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A Simulation Estimator for Testing the Time Homogeneity of Credit Rating Transitions

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A Simulation Estimator for Testing the Time Homogeneity of Credit Rating Transitions*

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Abstract

The measurement of credit quality is at the heart of the models designed to assess the reserves and capital needed to support the risks of both individual credits and portfolios of credit instruments. A popular specification for credit-rating transitions is the simple, time-homogeneous Markov model. While the Markov specification cannot really describe processes in the long run, it may be useful for adequately describing short-run changes in portfolio risk.

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In this specification, the entire stochastic process can be characterized in terms of estimated transition probabilities. However, the simple homogeneous Markovian transition framework is restrictive. We propose a test of the null hypotheses of time-homogeneity that can be performed on the sorts of data often reported. We apply the tests to 4 data sets, on commercial paper, sovereign debt, municipal bonds and S&P Corporates. The results indicate that commercial paper looks Markovian on a 30-day time scale for up to 6 months; sovereign debt also looks Markovian (perhaps due to a small sample size); municipals are well-modeled by the Markov specification for up to 5 years, but could probably benefit from frequent updating of the estimated transition matrix or from more sophisticated modeling, and S&P Corporate ratings are approximately Markov over 3 transitions but not 4.

JEL Classification: G12,G21,G32,C12,C15,C52

Keywords: Ratings transitions; Risk measurement; Indirect inference; Specification testing; Risk dynamics

1 Introduction

The measurement of levels and changes in credit quality is at the heart of the development of a whole spectrum of models designed to assess and support the dynamic management of credit risky assets. Several of these models are specifically designed to estimate the reserves and capital needed to support the risks of both individual credits and portfolios of credit instruments as functions of credit quality. Credit quality can be expressed in a variety of ways, but it is increasingly common to have credits assigned one or more ratings that summarize, over some specified horizon, a credit's probability of default, rate of loss given default, or both. For example, the proposed Basel II agreement (2004) requires institutions to rate assets by their 1-year probability of default and by their expected loss severity given default.

Banks, supervisors and other institutions are relying upon these systems to produce accurate, stable, representations of the risks of credit loss for current and future populations of similar credit exposures. Financial institutions use this information in portfolio selection. Historical information on the transition of credit exposures from one quality level, or rating, to another is used to estimate various models that describe the probabilistic evolution of credit quality.

A popular specification is the simple, time-homogeneous Markov model. With this specification, the stochastic processes can be specified completely in terms of transition probabilities. These correspond nicely to summary data that are often available and reported (though not without problems, as we note below). In particular, details on the history of individual assets are not required under this specification. As an example, we apply our methods to migration matrices for commercial paper over horizons of 30, 60, 90, 120, 180 and 270 days. Suppose we accept the simple Markov chain specification for describing credit rating transitions. Is there anything we can say about how to check the simple Markov structure using the summary data that are commonly published? Important restrictions on credit ratings transitions are implied by the simple Markov specification, and these restrictions suggest methods by which specification tests of the model can be made.

Given the relative ease with which the simple Markov model can be estimated or manipulated, it is not surprising that several practitioners have embedded the Markov framework into their credit quality tracking and risk assessment/management models. However, the adoption of the simple Markovian transition framework is not without cost; restrictive assumptions on the nature of credit transitions are required for the validity of this simple model. In fact, many of these assumptions are unrealistic, and are likely to be violated by the types of credit transitions considered by practitioners. Therefore, a diagnostic test for the validity of the simple Markov chain model specification would be a valuable tool to add to the credit risk modeler's toolkit. In the next section, we illustrate how such a test can be constructed, and we discuss its statistical properties.

We consider testing time-homogeneity. This is not required by the general Markov specification, but seems to be a featured assumption in practice (and is an assumption that makes much empirical work possible!). Time-homogeneity means that the transition matrix P, whose $ijth$ element is the probability that a loan is in state j next period given it is in i this period, is constant over time. This is a strong assumption. For example it might be thought that these transition probabilities would depend on macroeconomic conditions, or conditions specific to the industrial sector of the loan. Time-homogeneity is sometimes referred to as the property of having "stationary transition probabilities." This is probably bad terminology, as it may cause confusion with stationarity of the stochastic process determined by these transition probabilities. Markov chains are not usually stationary, in the sense that the joint distribution of N successive observations may be different depending on where they are taken. Nevertheless, a test for time homogeneity of ratings transitions is one diagnostic test that can be used to evaluate the adequacy of the simple Markov specification.

This paper proceeds as follows: section two introduces the simple Markov chain model. Section three describes how its parameters might be estimated with the sorts of data frequently available. Section four develops a test of the null hypotheses of time-homogeneity that is implied by the simple Markov specification. The performance of the proposed statistic is considered using a simple Monte Carlo simulation. In Section 5, the test is applied to several ratings transition datasets. Before proceeding, it is useful to put the goals in perspective. First, credit rating transitions cannot "really" be Markovian. To see this, note that a key probability is the transition into default. Default is an absorbing state in any sensible specification. That is, there are no transitions out of default. Strictly speaking, this implies that in the long run, all assets are in default. This is ridiculous. However, this is not the issue. The issue instead is whether the Markovian specification adequately describes credit rating transitions over rather short periods. Our test will allow researchers to determine which assets look Markovian and over what periods.

2 The Markov Chain Model

One relatively simple probability model for ratings transitions that is increasingly being used by financial practitioners is the Markov chain model. In the simple, discrete Markov chain model, the states that a stochastic process X_t may occupy at (discrete) time t form a countable or finite set. It is convenient to label the states by the positive integers and indicate particular values by i, j, k, etc.. We will denote the number of unique states in a finite-dimensional Markov chain by the integer K. The probability of $X_{t+1} = j$, given that $X_t = i$, called a one-step transition probability, is denoted $p_{ij}(t)$. This representation emphasizes that transition probabilities in general will be functions of both the initial and the final states, and the time of transition. When the transition probabilities are independent of the time variable (the usual case in financial applications), the Markov process is said to have stationary transition probabilities. In this case we may write $p_{ij}(t) = p_{ij}$. It will be convenient to use index notation to develop our test statistic. The data we will typically have available consist of summary statistics on multistep transitions. The number of steps will be indexed by $r, s, t, \ldots$. Thus, $p_{ijr} = \Pr(X_r = j | X_0 = i)$. Note that this is different from $p_{ij}(r)$, the one-step transition probability associated with the transition from i to j between periods $r-1$ and r. Although the hypothesis $p_{ij}(r) = p_{ij}(s)$ for all r, s is of interest, we will not use this notation since we do not typically have data on one-step transitions by time index. However, under the null hypothesis that the transition rates are invariant, we have the relation

$$p_{ijr} = p_{ik1} p^{kj(r-1)}. \tag{1}$$

Here we are using the index convention that indices repeated in super and subscripts are summed. Thus $p_{ijr} = \sum_k p_{ik1} p_{kj(r-1)}$. With this notation, superscripts are always indices and never indicate powers, which are indicated by repeating symbols. All probabilities under the null are generated by the array p_{ij1} and the above recurrence relation. It will be convenient to parametrize the null hypothesis by the unknown elements θ of p_{ij1}, and the elements of θ will be indexed by a, b, c, etc.

