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Capital Gains

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Abstract

Capital gains play an important, positive role in the inter-temporal allocation of resources, but they can also be a source of economic instability. We analyze a simple overlapping-generations economy with two capital goods and irreversible investment. For each vector of initial capital/labor ratios, there is one and only one trajectory on which expectations are realized at every date. If there is any deviation from this trajectory, then there is a bubble which must burst in finite time.

Key Words: bubbles, capital gains, heterogeneous capital, irreversible investment, overlapping generations, Tobin's q

1 Introduction

Capital gains play an essential role in capitalist economies. Changes in asset prices signal anticipated changes in relative scarcities. Capital gains can, however, fuel self-perpetuating bubbles, some of which will eventually burst.

We need a dynamic general-equilibrium model with at least two assets in order to analyze the effects of capital gains. We follow the two-capital growth model of Shell and Stiglitz (1967),[1] where given the initial endowment of capitals and labor there is one and only one assignment of initial prices that is consistent with long-run balanced growth, whenever the

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[1] See also Shell, Sidrauski, and Stiglitz (1969), Caton and Shell (1971), Burmeister, Caton, Dobell, and Ross (1973), Shell (1972), and Burmeister and Graham (1974).

momentary equilibrium is not unique there is one and only one allocation of investment consistent with long-run balanced growth, and on trajectories not tending to the balanced growth path the price of one of the capital goods becomes zero in finite time.

Shell and Stiglitz made an assumption that is now a bit old-fashioned: an aggregate consumption function ungrounded in consumer optimization. In the present paper, we update their model by positing instead utility-maximizing individuals in an overlapping-generations (OG) model *a la* Diamond (1965) but extended to allow for the two capital goods. We also assume that the capitals, once installed, cannot be directly consumed or changed into the other type of capital. Therefore, investments are irreversible allowing for the prices of used machines to fall below their reproduction costs, i.e. for a Tobin's q which is less than 1.

We believe that the OG model is better suited for the analysis of capital gains and bubbles in decentralized economies than is the infinite-lifetime representative-agent (ILRA) model often used in macroeconomics. The ILRA model (and other homogenous-agent models) is essentially a planning model, in which prices, and hence capital gains, are merely dual variables to the optimization problem.[2] The OG structure highlights how prices today depend on expectations of future beliefs, including the beliefs of unborn generations.

In the 2-capital, discrete-time OG model, we show that for each initial endowment of capitals and labor, there is a unique competitive-equilibrium path on which expectations are fulfilled every period. On every other path, there is a bubble in that one of the capitals is overvalued relative to the other. The bubble must burst in finite time. Hence, even though Shell and Stiglitz (1967) assume *ad hoc* consumption behavior, their basic results do not depend on this assumption. However, because of their consumption function, Shell and Stiglitz did not allow for cases in which gross investments are both zero. In the OG model, we show that both investments are zero whenever capital-labor ratios are large. This defines

[2] When appropriate, transversality conditions close these planning models. See Shell (1969), for cases in which the so-called transversality conditions are not appropriate. The point here is that there are no natural transversality conditions in OG and other heterogeneous-agent models.

a region in which Tobin's q is less than 1. We show that once the economy achieves $q = 1$ it will not return to the $q < 1$ regime.

We compute some trajectories for an example in which the technological parameters, the depreciation rate, and the consumer time-discount rate are assigned reasonable values. We assign initial capitals so that one is much scarcer (based on relative marginal products) than the other and so that the economy is initially wealthy.

On the path in which expectations are always realized, gross investments are zero in the first few periods because the economy is rich in capital, which are followed by a few periods in which investment is specialized to the scarcer capital good. After these two stages, the marginal products of the two capitals are forever equalized. Asymptotically the economy tends to the steady-state just as it does in the Diamond model. This is the bubble-free path.

We also compute two bubble trajectories for the same parameters and initial endowments, but with initial prices that are slightly different from those on the bubble-free trajectory. For the first few periods, gross investments are zero as on the bubble-free path, but eventually investment is specialized to the "wrong" (lower marginal product) capital good. In about 120 years or so, the bubble bursts and it is revealed that this is a disequilibrium path in that expectations are ultimately unfulfilled.

What do we make of this? On the competitive equilibrium path in which expectations are always fulfilled, the allocation of investment is correct and there are no bubbles. On other paths, where the allocation of investment is wrong, short-run markets clear and expectations are fulfilled for a while (100-200 years), and there is a bubble that must eventually burst. This suggests to us that the long-run perfect-foresight equilibrium concept might be too rigid. Bursting bubbles should not be ruled out entirely.

2 The Model

In each period, there is a generation of identical, old consumers and a generation of identical, young consumers. Each young consumer inelastically supplies one unit of labor. The old do not work. The labor force L grows at the rate $n \geq 0$, so we have

$$L_{t+1} = (1+n)\, L_t, \tag{1}$$

where L_t is the number of consumers born in year $t = 0, 1, \ldots$. Consumers have identical utility functions

$$u(x_t^y, x_t^o) = \log x_t^y + \beta \log x_t^o,$$

where x_t^y is Mr. t's consumption when young and x_t^o is his consumption when old.

Production is given by the 1-sector, 3-output, 3-input model:

$$C_t + Z_t^1 + Z_t^2 = Y_t = \left(K_t^1\right)^{\alpha_1} \left(K_t^2\right)^{\alpha_2} L_t^{\alpha_3} \ , \tag{2}$$

where $\alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $K_t^i > 0$ is the capital of type i, $Y_t > 0$ is undifferentiated output, $C_t \geq 0$ is consumption, $Z_t^i \geq 0$ is gross investment in Capital i, all at time t, $i = 1, 2$. Investment is irreversible and capital goods are non-malleable (i.e. machines of one type cannot be turned into machines of the other type): $Z_t^i \geq 0$. Let $\mu > 0$ be the common rate of depreciation on each type of machinery. Capital accumulation is given by

$$K_{t+1}^i = (1-\mu) K_t^i + Z_t^i \tag{3}$$

for $i = 1, 2$. Denote by lower case letters quantities normalized by L, e.g., $k_t = K_t / L_t$, so we have

$$c_t + z_t^1 + z_t^2 = y_t = \left(k_t^1\right)^{\alpha_1} \left(k_t^2\right)^{\alpha_2} \tag{4}$$

and

$$(1+n) k_{t+1}^i = (1-\mu) k_t^i + z_t^i \tag{5}$$

for $i = 1, 2$. Under competition, factors are rewarded by their marginal products, so we have

$$r_t^i = \alpha_i \left(k_t^i\right)^{\alpha_i - 1} \left(k_t^j\right)^{\alpha_j} > 0, \tag{6}$$

for $i = 1, 2$, and

$$w_t = \alpha_3 \left(k_t^1\right)^{\alpha_1} \left(k_t^2\right)^{\alpha_2}, \tag{7}$$

where r_t^i is the rental rate on type-i capital and w_t is the wage rate.

