CAE Working Paper #04-16

On the Continuity of Ethical Social Welfare Orders

by

Kuntal Banerjee
and
Tapan Mitra

December 2004

.

On the Continuity of Ethical Social Welfare Orders

Kuntal Banerjee* and Tapan Mitra †

December 18, 2004

Abstract

In this paper we study the extent to which ethical social welfare orders on infinite utility streams can be continuous. For a class of metrics, we show that ethical preferences can be continuous if and only if the continuity requirement is in terms of a metric which satisfies a simplex condition. This condition requires that the distance from the origin to the unit simplex in the space of utility streams be positive. We use this characterization result to establish that the metric used by Svensson (1980) induces the weakest topology for which there exist continuous ethical preferences.

Journal of Economic Literature Classification Numbers: D60, D70, D90.

Keywords and Phrases: Intergenerational Equity, Pareto Axiom, Metrics, Topologies, Continuous Social Welfare Orders, Simplex Condition.

*Department of Economics, Uris Hall 4th floor, Ithaca 14853. kb236@cornell.edu

†Department of Economics, Uris Hall 4th floor, Ithaca 14853. tm19@cornell.edu

1 Introduction

In comparing social states which are specified by infinite utility streams, there are two widely accepted guiding principles. The equal treatment of all generations, proposed by Ramsey (1928), is formalized in the Anonymity Axiom. The positive sensitivity of the social preference structure to the well-being of each generation is reflected in the Pareto Axiom. A social welfare ordering (a binary relation on the social states, satisfying completeness and transitivity) satisfying both axioms is said to reflect *ethical preferences*.

Diamond (1965) showed that any social welfare ordering for which the lower contour set of each state is closed in the sup metric (a weak continuity requirement) cannot satisfy the Anonymity and Pareto Axioms simultaneously.[1]

Since the continuity requirement is a technical restriction, this result raises the question whether there exists any social welfare ordering respecting the Anonymity and Pareto axioms. Svensson (1980) was the first to show that such an ordering does exist.

Taken together, the impossibility result of Diamond and the possibility result of Svensson form the basis for all subsequent research on this topic[2]. They naturally lead us to enquire, loosely speaking, *how much* continuity of a social welfare ordering brings the Anonymity and Pareto axioms in conflict with each other. Formally, one would like to identify the weakest topology on the space of infinite utility streams under which ethical preferences can be continuous.

Svensson (1980) raised this question (p.1254, lines 14-15), and while he did not provide a complete answer to it, he did construct a social welfare ordering, satisfying the Anonymity and Pareto axioms, which satisfies continuity in a metric, which is based on the ℓ_1 norm (called a bounded ℓ_1 metric in what follows). Although several attempts have been made at studying continuity in this setting, there is still no satisfactory answer to Svensson's question. Lauwers (1997) and Shinotsuka (1998) have analyzed the implications, for the existence of ethical preferences, of imposing continuity with respect to specific metrics. Sakai (2004) has addressed the issue of restricting

[1] Actually, the statement of Diamond's result (p.176, lines 19-21, which he attributes to Yaari) is weaker, since he imposes continuity of the social welfare ordering in the sup metric. However, the proof clearly shows that the stronger result stated here is valid.

[2] See, among others, Campbell (1985), Fleurbaey and Michel (2003), Lauwers (1997), Shinotsuka (1998) and Sakai (2003, 2004).

the domain of preferences to escape the Diamond impossibility theorem. He finds that, if one fixes sup-norm continuity as the "natural" continuity axiom, the existence of ethical preferences implies a domain with empty interior in the sup metric.

In this paper we address Svensson's question in the framework used by Diamond (1965)[3]. For a class of metrics (which is general enough to accommodate most of the commonly used metrics in this literature) we identify a (simplex) condition that completely characterizes the possibility-impossibility divide. Specifically, we show that a social welfare order can simultaneously satisfy the Anonymity, Pareto and Continuity axioms *if and only if* one imposes the continuity requirement in terms of a metric, such that the distance from zero to the unit simplex in X (the space of utility streams) is positive. In other words, in order for ethical preferences to be continuous, utility sequences lying in the unit simplex of X (and therefore bounded away from zero in the metric generated by the ℓ_1 norm) must also be bounded away from zero in the metric in terms of which the continuity requirement is imposed.

We use our characterization result to show that for our class of metrics, the bounded ℓ_1 metric used by Svensson induces the weakest topology under which there exists a social welfare ordering satisfying the Anonymity, Pareto and Continuity axioms simultaneously. This settles a long-standing open question in this literature.

2 Preliminaries

2.1 Notation and Definitions

Let $\mathbb{N}$ denote, as usual, the set of natural numbers $\{1, 2, 3, ...\}$, and let $\mathbb{R}$ denote the set of real numbers. Let Y denote the closed interval $[0, 1]$, and let X denote the set $Y^{\mathbb{N}}$. Then, X is the domain of utility sequences (also, referred to as "utility streams") that we are interested in. Hence, $x \equiv (x_1, x_2, \ldots) \in X$ if and only if $x_n \in [0, 1]$ for all $n \in \mathbb{N}$. The sequence $(0, 0, 0, ...) \in X$ will be denoted by 0, and the sequence $(1, 1, 1, ...)$ will be denoted by e.