Because the state must either remain unchanged (with probability p_{ii}), or move to one of the alternative states, the sum over destination states of transition probabilities must equal one, $\sum_{j=1}^{K} p_{ij} = 1$. This is one of the reasons the parameter vector θ is introduced; not all elements of p_{ij1} are unknown.

The simplicity of this recurrence relation emphasizes why the simple Markov chain model has proven so attractive a framework for those seeking to describe the ratings transition behavior of credit. Note the interpretations possible: If X_t identifies the risk category of a single asset, for example $X_t = (0, 0, 1, 0, ..., 0)$, say, indicating the asset is in the third risk group, then $X_{t+1} = (p_{311}, p_{321}, ..., p_{3K1})$ is the probability distribution across risk states in time $t+1$ for an asset in risk state 3 at time t in the time-homogeneous case. Alternatively, if X_t is itself the distribution of assets across risk categories at time t, then X_{t+1} is the corresponding distribution in time $t+1$. The same probabilistic framework applies to single assets and the portfolio as a whole.

In the non-homogeneous case the transition probabilities should be indexed by t, P_t and consequently so should the p_{ijr}, i.e. the r-step transition probabilities depend on the time period in which the initial transition occurs. In this case the model remains Markovian in that the distribution across states in the next period depends only on the distribution in the current period (and not on the distributions in previous periods). That is, the transitions are memoryless. History does not matter given the current state. In the credit ratings context, this result is equivalent to saying that given a credit's current rating, the likelihood that the credit will move to any other rating level, or keep its current one, is independent of its past ratings history. While an interesting characteristic of Markovian transitions, this property seems to run against the widely held view that actual credit ratings changes exhibit "momentum." Conditional on the current rating level, a future credit downgrade is perhaps more likely if the credit had experienced a downgrade in the previous period, than if it had experienced an upgrade or no change in rating; see, for example Carty and Fons (1993) or Bangia, Diebold, Kronimus, Schagen, and Schuermann (2002).

3 Estimation

Estimation of the transition probabilities in a simple Markov chain can be carried out simply by counting the number of changes from one state to another that occur during a specified sample period and using sample fractions as estimators. These are method of moments estimators and MLEs. The estimation and inference problem here is classical; leading early papers include Anderson and Goodman (1957), Billingsley (1961) and the application-focused paper by Chatfeld (1973). Both maximum likelihood and minimum chi-squared (method of moments) approaches have been studied and used as a basis for inference as well as estimation. The likelihood method is treated recently in the credit transition application by Thomas, Edelman and Crook (2002). A variety of methods for estimating continous-time transition matrices are studied in Lando and Skodeberg (2002). Continuous time models can be fit on the basis of discrete data with appropriate assumptions. Jafry and Schuermann (2004) provide a comparison of different estimators suggested by Lando and Skodeberg (2002). Continuous observation offers real advantages, in that the discrete-time approximation need not be maintained. However, these methods require data configurations not typically available in the summary statistics provided by financial institutions or rating services.

Our basic data consist of the empirical multistep average transition rates e_{ijr} together with the sample sizes n_{ri}, indicating the number of assets in the initial state i used to compute the e_{ijr} across the destination states j and over r steps. This data distinguishes our approach, since others tests (Thomas, et.al.) have been formulated based upon raw data or (equivalently) sufficient statistics. The sufficient statistics are, in a natural notation, $e_{ij}(r)$, the one-step transition rates in each starting period. As a practical matter, these are rarely available.

Although an unrestricted parametrization in terms of the $p_{ij}(t)$ may seem natural, the equivalent parametrization in terms of p_{ijr} (without imposing the restricted parametrization in terms of θ or the recurrence relations relevant to the time-homogeneous case) is substantially more convenient, in that the unrestricted estimators are simply $\hat{p}_{ijr} = e_{ijr}$. For this reason we will no longer use the notation $\hat{p}_{ijr}$ or distinguish the unrestricted estimator from the data e_{ijr}. The notation p_{ijr} will refer to the restricted specification. To estimate the restricted model we will minimize the deviations $d_{ijr} = d_{ijr}(\theta) = e_{ijr} - p_{ijr}(\theta)$ in an appropriate metric, where the dependence of the deviations on θ is implied where not necessarily explicitly indicated.

Our approach admits an indirect inference interpretation, along the lines of that described by Gourieroux, Monfort, and Renault (1993), and Gallant and Tauchen (1996). The method can also be viewed within the framework of asymptotic likelihood (Jiang and Turnbull (2004)), which suggests satisfactory practical performance for the estimator.

We estimate θ by minimizing the weighted sum of squares

$$TS_{\tau}(\theta) = d_{ijr}(\theta)d_{k1s}(\theta)\tau^{ijrkls} \tag{2}$$

where τ is an array of weights assigned to the squares and cross-products of the deviations. We consider different specifications of τ, noting that the optimal weights consist of the precisions of the deviations (precisely, the inverse covariance of the e_{ijr}).

First, we consider the simplest specification, $\tau_{ijrkls} = \delta_{ij}\delta_{jk}\delta_{kl}\delta_{rs}$, where the Kronecker $\delta_{ij} = 1$ if $i = j$ and zero otherwise, leading to minimization of the sum of squares

$$TS_{l}(\theta) = d_{ijr}(\theta)d^{ijr}(\theta) \tag{3}$$

which is easily seen to yield a consistent estimator of θ. In fact, this is useful in obtaining quick initial estimates of θ to use in constructing the weight matrix for an efficient estimator. A simple improvement weights by $1/n_{ir}$, leading perhaps to a modest efficiency gain and certainly to a well-defined asymptotic distribution theory. We next consider the sum of squares weighted by the associated precisions using $\tau_{ijrkls} = \delta_{ij}\delta_{jk}\delta_{kl}\delta_{rs}n_{ir}/((1 - p_{ijr})p_{ijr})$. Implementation requires some initial estimate of p_{ijr}. It is natural to use the unrestricted estimator e_{ijr}, leading possibly to a high-order efficiency loss (but no first-order loss). Since this corresponds to a diagonal covariance specification, we refer to the associated criterion function as TS_D. A further refinement is possible by introducing approximate covariances across empirical transition rates from the same initial state. Let ω, an array of variances and covariances correspond to the inverse of the precision τ. Then let

$$\begin{aligned} \omega_{ijrkls} &= \delta_{ij}\delta_{jk}\delta_{k1}\delta_{rs}((1-p_{ijr})p_{ijr})/n_{ir} + (1-\delta_{ij})\delta_{ik}\delta_{rs}(-p_{ijr}p_{ilr})/n_{ir} \\ &\quad +(1-\delta_{ij})\delta_{ik}(1-\delta_{rs})(p_{ijr}p_{jl(s-r)} - p_{ijr}p_{kls})/n_{is} \end{aligned} \tag{4}$$

for $s \geq r$ and fill in the $s < r$ case by symmetry. Note the symmetry: $\omega_{ijrklr} = \omega_{k1rijr}$ (symmetry for fixed steplength, and $\omega_{ijrkls} = \omega_{ijsklr}$ (symmetry across steplengths for fixed transitions.). As above, the unrestricted estimators e_{ijr} can be substituted for the p_{ijr} in calculating the weights without affecting the first-order efficiency. The resulting weighted sum of squares is denoted TS_{BD}. Of course, redundant elements of d_{ijr} must be deleted, eg. those reflecting the restriction on row sums, or else (and equivalently) a generalized inverse must be used.