We assume that individuals possess perfect foresight about price changes. Hence equilibrium in the used machinery market requires that the rate of return (including capital gains) on each type of capital must be the same, or

$$\frac{(1-\mu)p_{t+1}^1 + r_{t+1}^1}{p_t^1} = \frac{(1-\mu)p_{t+1}^2 + r_{t+1}^2}{p_t^2} = \rho_{t+1}, \tag{8}$$

where $p^i \geq 0$ is the current price of machine i in terms of the consumption good and ρ is the (common) rate of return. Equation (8) is the perfect-foresight asset-market-clearing equation.

Mr. t chooses consumptions (x_t^y, x_t^o) and savings $s_t \geq 0$ to maximize

$$u(x_t^y, x_t^o) = \log x_t^y + \beta \log x_t^o$$

subject to

$$x_t^y = w_t - s_t$$

and

$$x_t^o = \rho_{t+1} s_t,$$

where $0 < \beta < 1$ is the discount factor, "log" denotes the natural logarithm, and s_t is savings. The consumer's problem can be stated more succinctly as

$$\max_{s_t} \ \log(w_t - s_t) + \beta \log(\rho_{t+1} s_t) \tag{9}$$

subject to $0 \leq s_t \leq w_t$.The solution s_t to this problem is interior and given by

$$s_t = \frac{\beta}{1+\beta} w_t. \tag{10}$$

3 Equilibrium

Young consumers use their savings to buy capital that they will rent in period t and sell in period $t+1$. In equilibrium, the value of the supply of machinery must equal the value of savings, or

$$(1+n)\left(p_t^1 k_{t+1}^1 + p_t^2 k_{t+1}^2\right) = \frac{\beta}{1+\beta} w_t = \frac{\beta}{1+\beta} \alpha_3 \left(k_t^1\right)^{\alpha_1} \left(k_t^2\right)^{\alpha_2}. \tag{11}$$

Consumption is always positive, so we can normalize prices by the price of current consumption. Under competition, firms will only produce goods with the highest market price. Hence we have

$$\max(p_t^1, p_t^2) \leq 1.$$

If $\max(p_t^1, p_t^2) < 1$, then $z_t^1 = z_t^2 = 0$. If $\max(p_t^1, p_t^2) = 1$, then the machine with the lower price will not be produced. If $p_t^1 = p_t^2 = 1$, then the composition of investment is indeterminate. Define aggregate gross investment per worker z by

$$z_t = z_t^1 + z_t^2$$

and the allocation-of-investment fraction σ by

$$\sigma_t = z_t^1 / z_t.$$

Then σ is the upper hemi-continuous correspondence given by

$$\sigma_t \begin{cases} = 1 & \text{if} \quad p_t^1 > p_t^2 \quad \text{and} \quad z_t > 0 \\ \in [0,1] & \text{if} \quad p_t^1 = p_t^2 \quad \text{and} \quad z_t > 0 \\ = 0 & \text{if} \quad p_t^1 < p_t^2 \quad \text{and} \quad z_t > 0 \\ \text{undefined} & \text{if} \quad \qquad\qquad\qquad\quad z_t = 0 \end{cases}. \tag{12}$$

Definition 1 *Given initial per capita capital stocks* (k_0^1, k_0^2)*, a long-run perfect-foresight equilibrium is given by the sequence of allocations* $\left\{k_{t+1}^1, k_{t+1}^2, s_t, x_t^y, x_t^o\right\}_{t=0}^{\infty}$ *and the sequence of non-negative prices* $\{r_t^1, r_t^2, p_t^1, p_t^2\}_{t=0}^{\infty}$ *such that equations (7), (6) and (10), and the market-clearing conditions (8) and (11) are satisfied.*

4 Steady State Growth

In the steady state, both capitals are produced,

$$z^i = \lambda k^i \qquad for\ i = 1, 2\ where\ \ \lambda = n + \mu, \tag{13}$$

prices must be the same,

$$p^1 = p^2 = 1, \tag{14}$$

and

$$y = (k_1)^{\alpha_1} (k_2)^{\alpha_2} \tag{15}$$

To have $p^1 = p^2 = 1$, we must have $r^1 = r^2$ and hence $k^1/k^2 = \alpha_1/\alpha_2$. This, together with equation (11), yields

$$k^1 = \left(\frac{\beta}{1+\beta}\frac{\alpha_3}{\alpha_1+\alpha_2}\frac{1}{1+n}\right)^{\frac{1}{\alpha_3}} \alpha_1^{\frac{1-\alpha_2}{\alpha_3}} \alpha_2^{\frac{\alpha_2}{\alpha_3}}, \tag{16}$$

$$k^2 = \left(\frac{\beta}{1+\beta}\frac{\alpha_3}{\alpha_1+\alpha_2}\frac{1}{1+n}\right)^{\frac{1}{\alpha_3}} \alpha_1^{\frac{\alpha_1}{\alpha_3}} \alpha_2^{\frac{1-\alpha_1}{\alpha_3}}, \tag{17}$$

and

$$\sigma = \frac{\alpha_1}{\alpha_1 + \alpha_2}. \tag{18}$$

The following proposition summarizes the results of this section.

Proposition 1 *In the steady state, the capital to labor ratios* k^1 *and* k^2*, output per worker* y*, and the fraction* σ *of gross investment directed to machinery of type-1 are uniquely determined.*

5 Dynamic Analysis

We assumed that once capital is installed it cannot be consumed. At the end of each period t, the value of the capital stock per worker is $p_t^1 (1-\mu) k_t^1 + p_t^2 (1-\mu) k_t^2$. The savings of young workers, $(\beta / (1+\beta)) \alpha_3 (k_t^1)^{\alpha_1} (k_t^2)^{\alpha_2}$, must be sufficient to buy the existing capital stock. For $z_t \geq 0$ to hold, we must have

$$p_t^1 (1-\mu) k_t^1 + p_t^2 (1-\mu) k_t^2 \leq \frac{\beta}{1+\beta} \alpha_3 \left(k_t^1\right)^{\alpha_1} \left(k_t^2\right)^{\alpha_2}. \tag{19}$$

For the time being, we will assume that this constraint is not binding. If $\max(p_t^1, p_t^2) \leq 1$, a sufficient condition for (19) to hold is

$$(1-\mu) \left(k_t^1 + k_t^2\right) \leq \frac{\beta}{1+\beta} \alpha_3 \left(k_t^1\right)^{\alpha_1} \left(k_t^2\right)^{\alpha_2}. \tag{20}$$

We will use this condition for now, but we relax it later.

Given our temporary assumption, there are three different regimes in which we can find the economy:

Regime 1. $1 = p_t^1 > p_t^2$,

Regime 2. $1 = p_t^2 > p_t^1$,

or

Regime 3. $1 = p_t^1 = p_t^2$

It is redundant to analyze both Regime 1 and Regime 2. We focus on Regime 1 and Regime 3.

