \quad (34)$$

Because $q_s(1) = \hat{q}$, setting $x = 1$ in (34) and by (8), we have

$$(1 - \theta)[u(\hat{q}) + \beta v(1) - \beta v(2)] = \theta u'(\hat{q})[-\hat{q} + \beta v(1) - \beta v(0)]. \quad (35)$$

Comparing (34) and (35), we have

$$q_s(x) \leq \hat{q} \text{ and strict if } l_s(x) < \bar{x}. \quad (36)$$

Next, let the right derivative of v at $x \geq 0$ and left derivative of v at $x > 0$ be denoted by $v'_+(x)$ and $v'_-(x)$, respectively. By concavity of v , those derivatives are well defined; $v_-(x_1) \geq v_-(x_2) \geq v_+(x_2)$ if $x_1 < x_2$; and $v'_+(x) > 0$ if $x < \bar{x}$. Setting $x = 1$ in (6) and by (8), we have

$$u'(\hat{q})v'_-(1) \geq v'_+(1). \quad (37)$$

Now we are ready to prove the claim. By (5) and continuity of v , $q_b(x) \rightarrow 0$ as $x \rightarrow 0$. So given $u'(0) > 1$, $\exists z < \bar{x}$ s.t. $u'[q_b(x)] > 1 \forall x \leq z$. Fix such z and fix $x \leq z$. If $l_s(x) < \bar{x}$, then by (6) and $l_s(1) = 1$, we have

$$u'[q_s(x)]v'_+[x + l_s(x) - l_b(x)] \leq v'_-[2 - l_s(x)]. \quad (38)$$

Comparing (37) and (38), and using $l_s(x) < \bar{x}$ and (36), we have

$$x + l_s(x) - l_b(x) \geq \bar{x}, \quad (39)$$

and hence $l_b(x) < x$. By (5), $l_b(x) < x$ implies

$$u'[q_b(x)]v'_+[l_b(x)] \leq v'_-[x + l_s(x) - l_b(x)], \quad (40)$$

but since $u'[q_b(x)] > 1$, $l_b(x) < x$, (39), and (40) are incompatible. So it must be $l_s(x) \in [\bar{x}, 1]$; now if $l_b(x) < x$, again we have (39) and (40), so it must be $l_b(x) = x$. Then comparing (34) and (35), we have $q_s(x) = \hat{q}$ and (10). ■

The proof of Proposition 3

Proof. The proof is similar to the proof of Proposition 4 (some minor difference is to deal with issues from finiteness of the household). Fix \hat{q} . First, let v be defined by

$$v(x) = u(x\hat{q}) - \hat{q} + \beta v(1) \text{ if } x \leq 1 \text{ and } v(x) = v(1) \text{ if } x > 1. \quad (41)$$

Let $\hat{x} \equiv \min\{x, 1\}$, and it follows that

$$v(x) - v(0) = u(\hat{x}\hat{q}). \quad (42)$$

Regarding p_1^* , let

$$\sigma_1^*(x)\{(q, l) : q = \hat{q}, l = 1\} = 1 \quad \forall x, \quad (43)$$

and $\forall (x, \sigma)$ with $\sigma\{(q, l) : l \leq x\} = 1$, let

$$\lambda_1^*(x, \sigma) = 1 \text{ if } \sigma_x\{(q, l) : q \leq \hat{x}\hat{q}, l = x\} = 1, \quad \lambda_1^*(x, \sigma) = 0 \text{ otherwise.} \quad (44)$$

Regarding p_x^* for general x , we only describe $\sigma_x^*(1)$ and $\lambda_x^*(1, \sigma)$. Let

$$\sigma_x^*(1)\{(q, l) : q = \hat{x}\hat{q}, l = x\} = 1, \quad (45)$$

and $\forall (1, \sigma)$ with $\sigma\{(q, l) : l \leq 1\} = 1$, let

$$\lambda_x^*(1, \sigma) = 1 \text{ if } \sigma\{(q, l) : q \leq l\hat{q}, l = 1\} = 1, \quad \lambda_x^*(1, \sigma) = 0 \text{ otherwise.} \quad (46)$$

Now we verify that v in (41) and p_x^* in (43)-(46) constitute a commitment equilibrium. First, p_1^* satisfies (22). Next, given λ_x^* and (42), $\sigma_1^*(x)$ solves the problem in (23). Next, given σ_x^* and (42), by $\beta u(\hat{q}) > \hat{q}$ (recall $\beta u'(\hat{q}) \geq 1$), $\lambda_1^*[x, \sigma_x^*(1)]$ solves the problem in (24). Next, as is clear, $v(x) = f(p_x^*, p_1^*)$. Finally, we show that given p_1^* and v , p_x^* solves the problem in (21). It suffices to show $f(p_x^*, p_1^*) \geq f(p_x, p_1^*)$ for p_x in which μ is the probability that the buyer offers $(\hat{x}\hat{q}, x)$ to a regular seller, $1 - \mu$ is the probability that the buyer offers $(0, 0)$ to a regular seller, and ρ is the probability that the seller accepts $\sigma_1^*(x)$ from a regular buyer. So it suffices to show

$$(1, 1) \in \arg \max_{0 \leq \mu \leq 1, 0 \leq \rho \leq 1} \mu \rho [u(\hat{x}\hat{q}) - \hat{q} + \beta v(1)] + (1 - \mu) \rho [-\hat{q} + \beta v(1 + x)] \\ + \mu(1 - \rho) [u(\hat{x}\hat{q}) + \beta v(0)] + (1 - \mu)(1 - \rho) \beta v(x).$$

Notice that $\mu = 1$ is optimal as $x = 0$, and that by (42), $\mu = 1$ is optimal as $x > 0$. Then by $v(1 + x) = v(1)$, (42), and $\beta u(\hat{q}) > \hat{q}$, $\rho = 1$ is optimal. ■

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