These approximate variance and covariance calculations are precise if we observed the 1 and multi-step transitions starting from a given time. In fact, the exact calculations are complicated by the fact that the reported summary statistics are typically average 1-step transition rates across assets and over time, average 2-step (same assets but 1 fewer step), etc. Add to this already complicated situation the fact that new assets enter the sample, and others leave, over the sample period and the precise variance calculation becomes problematic. We can expect covariance between elements of different rows, as well as a modified structure across columns within a row.

To obtain a better approximation to the optimal weighting matrix, and thus not only more efficient estimates but a better approximation to the sampling distribution of the test statistic, we suggest a simulation estimator for ω, and hence τ. For the simulation estimator, we begin with a consistent estimator for θ, possibly obtained by minimizing TS_Ior TS_D (our actual choice in practice). We then generate assets in the numbers given by n_{i1}, the observed number of assets experiencing a one-step transition beginning in state i, and follow them over time as they move between states according to the transition probabilities given by our initial θ. We calculate the empirical transition rates corresponding to the basic data e_{ijr}. We do this exercise M times and then calculate the sample covariance of the simulated e_{ijr}. That sample covariance (deleting the redundant elements) is our ω and its inverse gives the weights θ. M is important in the sampling distribution but not in the asymptotics. Increasing M is simply a matter of (small amounts of) computer time, so this does not pose a real constraint. The weighted sum of squares using this simulated weight matrix is TS_s.

We are now in a position to estimate the restricted model and test the hypothesis that the transition data are generated by a time-homogeneous Markov chain. In many applications to ratings transitions, the worst state is the absorbing state, default. With the convention that state 1 is the best (AAA in S&P and Fitch, Aaa in Moody's, etc.) and state K the worst, we have that the $p_{Kjr} = e_{Kjr} = \delta_{Kj}$; there are no transitions out of default. Thus, the array e_{ijr} has $(K-1)^2 \times T$ non-redundant elements in the unconstrained case, where T is the maximum steplength and we note the adding up constraint on the probabilities out of each state. If there is no default state, then there are $(K-1) \times K \times T$ nonredundant transition probabilities. In the constrained model all the transition probabilities can be generated from the one-step transition matrix through the recursions above. The point is that the number of parameters is not unduly large in practical applications and it does not increase with the number of periods available in the time-homogeneous case. There are other restrictions as well ($p_{ij} \in [0,1]$) which make a reparameterization convenient before maximization.

We have found it useful to reparameterize with a logit transform to new parameters

$$\theta_{ij} = \ln(p_{ij}/(1 - \sum_{j=1}^{K-1} p_{ij})), \tag{5}$$

noting that this specification implies that $\theta_{iK} = 0$. We then maximize numerically with respect to the $(K-1)^2$ or $(K-1) \times K$ (depending on whether there is an absorbing state) unrestricted parameters θ_{ij}. This is a fairly easy maximization using any of many available software packages. The reparametrization is useful for computation since, due to the nature of our data, multistep transition matrices rather than sequences of one-step transition statistics, nonlinear optimization is required to estimate the constrained model. With the transformation, the parameters in the optimization are unconstrained. Convenient starting values are provided by the logit transforms of the sample fractions making up the 1-period transition data (e_{ij1}). Finally, we rule out zeros among the transition probabilities, taking the position that no transition, however unlikely, is really impossible. To this end, we bound the estimated transition probabilities from below by 10^{-8}. As this corresponds to a transition on average once every 100 million periods, we do not regard this as a restriction likely to be binding.

Using standard arguments for a GMM interpretation,

$$\theta_{GMM} = \arg\min\{d_{ijr}(\theta)d_{kls}(\theta)\tau^{ijrkls}\} \tag{6}$$

can be minimized directly or by solving the FOC

$$d_{ijr/a}(\theta)d_{kls}(\theta)\tau^{ijrkls} = 0 \tag{7}$$

Note that differentiation in this notation is indicated though use of a "/" followed by the appropriate index.

Any weights (almost) will give a consistent estimator. Efficiency, within this class, requires that the weights are the precisions. Then,

$$n^{1/2}(\theta - \theta_{GMM}) \to N(0, Q^{-1}) \tag{8}$$

with

$$Q_{ab} = d_{ijr/a}(\theta)d_{kls/b}(\theta)\tau^{ijrkls}. \tag{9}$$

To obtain an asymptotic likelihood interpretation, note that since e_{ijr} is a sample mean,

$$n^{1/2}(e_{ijr} - p_{ijr}(\theta)) \to N(0, \omega) \tag{10}$$

which suggests an indirect likelihood, approximate likelihood, or asymptotic likelihood approach using this normal distribution as a likelihood function

$$L(\theta) = c(t)\exp\{-1/2d_{ijr}(\theta)d_{kls}(\theta)\tau^{ijrkls}\} \tag{11}$$

to be maximized over θ for a fixed τ, giving the estimator θ_{AL}. Standard arguments give

$$n^{1/2}(\theta - \theta_{AL}) \to N(0, Q^{-1}) \tag{12}$$

and the asymptotic chi-squared distribution for the likelihood ratio statistic.

Jiang and Turnbull (2004) show that θ_{AL} has minimum asymptotic variance among all CAN estimators based on e. Indeed, it can be shown that the estimators (the GMM and the AL estimators are the same) of θ are consistent and asymptotically normally distributed, with asymptotic covariance matrix given by $Cov(\theta_a, \theta_b) = g_{ab} = d_{ijr/a} d_{kls/b} \tau^{ijrkls}$.

4 Testing Time-Homogeneity

The test statistic we propose is the minimized value of TS_S, which is easily seen by standard arguments to have asymptotically a $\chi^2((K-1)^2(T-1))$ distribution when there is an absorbing state and a $\chi^2(K(K-1)(T-1))$ distribution when there is no absorbing state. We have examined the other minimized TS criterion functions for use as test statistics, but it seems that the various approximations to the actual "optimal" weighting matrix are poor and the associated approximate asymptotic distribution theory provides poor approximations to their sampling distributions.

It is useful to consider whether the test has power against other departures from the time-homogeneous Markov model. Departures of interest include alternatives with more dependence than the Markov specification allows. For example, does next period's distribution of assets across risk categories depend not only on the current distribution but on previous distributions as well? If so, the process is not Markovian. Our test will have power against these alternatives if these processes can be better approximated by a time-inhomogeneous process than by a time-homogeneous process. To see the argument, note that a misspecified model will have parameters (sometimes called quasi-true values) that minimize the information distance between the misspecified parametric family and the true model. If both the constrained (homogeneous) and the unconstrained Markov models are incorrect, but the unconstrained is closer in information distance to the true model, our test will see this difference and provide evidence against the constrained model. Thus it can be expected that our test has power against a fairly broad class of alternatives to the time-homogeneous Markov specification.

5 Sampling Performance

In order to assess our asymptotic approximations to the null distribution of our test statistic, we carry out a series of simulations. A representative subset is reported here. Figure 1a reports the null sampling distribution for the test statistic based on the diagonal approximation to the covariance matrix, TS_d for a 5-state model with 1-step and 2-step transitions and 1000 assets. The data generating process used the transition matrix

$$\begin{pmatrix} 0.4 & 0.2 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0.4 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.4 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.2 & 0.4 & 0.2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The sampling distributions were calculated on the basis of 2000 realizations (a realization is a transition history) throughout. The cdf of the test statistic is substantially to the right of the chi-square(16) distribution, perhaps indicating that the diagonal is a poor approximation to the covariance matrix of the auxiliary statistic (we know it is not a correct specification; now we know it is not a satisfactory approximation). Figure 1b reports the same statistic with 1,2,3,4, and 5-step transitions. Here the null distribution is chi-square(64). The cdf is still to the right of the desired null distribution. The statistic based on a diagonal approximation would apparently lead to overrejection. relative to the nominal size.