Regime 1: $1 = p_t^1 > p_t^2$. Only capital of type 1 is produced, so we have $z_t^2 = 0$. Using the motion equations and the arbitrage condition, we have

$$k_{t+1}^2 = \frac{(1-\mu)}{(1+n)} k_t^2, \tag{21}$$

$$k_{t+1}^1 = \frac{\beta}{1+\beta} \frac{\alpha_3}{1+n} \left(k_t^1\right)^{\alpha_1} \left(k_t^2\right)^{\alpha_2} - p_t^2 \frac{1-\mu}{1+n} k_t^2, \tag{22}$$

and

$$p_{t+1}^2 = p_{t+1}^1 p_t^2 + \frac{r_{t+1}^1 p_t^2 - r_{t+1}^2}{1-\mu} \tag{23}$$

From equation (23), we know that if $r_{t+1}^1 p_t^2 < r_{t+1}^2$, then the price of Capital 2 will decrease (and hence we must have $p_{1,t+1} = 1$). In period $t+1$, the price of Capital 2 will decrease at a faster absolute rate, because only Capital 1 is produced, and the marginal productivity of Capital 1 relative to Capital 2 will have decreased. With the decrease in the price of Capital 2, the value of $r_{t+1}^1 p_t^2 - r_{t+1}^2$ will remain negative. It is easy to check that in finite time the price of Capital 2 will become negative. So this trajectory cannot be a long-run equilibrium path, one on which expectations are realized at every date. So we can easily conclude that

Proposition 2 *If we have $r_{t+1}^1 < r_{t+1}^2$ there is no pair of prices (p_1^t, p_2^t) satisfying $1 = p_t^1 > p_t^2$ that can support a long-run competitive equilibrium in which expectations are always fulfilled.*

If $r_t^1 \leq r_t^2$ and $1 = p_t^1 > p_t^2$, all new investment is directed towards k^1, and hence we have again, $r_{t+1}^1 \leq r_{t+1}^2$. This simple observation leads to the next corollary.

Corollary 1 *If $r_t^1 \leq r_t^2$ there is no pair of prices (p_1^t, p_2^t) satisfying $1 = p_t^1 > p_t^2$ that can support a long-run competitive equilibrium in which expectations are always fulfilled.*

These results tell us that to be on a long-run equilibrium path it must be the case that the price of the relatively scarce type of machines cannot be lower than the price of the relatively abundant type of machines.[3]

Regime 3: $1 = p_t^1 = p_t^2$. In this case we have:

$$k_{t+1}^1 = \frac{1-\mu}{1+n} k_t^1 + \frac{1}{1+n} \sigma_t z_t, \tag{24}$$

[3] We use the term "scarce" as a synonym for higher marginal productivity.

$$k_{t+1}^{2}=\frac{1-\mu}{1+n}k_{1}^{2}+\frac{1}{1+n}\left(1-\sigma_{t}\right)z_{t}, \tag{25}$$

and

$$\left(1-\mu\right)p_{t+1}^{1}+r_{t+1}^{1}=\left(1-\mu\right)p_{t+1}^{2}+r_{t+1}^{2}. \tag{26}$$

If $r_{t+1}^{1}=r_{t+1}^{2}$, we will have $p_{t+1}^{1}=p_{t+1}^{2}=1$. If the economy stays in this regime, it will then converge to the steady state. If $r_{t+1}^{i}>r_{t+1}^{j}$ then $p_{t+1}^{i}<p_{t+1}^{j}=1$, with $i,j=1,2$ and $i\neq j$. By the previous corollary we know that this is not compatible with long-run equilibrium in which expectations are always fulfilled. Hence we have the following result.

Proposition 3 *If $1=p_{t}^{1}=p_{t}^{2}$, only $r_{t+1}^{1}=r_{t+1}^{2}$ is compatible with a long-run competitive equilibrium trajectory.*

So far we have argued that the price of the relatively scarce type of capital must be equal to unity, so if we have $r_{t}^{2}\leq r_{t}^{1}$ we must have $1=p_{t}^{1}\geq p_{t}^{2}$.

We also know that $1=p_{t}^{1}\geq p_{t}^{2}$ and $r_{t+1}^{1}<r_{t+1}^{2}$ are not compatible with long-run competitive equilibrium, so if the initial conditions are such that Capital 1 is scarcer, it will remain so, unless, of course, their marginal productivities become equal. So if $r_{t}^{2}<r_{t}^{1}$, unless $r_{t+1}^{2}=r_{t+1}^{1}$, we must have $1=p_{t}^{1}>p_{t}^{2}$. If $r_{t+1}^{2}=r_{t+1}^{1}$, then $1=p_{t+1}^{2}=p_{t+1}^{1}$.

Once the economy is in a situation with $1=p_{t}^{2}=p_{t}^{1}$ and $r_{t}^{2}=r_{t}^{1}$ so that $k_{t}^{1}/k_{t}^{2}=\alpha_{1}/\alpha_{2}$, σ_{t} should be such that the ratio of Capital 1 to Capital 2 remains constant, $\sigma_{t}=\alpha_{1}/\left(\alpha_{1}+\alpha_{2}\right)$. Once the economy is in this path, with $k_{t}^{1}=\left(\alpha_{1}/\alpha_{2}\right)k_{t}^{2}$, the analysis is basically as in the typical Diamond OG economy. Simplifying equation (11), one can see that the dynamics are reduced to the study of the difference equation $k_{t+1}^{2}=A\left(k_{t}^{2}\right)^{\alpha}$,[4] a well-known difference equation. Hence, we know that the economy will converge to the unique steady-state.

Suppose that in period zero we have $k_{0}^{1}/k_{0}^{2}<\alpha_{1}/\alpha_{2}$. Finding the initial prices that are compatible with the long-run equilibrium trajectory is now reduced to the problem of finding

[4] Where $A=\beta/1+\beta\left(\alpha_{3}/\left(\left(1+n\right)\left(1+\alpha_{2}/\alpha_{1}\right)\right)\right)\left(\alpha_{2}/\alpha_{1}\right)^{\alpha_{2}}>0$ and $\alpha=\alpha_{1}+\alpha_{2}<1$.

the initial prices that guarantee that in some period $t^* = 0, 1, \ldots,$ we have $p^2_{t^*} = p^1_{t^*} = 1$ and that in the next period we have $r^2_{t^*+1} = r^1_{t^*+1}$. Suppose that $t^* > 0$. We would expect that, in equilibrium, as Capital 1 becomes relatively less scarce, the price of Capital 2 increases. This is easily confirmed. If, for $t < t^*$, $p^2_{t+1} \leq p^2_t$ we know that $r^1_{t+2}/r^2_{t+2} \leq r^1_{t+1}/r^2_{t+1} \leq 1/p_{2,t} \leq 1/p^2_{t+1}$. But $r^1_{t+2}/r^2_{t+2} \leq 1/p^2_{t+1}$ implies that $p^2_{t+2} \leq p^2_{t+1}$, so the price of Capital 2 cannot approach 1, contradicting our initial assumption. Therefore, if an equilibrium exists we will have $p^2_{t^*} > p^2_{t^*-1} > \cdots > p^2_0$.