Figure 1a:
TS_Diag Distribution with k=5, T=2, and N=2000

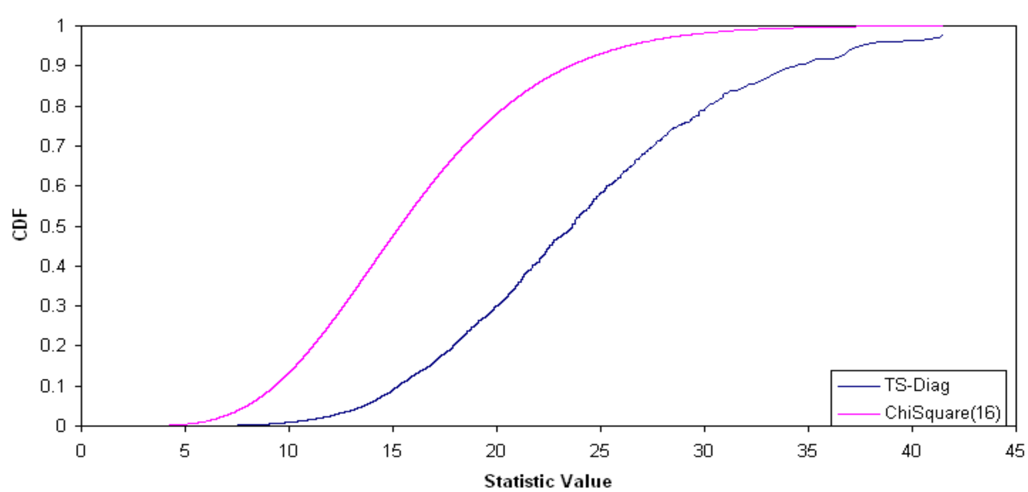
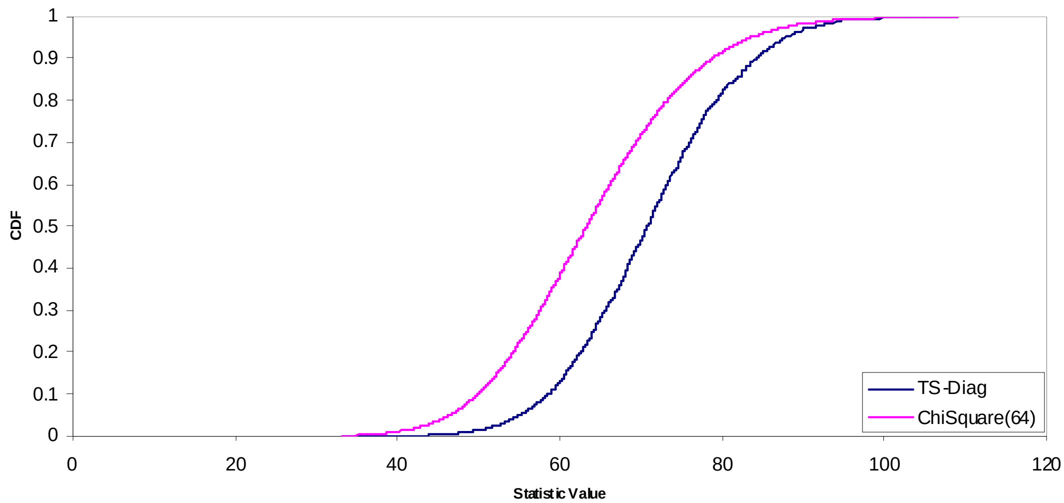


Figure 1b:
TS_Diag Distribution with k=5, T=5, and N=2000



Figures 2a and 2b report the same experiment with the block-diagonal approximate covariance matrix. This approximation adjusts for covariances across the elements of the within-period transition matrix but does not account for the averaging involved in the development of the auxiliary statistic. It too overrejects relative to the chi-squared baseline.

Figure 2a:
TS_BlockDiag Distribution with k=5, T=2, and N=2000

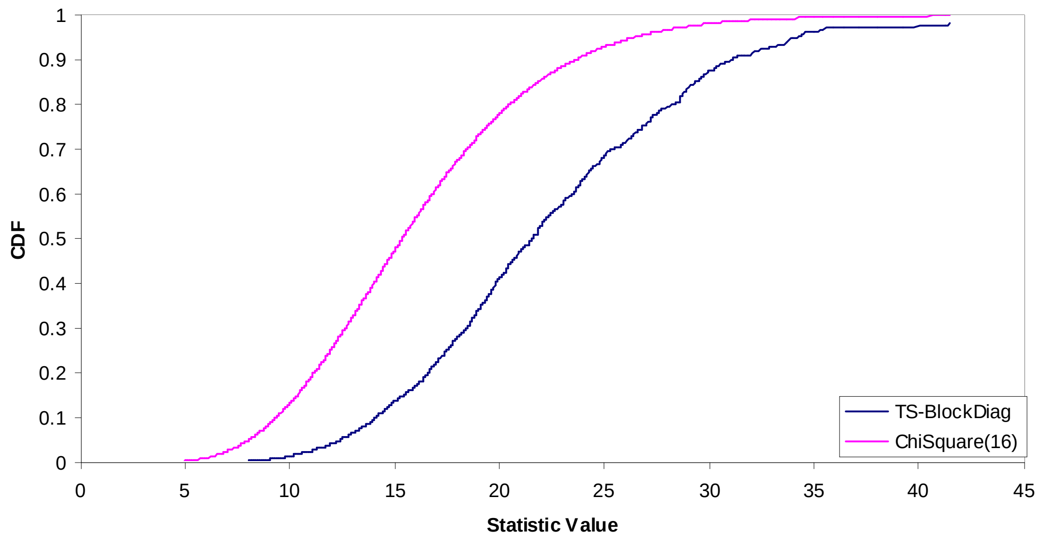
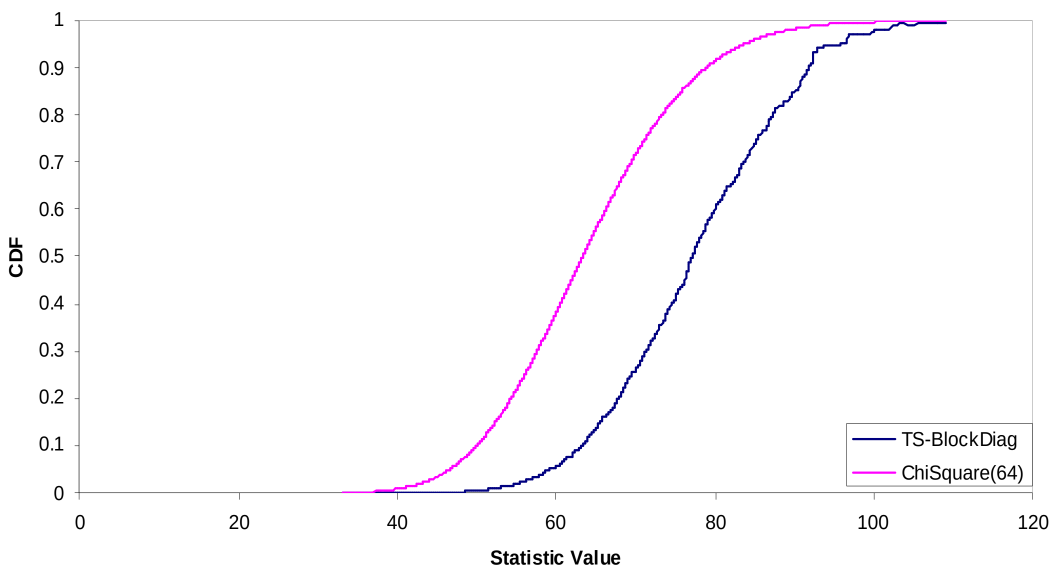


Figure 2b:
TS_BlockDiag Distribution with k=5, T=5, and N=2000



We turn next to the results based on the simulated covariance matrix. These work remarkably well, as shown in Figures 3a and 3b (the same experimental specification). In the simulation we used $M = 2000$. The actual distributions of the test statistic match well the asymptotic chi-square distribution. We conclude that the simulation estimator gives an asymptotic distribution with which we can control size accurately. We did not do power simulations, though the statistic does reject in actual applications, so there is evidence of some power. Note that the likelihood ratio statistic depends in general on data not available to us (recall that the sufficient statistic is the sequence of one-step transitions for each period) and therefore our test cannot have maximal power. It can have maximum asymptotic power for tests based on our auxiliary statistic, and does (Jaing and Turnbull, 2004).

Figure 3a:
TS_Sim Distribution with k=5, T=2, and N=2000

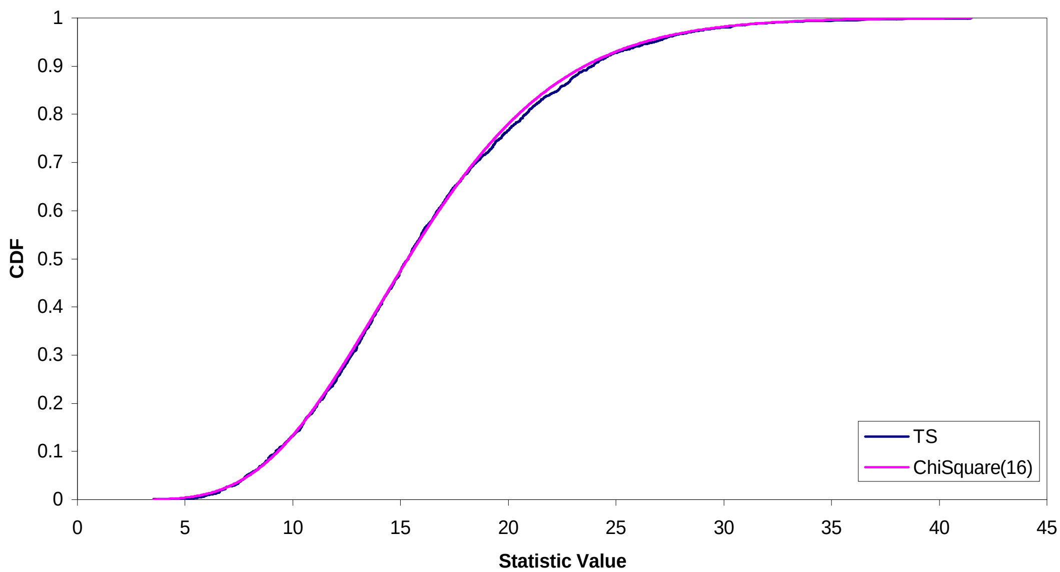
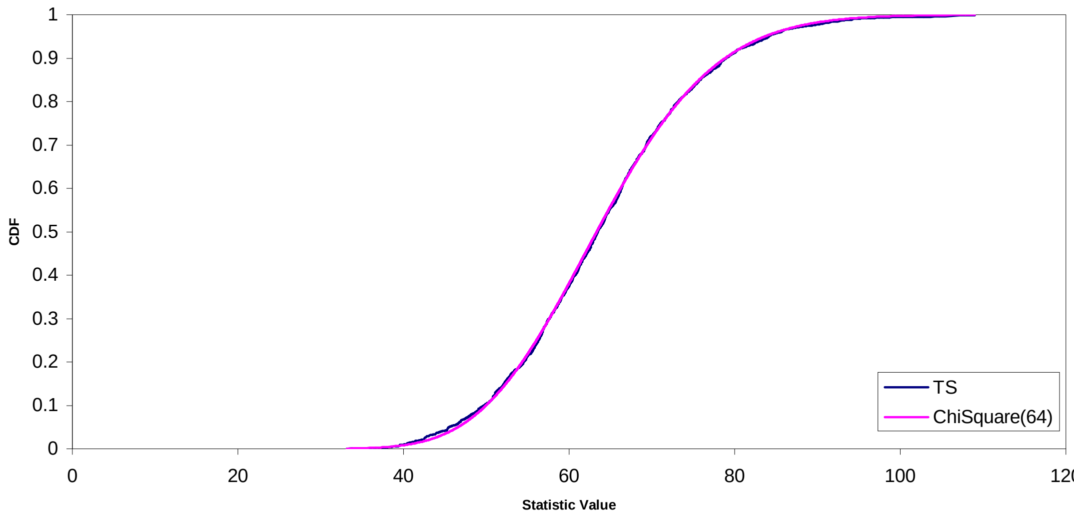


Figure 3b:
TS_Sim Distribution with k=5, T=5, and N=2000

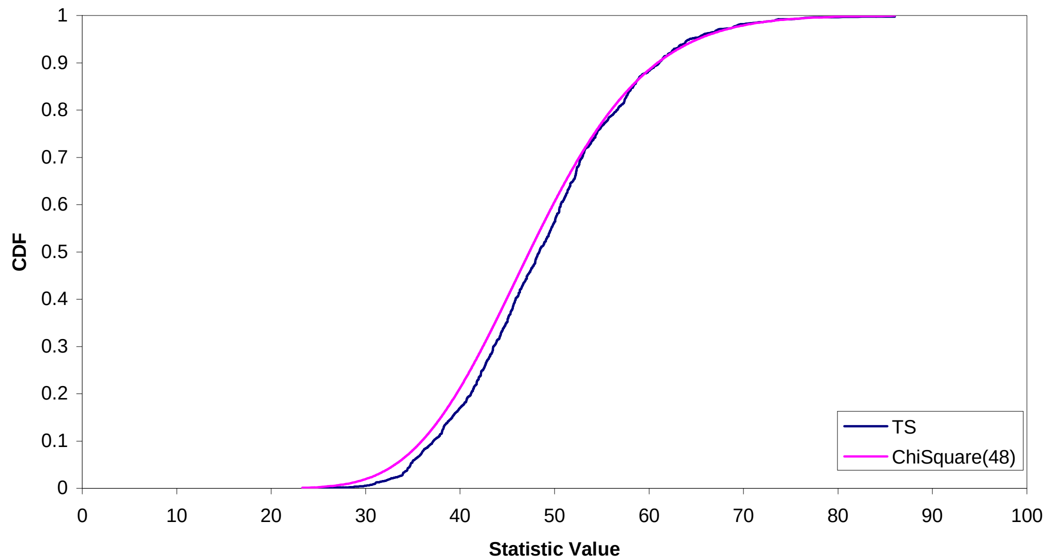


Simulations were carried out for a variety of other specifications. One feature of the data sets seen in practice is that the diagonals are typically large and the off-diagonals small. This is reflected in our specification of the DGP above. Simulation results were also produced for a transition matrix that more closely resembles the one period transition of rated corporate bonds:

$$\begin{pmatrix} 0.97 & 0.02 & 0.005 & 0.0045 & 0.0005 \\ 0.1 & 0.87 & 0.015 & 0.0135 & 0.0015 \\ 0.05 & 0.2 & 0.6595 & 0.0405 & 0.05 \\ 0.05 & 0.12 & 0.2 & 0.53 & 0.1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 4 illustrates that the TS-Sim distribution for this exercise yielded results nearly identical to those reported in Figures 3a and 3b.