This leads to the next proposition.

Proposition 4 *Let $k^1_0 < (\alpha_1/\alpha_2)\, k^2_0$. If there is a long-run equilibrium trajectory, it will be unique.*

Proof. Consider two equilibrium price sequences $\bar{p} = \{(\bar{p}^1_0, \bar{p}^2_0), (\bar{p}^1_1, \bar{p}^2_1), \ldots\}$ and $\tilde{p} = \{(\tilde{p}^1_0, \tilde{p}^2_0), (\tilde{p}^1_1, \tilde{p}^2_1), \ldots\}$

1. First we show that if $\bar{p}^2_0 > \tilde{p}^2_0$ and $\bar{p}^2_0 < \bar{p}^2_1 < \cdots < \bar{p}^2_{t^*-1} < \bar{p}^2_{t^*} = 1$, then we have $\bar{p}^2_t > \tilde{p}^2_t$ for $t \leq t^*$. In period 0, the relevant arbitrage condition is $p^2_{t+1} = p^2_t + \left(r^1_{t+1} p^2_t - r^2_{t+1}\right) / (1-\mu)$. All the new investment is devoted to Capital 1. The motion equations for capital are

$$\begin{aligned} k^2_1 &= \frac{(1-\mu)}{(1+n)} k^2_0 \\ k_{1,1} &= \frac{\beta}{1+\beta} \frac{(\alpha_3)}{(1+n)} k^{\alpha_1}_{1,0} k^{\alpha_2}_{2,0} - p_{2,t} \frac{(1-\mu)}{(1+n)} k_{2,0} \end{aligned} .$$

   If $\bar{p}^2_0 > \tilde{p}^2_0$ we have $\left\{ \begin{array}{l} \bar{k}^2_1 = \tilde{k}^2_1 \\ \bar{k}^1_1 < \tilde{k}^1_1 \end{array} \right.$ which implies $\left\{ \begin{array}{l} \bar{r}^2_1 < \tilde{r}^2_1 \\ \bar{r}^1_1 > \tilde{r}^1_1 \end{array} \right.$ which yields $\bar{r}^1_1 \bar{p}^2_0 - \bar{r}^2_1 > \tilde{r}^1_1 \tilde{p}^2_0 - \tilde{r}^2_1$, so we must have $\bar{p}^2_1 > \tilde{p}^2_1$. The same happens in the succeeding periods.

2. We have shown before that unless $k^1_1 = (\alpha_1/\alpha_2)\, k^2_1$, only $1 = p^1_0 > p^1_0$ is compatible with long-run equilibrium. If $k^1_1 = (\alpha_1/\alpha_2)\, k^2_1$, then we have $1 = p^1_0 = p^2_0$. Therefore, focus on the first case. Assume that $\bar{p}_{2,0} < \bar{p}_{2,1} < \cdots < \bar{p}_{2,t^*} = 1$ is compatible with

the long-run equilibrium. We know that for this to be a part of a long-run equilibrium trajectory we must have $\bar{k}^1_{t^*+1} = (\alpha_1/\alpha_2)\,\bar{k}^2_{t^*+1}$. Also note that at time t^*-1 we have $\bar{\sigma}_{t^*-1} = 1$.

3. Suppose that the alternative sequence, $\tilde{p}^2_0 < \tilde{p}^2_1 < \cdots < \tilde{p}^2_{t^{**}} = 1$, with $1 > \tilde{p}^2_0 > \bar{p}^2_0$, is also an equilibrium. Because of Step 1, we know that $t^{**} < t^*$. Since, as long as $p^2 < 1$, there is no new investment in Capital 2, at t^{**} we have $\tilde{k}^2_{t^{**}} = \bar{k}^2_{t^{**}}$. We also have $\tilde{k}^1_{t^{**}} < \bar{k}^1_{t^{**}}$. But we also know that $\bar{k}^1_{t^{**}+1} \leq (\alpha_1/\alpha_2)\,\bar{k}^2_{t^{**}+1}$. Since $\bar{k}^2_{t^{**}+1} \leq \tilde{k}^2_{t^{**}+1}$, and $\tilde{k}^1_{t^{**}+1} < \bar{k}^1_{t^{**}+1}$, we have $\tilde{k}^1_{t^{**}+1} < (\alpha_1/\alpha_2)\,\tilde{k}^1_{t^{**}+1}$ implying that the price sequence with $\tilde{p}^2_0 < \tilde{p}^2_1 < \cdots \tilde{p}^2_{t^{**}} = 1$ cannot be an equilibrium sequence.

■

With this result, we know that for any initial conditions if we find a long-run equilibrium path it will be unique. Again, suppose, without loss of generality, that we have $k^1_0 \leq (\alpha_1/\alpha_2)\,k^2_0$. If only capital of type 1 is produced, it is easy to check that eventually this inequality will be reversed. Given our previous results, we know that the equilibrium prices must be such that exactly in the period before the inequality is reversed, say at t', prices are both equal to unity. therefore, σ_t may take any value between zero and one, and can be appropriately chosen so that $k^1_{t'+1} = (\alpha_1/\alpha_2)\,k^2_{t'+1}$.

Using equation (5), it is apparent that to have $k^1_{t'+1} = (\alpha_1/\alpha_2)\,k^2_{t'+1}$ we must have at t'

$$(1-\mu)\,k^1_{t'} = \frac{\alpha_1}{\alpha_2}(1-\mu)\,k^2_{t'} + \left(\frac{\alpha_1}{\alpha_2}(1-\sigma_{t'}) - \sigma_{t'}\right) z_{t'}. \tag{27}$$

With $k^1_{t'} \leq (\alpha_1/\alpha_2)\,k^2_{t'}$, we would have $\sigma_{t'} \geq \alpha_1/(\alpha_1+\alpha_2)$. Therefore, $\sigma_{t'} \in [\alpha_1/(\alpha_1+\alpha_2), 1]$.

Consider an initial endowment of capital of type 2, say $k^2_0 = \bar{k}^2_0$, with $k^1_0 < (\alpha_1/\alpha_2)\,\bar{k}^2_0$. Is it possible to have $k^1_1 = (\alpha_1/\alpha_2)\,k^2_1$? Using equation (27), we can confirm that the lowest value that k^1_0 can take is $\underline{k^1_0} = [((\alpha_1+\alpha_2)/\alpha_2)((1+\beta)/\beta)((1-\mu)/\alpha_3)]^{1/\alpha_1}\left(\bar{k}^2_0\right)^{(1-\alpha_2)/\alpha_1}$. So if $k_{1,0} \in \left[\underline{k^1_0}, (\alpha_1/\alpha_2)\,k_{2,0}\right]$, $(p^1_0, p^2_0) = (1,1)$ is an equilibrium price.