Figure 4:
TS_Sim Distribution with k=5, T=3, and N=1000



6 Data and Applications

While several institutions have their own historical data and ratings assignments with which to estimate Markov transition models, several alternative sources exist from which credit rating transition matrices might be obtained. Transition matrices are regularly published by the major rating agencies, including Moody's and Standard and Poor's; they are also available from risk management consulting companies and vendors of credit risk models including RiskMetrics, KMV and Kamakura, Inc. Unfortunately, reporting standards in this area are fairly lax. Often, transition fractions are reported with no indication of the size of the underlying data set, or the initial distribution of assets across risk categories. Since many of these transitions are low-probability events, large samples are necessary for precise estimation. This information should be routinely provided as a matter of sound statistical practice.

We have assembled several data sets with which to illustrate our technique and demonstrate its usefulness.

6.1 Commercial Paper

These data are from a study by Moody's (2000). This study is also commendable in the detail provided. Commercial paper defaults are extremely rare, although there is some migration among rating categories (here P-1, P-2, P-3 and NP; P is for prime). Moody's goal is that commercial paper with any prime rating should never default. Since commercial paper, in contrast to the municipals studied above, are short term assets, the transitions examined are 30-day transitions. The Moody's study reports transitions over 30, 60, 90, 120, 180, 270 and 365 days. Note that the "missing" powers of P at 150, 210, 240, 300, and 330 days present no problem for our methods. The data up to the 180 day transitions exhibit expected patterns, for example the transitions into default (and generally transitions out of the initial state) are increasing with the length of the time span. However, for 270 and 365 days, the transitions from NP into default are zero. This appears problematic and is perhaps due to definitions Moody's has used and the fact that commercial paper is rarely extended for long periods. We have chosen simply not to use the data for 270 and 365 days. Another question arises as to the treatment of Moody's category WR – withdrawn. These occur because commercial paper is not rolled over so the asset size becomes negligible, or the market has otherwise lost interest in the offering. It is apparently not a synonym for default or a decline in creditworthiness. Consequently, we treat these as censored. Again, to illustrate the use of the test, we calculate the statistic based on increasing numbers of periods. Results are given in Table 1.

Table 1: Tests for Commercial Paper, 1972-1999

Transitions	*Chi-Square*	*DF*	*P-value*
1,2	1.44	16	1.0000
1,2,3	11.38	32	0.997
1,2,3,4	41.56	48	0.7346
1,2,3,4,6	74.55	64	0.1727

There is no serious evidence against the Markov specification. Recall that the time scale here is 30 days; commercial paper is generally short lived. The Markov model "works" over the period available (6 months). This is probably also the relevant period for applications. Note that the sample size here is enormous, so these results are pretty firm and it may be considered surprising that the tightly specified model is not soundly rejected for all transitions. The numbers of observations for the 1-step transition matrix, for the nondefault states, for example, are 264000, 561000, 6600, and 3300.

It is useful to look closely at the fit. We focus on the 4-step model. Table 2 shows the estimated (constrained) 4-step transition matrix (the fourth power of the 1-step matrix estimated on the basis of all of the transitions), the empirical 4-step matrix, and their difference.

Table 2: Commercial Paper 4-Step Rating Transition Matrices

		Estimated		
0.98791	0.01161	0.00019	0.00029	0.00001
0.02172	0.95808	0.01601	0.00393	0.00026
0.00591	0.05768	0.88110	0.05440	0.00091
0.00273	0.01117	0.02034	0.96231	0.00345
0	0	0	0	1
		Empirical		
0.98785	0.01164	0.00020	0.00031	0.00000
0.02206	0.95814	0.01557	0.00402	0.00021
0.00618	0.05976	0.88261	0.05033	0.00112
0.00207	0.00977	0.02208	0.96240	0.00368
0	0	0	0	1
		Empirical-Estimated		
-0.000056	0.000025	0.000016	0.000019	-0.000005
0.000339	0.000061	-0.000441	0.000096	-0.000054
0.000270	*0.002079*	*0.001514*	*-0.004076*	0.000213
-0.000664	*-0.001395*	*0.001735*	0.000094	*0.00230*
0	0	0	0	0

Note: The numbers of assets for computing the 4-step transition matrix are (6560, 1394, 164, 82). Differences greater than 0.001 in absolute value are italicized.

Transitions are fairly rare: the minimum diagonal element is approximately 0.88 (constrained and unconstrained). There are a number of suggested summary measures for the amount of mobility in a transition matrix. These are discussed in Jafry and Schuermann (2004), who suggest

$$M = K^{-1} \sum_{i=1}^{K} \sqrt{\lambda_i((P-I)(P-I)^T)} \tag{13}$$

where P is the transition matrix in question and $\lambda_i(A)$ are the eigenvalues of A. For motivation see Jafry and Schuermann (2004); note that M=0 for the identity transition matrix and 1 for (strict) permutations. Note also that M and related mobility measures do not define metrics on the space of transition matrices. In this CP application, M =0.0449 for the estimated (constrained) 4-step transition matrix and 0.0445 for the data (the unrestricted frequency estimator). The fit is clearly quite good, in terms of the mobility measure and of the matrices themselves. Entries greater than 0.001 in absolute value are italicized in the third panel (the "residual"). It would be hard to argue that there are economically important differences between the constrained and unconstrained models. Although the fit is good, the data may be rich enough to support finer and more sophisticated modeling of this stochastic process. Such a modified model would address the finding that there are slightly more defaults in the 4-th step from the two lowest rating categories than expected on the basis of the constrained model. Further, flows out of the third risk category are higher than expected on the basis of the Markov model. Most of the unexpected outflow consists of upgrades. Comparing the 6-step empirical vs. estimated leads to the same conclusion: the differences are not economically important. The Markov model appears to perform well in explaining Commercial Paper ratings transitions over periods up to a maximum of 6 months.

6.2 Sovereign Debt

These data are from Standard & Poor's (2003). Again, suitable detail is provided. Here, it appears that overlapping cohorts were used to calculate the transition matrices. For example, in a sample of 15 years, there is one 15-year cohort, two 14-year, three 13-year, etc. Transitions among 8 rating categories including default are recorded. In the case of sovereign debt, default must be defined particularly carefully, as some of the debt to preferred creditors is often serviced after default. S&P give a discussion in Appendix 2, and 1, 5 and 7 year transition rates are reported. Our results are reported in Table 3.

Table 3: Tests for Sovereign Debt, 1975-2002

Transitions	*Chi-Square*	*DF*	*P-value*
1,3	20.5	49	1.0000
1,3,5	126.77	98	0.0629

The time-homogeneous Markovian specification appears fine for the 5-year period. It should be noted that the sample sizes are relatively small here (from the least risky, the number of assets in the average 1-step transition matrix by state are: 304, 179, 111, 123, 130, 75, and 6; the default transitions are constrained), and this is perhaps driving the statistical result, as the fit is not at all as good as that for the commercial paper. Table 4 gives the constrained and unconstrained 5-period transition matrices and again, in the third panel, their difference.

Table 4: 5-Year Sovereign Debt Rating Transition Matrices

Estimated							
0.92227	0.07671	0.00055	0.00022	0.00024	0.00001	0.00000	0.00000
0.05428	0.90885	0.01941	0.00828	0.00820	0.00071	0.00008	0.00020
0.00150	0.07333	0.79521	0.11205	0.00593	0.00979	0.00125	0.00094
0.00004	0.00628	0.19779	0.63704	0.09054	0.03234	0.01028	0.02569
0.00002	0.00092	0.01541	0.13874	0.64234	0.14562	0.00792	0.04903
0.00001	0.00075	0.02259	0.01308	0.16358	0.66295	0.04429	0.09277
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00599	0.99401
0	0	0	0	0	0	0	1
Empirical							
0.86556	0.11859	0.00001	0.00001	0.01580	0.00001	0.00001	0.00001
0.08590	0.85937	0.03130	0.01560	0.00780	0.00001	0.00001	0.00001
0.00001	0.12310	0.67688	0.16919	0.01540	0.01540	0.00001	0.00001
0.00001	0.00001	0.30515	0.45768	0.11862	0.05081	0.01690	0.05081
0.00001	0.00001	0.01490	0.19401	0.53734	0.17911	0.00001	0.07461
0.00001	0.00001	0.05000	0.00001	0.15000	0.54998	0.05000	0.19999
0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.99993
0	0	0	0	0	0	0	1
Empirical-Estimated							
-0.05671	*0.04188*	-0.00054	-0.00021	*0.01556*	0.00000	0.00001	0.00001
0.03162	*-0.04947*	*0.01189*	0.00732	-0.00040	-0.00070	-0.00007	-0.00019
-0.00149	*0.04977*	**-0.11833**	*0.05715*	0.00946	0.00561	-0.00124	-0.00093
-0.00003	-0.00627	**0.10737**	**-0.17936**	*0.02808*	*0.01847*	0.00662	*0.02512*
-0.00001	-0.00091	-0.00051	*0.05528*	**-0.10500**	*0.03349*	-0.00791	*0.02557*
0.00000	-0.00074	*0.02741*	*-0.01307*	*-0.01358*	**-0.11296**	0.00571	**0.10723**
0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	-0.00598	0.00592
0	0	0	0	0	0	00	0

Note: The numbers of assets for computing the 5-step transition matrix are (249, 127, 65, 56, 61,22, 1). Differences greater than 0.01 in absolute value are italicized; those greater than 0.1 are bolded.

Here, differences between the constrained and unconstrained transition probabilities (5-year) greater than .01 are italicized and differences greater than 0.1 are also in bold. The fit is obviously much worse than the fit for commercial paper (hence the difference in the highlighting threshold). That these differences are not statistically significant is no doubt due to the small sample sizes. The mobility measures M=0.3617 (constrained) and 0.4464 (unconstrained) are quite different in magnitude. The differences indicate that sovereign debt needs to be monitored carefully and probably on a case-to-case basis. The Markov model does not do a great job of capturing these transitions. Nevertheless, it is doubtful that a more complicated model would be supported by the data.

6.3 Municipal Bonds

These data are from a well-known Standard and Poor's study published on the web (Standard & Poor's, 2001). This data set contains information on rating transitions of municipal bonds through 8 rating categories and a nonrated category from 1986 through 2000. Average 1-year, 2-year, and through 15-year transition matrices are reported. This is a rare study that gives sufficient detail to be realistically useful. We have run tests based on the 1 and 2 year data, the 1,2 and 3 year data, and so on up to 5- year transitions. Results are shown in Table 5. We see here that the Markov specification is adequate for describing annual ratings transitions for periods up to about 5 years. The model is fairly reliable for 1,2,3, and even 4 year transitions, but begins to fail when looking 5 years out. The implication for practice is that municipal bond rating transitions can over reasonable periods be described by a simple time-homogeneous model, but the transition probabilities should probably be updated, and the new ones used, every few years. The updating should be done on a rolling basis, discarding older data as new years become available, as the older data are no longer relevant as far as the Markov approximation is concerned (though they would be useful in developing a richer model for longer-term modeling).

Table 5: Tests for Municipal Bonds 1986-2000

Transitions	*Chi-Square*	*DF*	*P-value*
1,2	56.59	49	0.2130
1,2,3	128.6	98	0.0207
1,2,3,4	173.5	147	0.0669
1,2,3,4,5	937.6	196	0.0000

In order to understand the rejection for the long-period prediction, we again calculate the estimated transition matrix (constrained) and compare it with the actual 5-period transition. For the first 4 ratings categories, there is substantial inertia. For the next two, there is substantial outflow, consisting mostly of upgrades. For the worst (nondefault) category, there are substantial defaults.

Table 6: 4-Year Municipal Bond Rating Transition Matrices

			Estimated				
0.95777	0.03954	0.00259	0.00010	0.00000	0.00000	0.00000	0.00000
0.01562	0.92465	0.05528	0.00430	0.00010	0.00002	0.00001	0.00000
0.00044	0.06507	0.87868	0.05234	0.00255	0.00048	0.00029	0.00014
0.00004	0.00683	0.09097	0.86197	0.02479	0.00766	0.00435	0.00341
0.00003	0.00440	0.02047	0.42443	0.45700	0.06415	0.00775	0.02177
0.00000	0.00130	0.02122	0.41031	0.01569	0.31772	0.06428	0.16949
0.00000	0.00039	0.00255	0.09432	0.09646	0.14310	0.26578	0.39741
0	0	0	0	0	0	0	1
			Empirical				
0.96155	0.03630	0.00210	0.00001	0.00001	0.00001	0.00001	0.00001
0.01560	0.92477	0.05530	0.00410	0.00001	0.00010	0.00010	0.00001
0.00030	0.06521	0.87808	0.05231	0.00270	0.00040	0.00030	0.00070
0.00020	0.00640	0.09172	0.86157	0.02460	0.00750	0.00450	0.00350
0.00001	0.00300	0.01470	0.40714	0.47784	0.07371	0.00300	0.02060
0.00001	0.00001	0.00001	0.37655	0.03899	0.36365	0.06489	0.15588
0.00001	0.00001	0.00001	0.21664	0.01670	0.08331	0.26665	0.41667
0	0	0	0	0	0	0	1
			Empirical-Estimated				
0.00378	-0.00324	-0.00049	-0.00009	0.00001	0.00001	0.00001	0.00001
-0.00002	0.00012	0.00003	-0.00020	-0.00009	0.00008	0.00009	0.00001
-0.00014	0.00014	-0.00060	-0.00003	0.00015	-0.00008	0.00001	0.00056
0.00016	-0.00043	0.00075	-0.00040	-0.00018	-0.00016	0.00015	0.00009
-0.00002	-0.00140	*-0.00577*	*-0.01729*	*0.02085*	*0.00955*	-0.00475	-0.00117
0.00001	-0.00129	*-0.02121*	*-0.03375*	*0.02330*	*0.04593*	0.00061	*-0.01361*
0.00001	-0.00038	-0.00254	*0.12232*	*-0.07975*	*-0.05979*	0.00087	*0.01926*
0	0	0	0	0	0	0	0

Note: The sample sizes for the one-step average transition matrices are, by state from the least risky, (781, 19650, 45771, 20060, 730, 203, 98)
Defaults are zero throughout. The numbers of assets for computing the 4-step

transition matrix are (558, 14450, 35715, 15481, 612, 174, 79). In the third panel, differences greater than .005 in absolute value between the empirical and predicted are italicized.