If $k_0^1 < \underline{k_0^1}$, we have to check if it possible to have $k_2^1 = (\alpha_1/\alpha_2)\,k_2^2$. Noting that $k_1^2 = [(1-\mu)/(1+n)]\,k_0^2$, and that we need $(p_1^1, p_1^2) = (1,1)$, we can use (27) again to conclude that $\underline{k_1^1} = [((\alpha_1+\alpha_2)/\alpha_2)((1+\beta)/\beta)((1-\mu)/\alpha_3)]^{1/\alpha_1}\left(\bar{k}_1^2\right)^{(1-\alpha_2)/\alpha_1}$. So for $(p_1^1, p_1^2) = (1,1)$ to be an equilibrium $k_1^1 \in \left[\underline{k_1^1}, (\alpha_1/\alpha_2)((1-\mu)/(1+n))\,\bar{k}_0^2\right]$. To find the values of k_0^1 that are compatible with $k_1^1 \in \left[\underline{k_1^1}, (\alpha_1/\alpha_2)((1-\mu)/(1+n))\,\bar{k}_0^2\right]$, we can use the arbitrage equation $p_1^2 = p_0^2 + (r_1^1 p_0^2 - r_1^2)/(1-\mu)$ and $p_1^2 = 1$ to solve for p_0^2.

$$p_0^2 = \frac{(1-\mu) + \alpha_2\left(k_1^1\right)^{\alpha_1}\left(k_0^2(1-\mu)/(1+n)\right)^{\alpha_2-1}}{(1-\mu) + \alpha_1\left(k_1^1\right)^{\alpha_1-1}\left(k_0^2(1-\mu)/(1+n)\right)^{\alpha_2}}. \tag{28}$$

For $k_1^1 = (\alpha_1/\alpha_2)((1-\mu)/(1+n))\,\bar{k}_0^2$ we have $k_0^1 = \underline{k_0^1}$ and $(p_0^1, p_0^2) = (1,1)$. It is immediate that if $k_1^1 < (\alpha_1/\alpha_2)((1-\mu)/(1+n))\,k_0^2$ we have p_0^2, and the lower is k_1^1 the lower will be p_0^2. It is a matter of algebra to check that for $k_1^1 = \underline{k_1^1}$ we have $\underline{\underline{k_0^1}} = \left(\left(\left(\underline{k_1^1} + p_0^2\frac{(1-\mu)}{(1+n)}\bar{k}_0^2\right)/\left(\bar{k}_0^2\right)^{\alpha_2}\right)((1+\beta)/\beta)((1+n)/\alpha_3)\right)^{1/\alpha_1}$ and to find the corresponding price $p_0^2 = \underline{\underline{p_0^2}}$.

Putting everything together, for $k_0^1 \in \left[\underline{k_0^1}, (\alpha_1/\alpha_2)\,k_0^2\right]$, we have $(p_0^1, p_0^2) = (1,1)$. If $k_{1,0} \in \left[\underline{\underline{k_0^1}}, \underline{k_0^1}\right)$ we have $p_0^1 = p_1^1 = 1$, $p_0^2 \in \left[\underline{\underline{p_0^2}}, 1\right)$ and $p_1^2 = 1$.

If $k_0^1 < \underline{\underline{k_0^1}}$, then using the same procedure we have to check if it is possible to have $k_3^1 = (\alpha_1/\alpha_2)\,k_3^2$,derive $\underline{\underline{\underline{k_0^1}}}$ and $\underline{\underline{\underline{p_0^2}}}$ and so on. We know that at some date, say $t'+1$, it will be possible to have the equality $k_{t'+1}^1 = (\alpha_1/\alpha_2)\,k_{t'+1}^2$ and $(p_{t'}^1, p_{t'}^2) = (1,1)$. Hence we have the following result.

Proposition 5 *For any initial vector (k_0^1, k_0^2) of capitals per worker there is one initial price vector (p_0^1, p_0^2) compatible with the long-run competitive equilibrium in which expectations are always fulfilled.*

6 Tobin's $q < 1$

So far we have assumed that savings are sufficient to buy the existing capital stock, namely,

$$p_t^1(1-\mu)\,k_t^1 + p_t^2(1-\mu)\,k_t^2 \leq \frac{\beta}{1+\beta}\alpha_3\left(k_t^1\right)^{\alpha_1}\left(k_t^2\right)^{\alpha_2}$$

at prices satisfying $\max\left(p_t^1, p_t^2\right) = 1$.

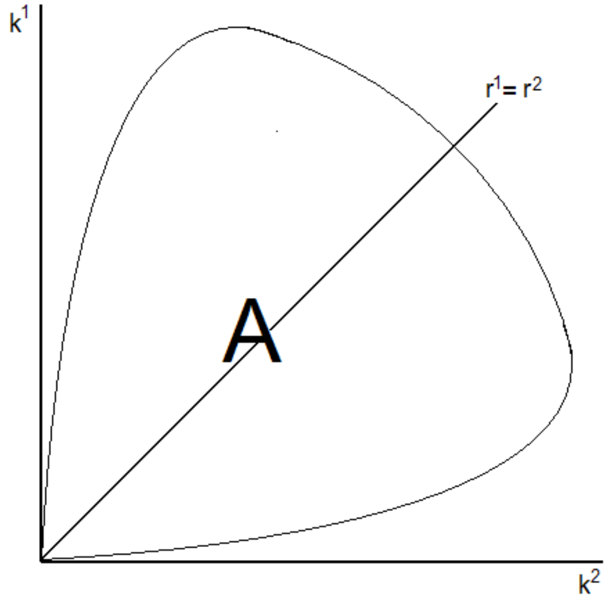

Figure 1: Region in which Tobin's $q = 1$

If the above constraint is not binding, we know that $\max\left(p_t^1, p_t^2\right) = 1$. A sufficient condition for the above inequality to hold with $\max(p^1, p^2) = 1$ is

$$k_t^1 + k_t^2 \leq \frac{\beta}{1+\beta}\frac{\alpha_3}{1-\mu}\left(k_t^1\right)^{\alpha_1}\left(k_t^2\right)^{\alpha_2}, \tag{29}$$

which implicitly defines the convex Region A in Figure 1. The slope of the frontier of A, when k^1 and k^2 are close to zero, is zero or infinity, depending on whether $k^2 > k^1$ or $k^2 < k^1$.