Outflows from the rating categories 4,6, and 7 are underpredicted. Defaults from the 6th category are overpredicted and from the 7th underpredicted. Here, the mobility measures M=0.3373 (constrained) and 0.3282 (unconstrained) are not substantially different, illustrating the importance of the comment that M , while a useful measure of mobility, is not a distance metric. Thus, "munis" are well-modeled by the time homogeneous Markov model only for a few (perhaps 4) years.

6.4 S&P Corporates

Data for this example consists of S&P-rated firms in the KMV North American Non-Financial Dataset , which for practical purposes mirrors the Compustat dataset. In order to deal with the sparseness of the transition matrix based upon individual S&P rating categories we used the following correspondence to map S&P ratings to a custom rating scale.

Table 7: Mapping of S&P Corporate Ratings

S&P Ratings	Bucket
AAA,AA+,AA,AA-	1
A+,A,A-	2
BBB+,BBB	3
BBB-,BB+	4
BB,BB-	5
B+,B,B-	6
CCC+,CCC,CCC-,CC,C	7
D	8

Note that while many possible aggregations exist, seven non-default buckets is required under the advanced internal ratings based approach of the Basel II capital accord. Our chosen mapping results in a more uniform distribution of assets across the rating buckets than would, say, a mapping based upon whole grades (AAA, AA, A, BBB, BB, etc.), and reflects a key goal that several internal bank rating systems are trying to achieve: the addition of granularity and risk differentiation for those obligors of medium to low credit quiality (rated and unrated) with whom the bank typically does business. Rating transitions for this custom scale were computed for cohorts of firms starting in September 1993 and running through September 2004; average cumulative multi-year credit-rating transition matrices were computed and were then tested for time-homogeneity. Results are reported in Table 8.

Table 8: Tests for S&P Corporates 1993-2004

Transitions	*Chi-Square*	*DF*	*P-value*
1,2	23.8747	49	0.9991
1,2,3	109.911	98	0.1933
1,2,3,4	3815.96	147	0.0000

The Markov constraint seems to hold adequately for 3 transitions but breaks down at 4, suggesting care in using the Markov model to forecast for more than a year or two (but supporting its use in forecasting over this short horizon.

Table 9: 4-Year S&P Corporate Rating Transition Matrices

Estimated							
0.72731	0.23032	0.03341	0.00391	0.00085	0.00043	0.00244	0.00132
0.02057	0.68179	0.2007	0.03428	0.00903	0.00789	0.02785	0.01789
0.00346	0.12538	0.38357	0.10727	0.00903	0.07043	0.13355	0.14112
0.00033	0.01773	0.11238	0.27357	0.03521	.15027	0.22416	0.08825
0.00004	0.00264	0.02344	0.12824	0.51377	0.21716	0.07249	0.04223
0.00000	0.00033	0.00387	0.02686	0.15018	0.58199	0.11800	0.11877
0.00000	0.00004	0.00071	0.00671	0.04884	0.38826	0.24111	0.31434
0	0	0	0	0	0	0	1
Empirical							
0.69440	0.27780	0.02780	0.0000	0.00000	0.00000	0.00000	0.00000
0.02030	0.74627	0.18272	0.04060	0.00760	0.00250	0.00000	0.00000
0.00000	0.14530	0.63850	0.13510	0.05070	.02030	0.00030	0.01010
0.00000	0.02730	0.20220	0.53010	0.18030	0.04370	0.01090	0.00550
0.00000	0.00910	0.05451	0.23182	0.43644	0.23182	0.02270	0.01360
0.00000	0.00000	0.00530	0.08470	0.25400	0.51320	0.09520	0.04760
0.00000	0.00000	0.00000	0.00000	0.00000	0.71430	0.28570	0.00000
0	0	0	0	0	0	0	1
Empirical-Estimated							
-0.0329	*0.0475*	-0.0056	-0.0039	-0.0009	-0.0004	-0.0024	-0.0013
-0.0003	*0.0645*	*-0.0180*	0.0063	-0.0014	-0.0054	*-0.0278*	*-0.0179*
-0.0035	*0.0199*	**0.2549**	*0.0278*	*0.0155*	*-0.0501*	**-0.1336**	**-0.1310**
-0.0003	0.0096	*0.0898*	**0.2565**	*0.0470*	**-0.1066**	**-0.2133**	*-0.0828*
0.000	0.0065	*0.0311*	**0.1036**	*-0.0773*	*00.0147*	*-0.0498*	*-0.0286*
0.0000	-0.0003	0.0014	*-0.03375*	**0.1038**	*-0.0688*	*-0.0228*	*-0.0712*
0.0000	0.0000	-0.0007	**0.12232**	*-0.0488*	**0.3260**	*0.0446*	**-0.3143**
0	0	0	0	0	0	0	0

Note: The numbers of assets for computing the 4-step transition matrix are (144, 394, 296, 183, 220, 189, 7). Differences greater than 0.01 in absolute value are italicized and greater than 0.1 are bolded.

Although the constrained and unconstrained 4-year transition matrices are quite clearly different (entries above 0.1 in absolute value are in bold) the mobility measure is not too different (though note we have not supplied a standard error): M=0.493

constrained, 0.426 unconstrained. This is consistent with the finding of Jafry and Schuermann (2004) that the time homogeneity assumption within a year did not dramatically affect the inference on the mobility measure; we find that the cross-year time homogeneity assumption also does not affect the mobility measure. Our finding that time-homogeneity fails over the longer period is also consistent with their results that there is a drift in mobility over time. Our analysis here serves primarily the purpose of demonstrating the feasibility of our techniques. Further analysis of this data set will focus on patterns across industries as well as over time.

7 Conclusion

The time-homogeneous Markov model for transitions among risk categories is widely used in areas from portfolio management to bank supervision and risk management. It is well known that these models can be overly simple as descriptions of the stochastic processes for riskiness of assets. Nevertheless, the model's simplicity is extremely appealing. We propose a likelihood ratio test for the hypothesis of time-homogeneity. Due to a convenient reparametrization, the test is simple to compute, requiring numerical estimation of only the restricted model, a (K-1)/2-parameter problem where K is the number of risk categories. The test can be based on summary data often reported by rating agencies or collected within banks. We recommend that the test be interpreted as determining whether or not transitions over particular periods can be adequately modeled as Markov chains. Specifically, we do not recommend interpreting the test as showing that the true underlying process is or is not Markovian. Not only does this interpretation confuse failing to reject with evidence in favor, it does not address the interesting issue. The transitions cannot truly be Markovian in the long run – the prediction would be that everything defaults. However, in some cases transitions can be usefully modeled as Markovian over periods of useful length.

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