In this section, we assume that we are outside region A. Again, without loss of generality, we assume $k_0^1 < (\alpha_1/\alpha_2)\, k_0^2$. If we determine p_0^2 using the algorithm described in the previous section and we get $(1-\mu)\, k_0^1 + p_0^2\,(1-\mu)\, k_0^2 \leq (\beta/\,(1+\beta))\, \alpha_3 \left(k_0^1\right)^{\alpha_1} \left(k_0^2\right)^{\alpha_2}$, then the results described before still apply. But, if instead, we conclude that

$$(1-\mu)\, k_0^1 + p_0^2\,(1-\mu)\, k_0^2 > \frac{\beta}{1+\beta}\alpha_3\left(k_0^1\right)^{\alpha_1}\left(k_0^2\right)^{\alpha_2} \tag{30}$$

holds, p_0^2 cannot be an equilibrium price. If no new investment can be made in period zero, then in period 1 we will have $k_1^i = \left((1-\mu)\,/\,(1+n)\right) k_0^i,\ for\ i = 1, 2$. If the same happens again, we will have $k_2^i = ((1-\mu)/(1+n))^2 k_0^i$, and so on. Eventually the inequality will be reversed (otherwise we enter in region A, where we know for sure that the inequality will be reversed).

Suppose that in period 1 the inequality is reversed, meaning that $\bar{p}_1^2$ is an equilibrium price and

$$(1-\mu) k_1^1 + \bar{p}_1^2 (1-\mu) k_1^2 \leq \frac{\beta}{1+\beta} \alpha_3 \left(k_1^1\right)^{\alpha_1} \left(k_1^2\right)^{\alpha_2}.$$

In period zero, which prices lead to $(p_1^1, p_1^2) = (1, \bar{p}_1^2)$?

In equilibrium, the Inequality (30) cannot hold, so prices will have to adjust, so that

$$\bar{p}_0^1 (1-\mu) k_0^1 + \bar{p}_0^2 (1-\mu) k_0^2 = \frac{\beta}{1+\beta} \alpha_3 \left(k_0^1\right)^{\alpha_1} \left(k_0^2\right)^{\alpha_2}, \tag{31}$$

with $\max(\bar{p}_0^1, \bar{p}_0^2) < 1$.

The arbitrage condition must hold, which implies

$$\begin{aligned} \frac{(1-\mu) + r_1^1}{\bar{p}_0^1} &= \frac{(1-\mu)\bar{p}_1^2 + r_1^2}{\bar{p}_0^2} \\ \frac{\bar{p}_0^2}{\bar{p}_0^1} &= \frac{(1-\mu)\bar{p}_1^2 + r_1^2}{(1-\mu) + r_1^1} < 1 \end{aligned} \tag{32}$$

Since we know that $k_1^i = ((1-\mu)/(1+n)) k_0^i,\ for\ i = 1, 2$, we can use equations (31) and (32) to uniquely determine $(\bar{p}_0^1, \bar{p}_0^2)$.

This analysis can be extended to an arbitrary number of periods. E.g., if only in period 2 Inequality (30) is reversed, then, using the same algorithm, we can determine the prices of period 1. Knowing these, we can determine the prices in period 0.

7 Computed Examples - How long before the bubble must burst?

Our numerical exercises are inspired in part by Atkinson (1969)[5]. The parameter values used in our experiments are given in Table 1.

[5] See pages 144-148.

Table 1: Assumed Parameter Values

α_1	α_2	β	μ	n	k_0^1	k_0^2
0.2	0.2	0.6	0.55	0	1	5

In the 2-period-lifetime OG model, we identify "youth" with the working years and "old age" with the retirement years. One period in the OG model corresponds to roughly 20 years, so $\beta = 0.6$ corresponds to an annual discount factor on the order of 97.5%, while $\mu = 0.55$ corresponds to an annual depreciation rate of about 4%.

For the economy described by Table 1, the unique bubble-free growth is displayed in Table 2 and Figures 2 – 4. By assumption $k_0^2 > k_0^1$ and hence $r_0^1 > r_0^2$, meaning that type-1 capital is scarcer then type-2 capital. This is reflected in the capital-goods prices: $p_t^1/p_t^2 > 1$ for $t = 0, \ldots, 5$. By assumption, $(k_0^1 + k_0^2)$ is large for this economy. This is reflected in Tobin's q: $q_t = \max(p_t^1, p_t^2) < 1$ and $z_t^1 = z_t^2 = 0$ for $t = 0, 1, 2$, $q_t = 1 = p_t^1 > p_t^2$ for $t = 3, 4, 5$, $q_t = p_t^1 = p_t^2 = 1$ for $t = 6, 7, \ldots$. In period 6, prices are equal $q_6 = p_6^1 = p_6^2 = 1$, but marginal products are unequal $r_6^1 > r_6^2$ since $k_6^2 > k_6^1$, and investments are positive but unequal. After period 6, we have balanced investment: $q_t = p_t^1 = p_t^2 = 1$, $z_t^1 = z_t^2$, $k_t^1 = k_t^2$, and $r_t^1 = r_t^2$ for $t = 7, 8, \ldots$.Asymptotically the economy tends to balanced growth with $k^1 = k^2 = 0.026234$. There are no bubbles.

Table 2: Bubble-free growth path

t	k_t^1	k_t^2	p_t^1	p_t^2	z_t^1	z_t^2
0	1	5	0.341401	0.069692	0.000000	0.000000
1	0.45	2.25	0.541100	0.114554	0.000000	0.000000
2	0.2025	1.0125	0.82821	0.194055	0.000000	0.000000
3	0.091125	0.455625	1.000000	0.322857	0.011875	0.000000
4	0.052882	0.205031	1.000000	0.543669	0.017076	0.000000
5	0.040873	0.092264	1.000000	0.902200	0.017850	0.000000
6	0.036243	0.041518	1.000000	1.000000	0.014355	0.011981
7	0.030665	0.030665	1.000000	1.000000	0.014113	0.014113
8	0.027913	0.027913	1.000000	1.000000	0.014321	0.014321
9	0.026882	0.026882	1.000000	1.000000	0.014384	0.014384
10	0.026481	0.026481	1.000000	1.000000	0.014405	0.014405
11	0.026322	0.026322	1.000000	1.000000	0.014414	0.014414
12	0.026259	0.026259	1.000000	1.000000	0.014417	0.014417
13	0.026234	0.026234	1.000000	1.000000	0.014418	0.014418

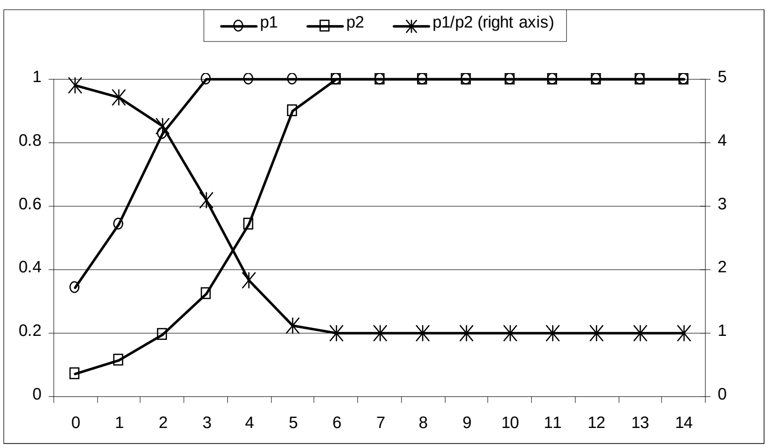

Figure 2: Prices on the bubble-free path

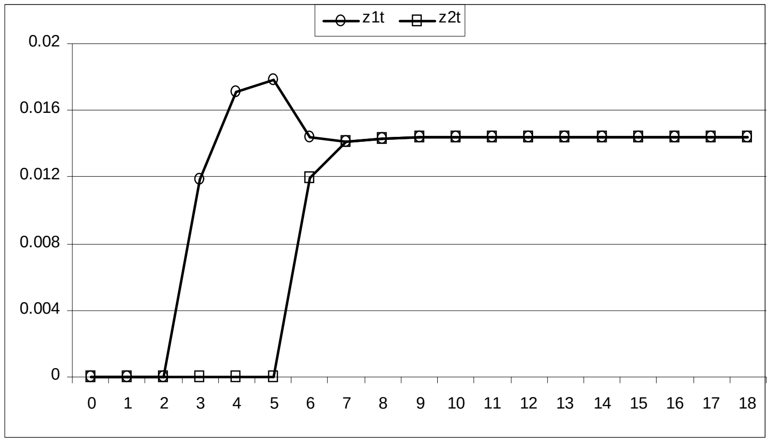

Figure 3: Savings and gross investments on the bubble-free path

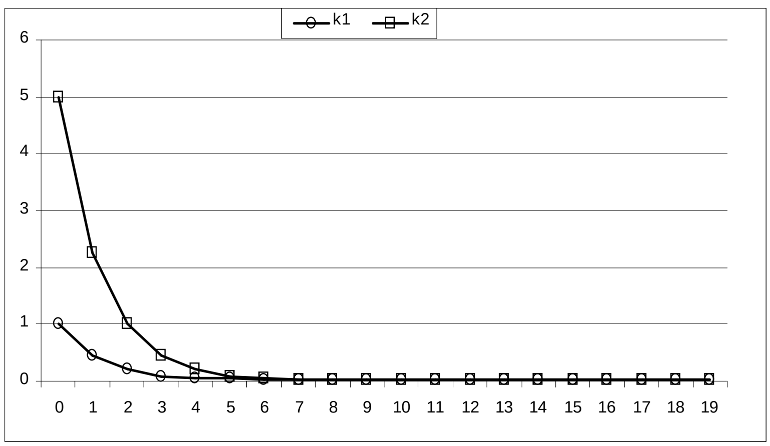

Figure 4: Capital/labor ratios on the bubble-free path

For Table 3 and Figure 5, we adopt the same economy as in the previous example (the one defined in Table 1), but we slightly perturb the initial prices from their bubble-free values. In particular p_0^1 is slightly larger than its bubble-free value. In the first 3 periods: $1 > q_t = p_t^1 > p_t^2$, $z_t^1 = z_t^2 = 0$ for $t = 0, 1, 2$ just as on the bubble-free path. In the next periods, Tobin's $q = 1$ and investment is specialized to type-1 capital: $1 = q_t = p_t^1 > p_t^2$, $z_t^1 > 0$, and $z_t^2 = 0$ for $t = 3, 4, 5$. By period 6, type-2 capital is scarcer, but the economy is investing only in type-1 capital: $k_6^1 > k_6^2$, $r_6^2 > r_6^1$, $1 = q_6 = p_6^1 > p_6^2$. The growth path cannot be extended to period 7, because $p_7^2 < 0$ would be impossible because with free disposal the rate of return on type-1 capital would exceed the rate of return on type-2 capital. The bubble must burst before period 7.

Table 3: Bubble-path-I

t	k_t^1	k_t^2	p_t^1	p_t^2	z_t^1	z_t^2
0	1	5	0.341441	0.069684	0	0.000000
1	0.45	2.25	0.541281	0.114518	0	0.000000
2	0.2025	1.0125	0.829027	0.193893	0	0.000000
3	0.091125	0.455625	1.000000	0.321334	0.012188	0.000000
4	0.053194	0.205031	1.000000	0.530791	0.018232	0.000000
5	0.042169	0.092264	1.000000	0.786965	0.022513	0.000000
6	0.041489	0.041519	1.000000	0.150000	0.041538	0.000000
7	0.060208	0.018683	1.000000	< 0	$-$	$-$

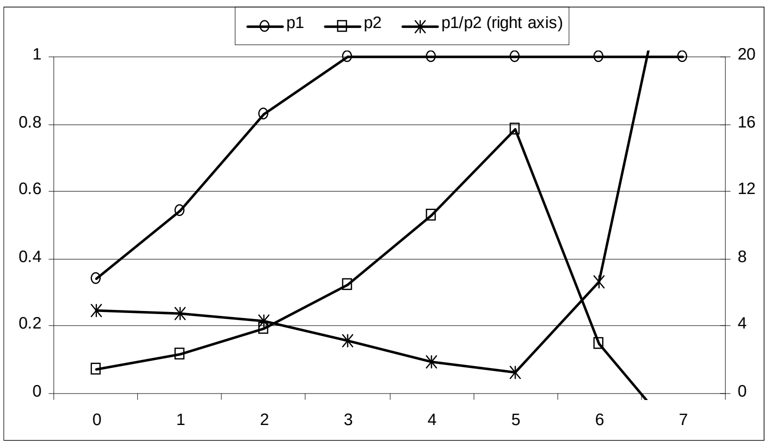

Figure 5: Prices on bubble-path-I

In Table 4 and Figures 6 – 7, we display Bubble-path-II. This is based on the same economy as analyzed in the previous examples (and described in Table 1) except that we set p_0^1 slightly below (rather than slightly above) its value on the bubble-free path. In the first 3 periods, we have $1 > q_t = p_t^1 > p_t^2$ and $z_t^1 = z_t^2 = 0$ for $t = 0, 1, 2$, just as on the bubble-free path. But here p_t^2 is increasing faster than on the bubble-free path. By period 5 (instead of period 6), we have $p_2^5 = 1 = q_5$, $p_5^1 < 1$, $z_2^5 > 0$, $z_1^5 = 0$, $k_5^2 > k_5^1$ and hence $r_5^1 > r_5^2$. In period 6, there is a switch in regimes from producing the relatively scarce capital good to producing the relatively abundant capital good. On this path, p_6^1 would become negative, which is impossible if there is free disposal of capital. Hence in period 5, the rate of return on machinery of type-1 would exceed that for machinery of type-2. Hence the bubble must burst before period 6.

Table 4: Bubble-path-II: Regime switching

t	k_t^1	k_t^2	p_t^1	p_t^2	z_t^1	z_t^2
0	1	5	0.341229	0.069727	0.000000	0.000000
1	0.45	2.25	0.540329	0.114709	0.000000	0.000000
2	0.2025	1.0125	0.824758	0.194747	0.000000	0.000000
3	0.091125	0.455625	1.000000	0.329376	0.010539403	0.000000
4	0.051545	0.205031	1.000000	0.600000	0.012016138	0.000000
5	0.035212	0.092264	0.677374	1	0.000000	0.019284287
6	0.015845	0.060803	< 0	1	–	–

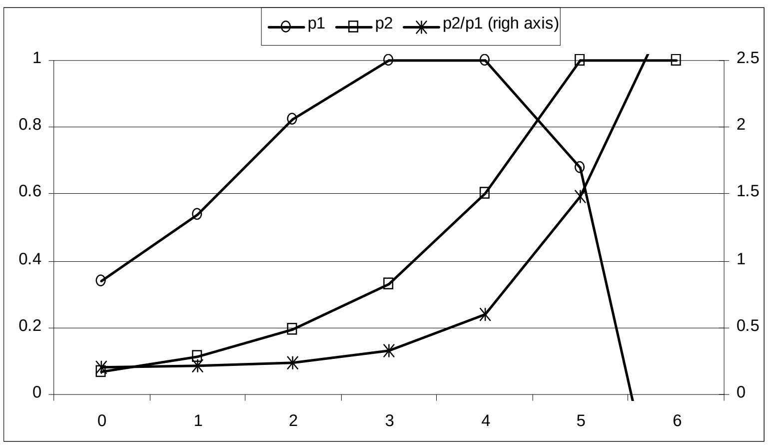

Figure 6: Prices on bubble-path-II

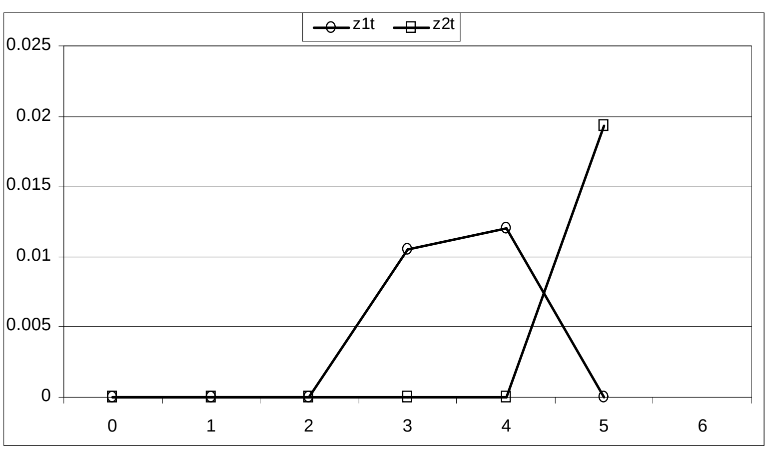

Figure 7: Investments on bubble-path-II

We have also investigated economies with parameter values different from those given in Table 1. Qualitatively, the results remain the same, although there are some differences. For example, the higher the depreciation rate the quicker the bubble will burst. The larger are the depreciation rates the smaller are the capital gains. Therefore, changes in the prices will have to be even greater to compensate for the differences in yields, leading to shorter-lived bubbles. If the depreciation rate were 100% , there would be no capital gains and hence there would be no perfect-foresight bubbles. Similarly, if expectations about prices were static, there would be no expected capital gains and hence no bubbles.

8 Summary

We have investigated asset prices and capital gains in a perfect-foresight economy. Our model is essentially a combination of the Shell and Stiglitz (1967) growth model with the Diamond (1965) OG model. We assume that investment is irreversible, allowing used machines to be sold for less than their replacement values: Tobin's q can be less than unity.[6] Just as the results of Shell, Sidrauski and Stiglitz (1967) for the money-and-single-capital growth model carry over in the Tirole (1985) OG model, the main results of Shell and Stiglitz are unchanged in the OG environment. There is a unique competitive-equilibrium path in which expectations are always fulfilled. Complete futures markets in machinery imply that this bubble-free path is the only one that will be pursued. Even if future markets are not complete (as in the real world), the bubble trajectories will be revealed to be disequilibrium paths, but only after some decades, or centuries, or more. Bubble trajectories are not equilibrium trajectories in the usual sense, but they test the usual definitions.

Comparing the analysis of the present paper with that of Shell and Stiglitz (1967) also reveals that introducing agent optimization and discrete time allows for a richer dynamics. For example, in Shell and Stiglitz the prices of the two capitals would be the same (and equal

[6]See Magill and Quinzii (2003).

to unity) only when their marginal productivities are the same. In the present model, prices become the same (and equal to unity) exactly one period before the marginal productivities are equalized.

The present paper is our second attempt to analyze capital gains in an OG economy with 2 capitals and perfect foresight. In Aguiar-Conraria and Shell (forthcoming), we focused on the degenerate case in which the 2 machines can be distinguished only by their colors (blue or red): their productivities and their replacement costs if newly produced are independent of their color, but their market prices are allowed to depend on color. We showed that, on the unique bubble-free trajectory, the prices of the 2 capitals are always equal.

9 Concluding Remarks

Capital gains are at the heart of the capitalist economy, but they are suspected of being a source of instability.

Keynes mistrusted them. He went so far as to suggest that capital ownership be made (like marriage) indissoluble except for grave cause in order to mitigate the effects of stock market speculation.

One interpretation of the Great Depression is that expected capital gains on holding money were very high (the general price level was falling) so that Tobin's q was driven below unity leading to drying up of investment.

The analysis of capital gains raises fundamental questions about the formation of expectations and the nature of temporary equilibrium. These are subjects in which Jean-Michel Grandmont is the master.[7] There may also be a role for sunspots. Our formal analysis shows that in our particular (non-monetary!) model, the only fully equilibrium path is bubble free. But on our calculated trajectories, the bubble is revealed only after several decades. In a technical sense, bubble paths are not perfect-foresight equilibrium paths, but bubbles that

[7] E.g., see Grandmont (1974, 1977, 1983, 1985) and Grandmont and Hildenbrand (1974).

burst in the far future beyond current lifetimes stretch the equilibrium concept. Perhaps the game-theoretic foundations of the expectations process should be re-examined.

Individuals perceive capital gains as part of income, as they do dividend and interest receipts. Individuals perceive capital gains as accretions to wealth and hence part of saving. Traditional measures of income and saving that do not include capital gains can be misleading.

Our model is special. It is non-monetary. Money allows for non-bursting bubbles. It exhibits saddlepoint dynamics. Not all multi-asset dynamics are of this type.[8] Our goal was to study a simple example in some detail to partially redirect the macro literature from the ILRA model to one (such as the heterogeneous agent, OG model) that might allow for destabilizing effects from capital gains.

[8] See, e.g. Benhabib and Rustichini (1994).

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