For $x \in X$, and $N \in \mathbb{N}$, we denote $(x_1, ..., x_N)$ by $x(N)$ and $(x_{N+1}, x_{N+2}, ...)$

[3] In his framework, the utility levels possible in any period belong to the interval $Y = [0, 1]$, and the social welfare order is defined on $X = Y^{\mathbb{N}}$, the space of infinite utility streams.

by $x[N]$. Thus, given any $x \in X$ and $N \in \mathbb{N}$, we can write $x = (x(N), x[N])$. If $x, y \in X$, and $N \in \mathbb{N}$, we write $z = (x(N), y)$ to denote the element $z \in X$, satisfying $z_k = x_k$ for all $k \in \{1, .., N\}$ and $z_{M+k} = y_k$ for all $k \in \mathbb{N}$. If $x \in X$ and $\sum_{n=1}^{\infty} x_n < \infty$, we define $I(x) = \sum_{n=1}^{\infty} x_n$. The *unit simplex* in X is the set $S = \{x \in X : I(x) = 1\}$.

For $y, z \in \mathbb{R}^{\mathbb{N}}$, we write $y \geq z$ if $y_i \geq z_i$ for all $i \in \mathbb{N}$; and, we write $y > z$ if $y \geq z$, and $y \neq z$. For $x, y \in X$, we will denote $(|x_1 - y_1|, |x_2 - y_2|, ...) \in \mathbb{R}_+^{\mathbb{N}}$ by $|x - y|$.

A *social welfare relation* (SWR) is a binary relation, $\succsim$ on X , which is reflexive and transitive (a quasi-ordering). We associate with $\succsim$ its symmetric and asymmetric components in the usual way. Thus, we write $x \sim y$ when $x \succsim y$ and $y \succsim x$ both hold; and, we write $x \succ y$ when $x \succsim y$ holds, but $y \succsim x$ does not hold. A *social welfare ordering* (SWO) is a binary relation, $\succsim$ on X , which is complete[4] and transitive (an ordering). Given an ordering $\succsim$ on X, for each $x \in X$ the *Lower and Upper Contour Sets* are defined as $LC(x) = \{y \in X : x \succsim y\}$ and $UC(x) = \{y \in X : y \succsim x\}$ respectively.

2.2 Ethical Preferences

The requirements that we would want any social welfare order $\succsim$ defined on X to satisfy are the Anonymity and Pareto axioms, which can be stated as follows.

Axiom 1 (Anonymity) *If x, y are in X, and there exist i, j in $\mathbb{N}$, such that $x_i = y_j$ and $x_j = y_i$, while $x_k = y_k$ for all $k \in \mathbb{N}$, such that $k \neq i, j$, then $x \sim y$.*

Axiom 2 (Pareto) *If $x, y \in X$, and $x \geq y$, then $x \succsim y$; further, if $x > y$, then $x \succ y$.*

The axiom of Anonymity requires an ordering to treat generations equally, however far out in time they are. The Pareto axiom requires that the ordering be positively sensitive to each generation's utility level. A social welfare order $\succsim$ satisfying the Anonymity and Pareto axioms is called an *ethical social welfare ordering*. [The term *ethical preferences* is used synonymously]. It is known (see Svensson (1980)) that ethical preferences on X exist.

[4]Since completeness implies reflexivity, a social welfare ordering is a social welfare relation, which is complete.

2.3 A Class of Metrics and the Continuity Axiom

Unlike the Pareto and Anonymity axioms, any Continuity axiom on X requires one to formalize the notion of utility streams being "close to each other". In this subsection, we do this in terms of a *metric* (the term *distance function* is used synonymously); that is, we introduce the Continuity axiom on a *metric space* (X, d), where d is a metric from $X \times X$ to $\mathbb{R}_+$.

We consider the class of metrics which satisfy the following four properties:

(M.1) If $x, y \in X$, then $d(x, y) = d(|x - y|, 0)$.

(M.2) If $x, y \in X$ be such that $x \geq y$ then $d(x, 0) \geq d(y, 0)$.

(M.3) If $x \in X$ and $M \in \mathbb{N}$, then $d((0(M), x), 0) \leq d(x, 0)$.

(M.4) If $\lambda^n \in [0, 1]$ for $n \in \mathbb{N}$, and $\lambda^n \to 0$ as $n \to \infty$ then, $d(\lambda^n(e(M), 0[M]), 0) \to 0$ as $n \to \infty$ for every $M \in \mathbb{N}$.

The condition (M.1) requires that the distance between two streams in X depends only on the component-wise differences between the streams[5]. The condition (M.2) is a weak monotonicity property. Condition (M.3) requires that the distance of a stream from zero does not increase if the stream in question is shifted forward in time by a finite number of periods, with the initial time period components set equal to zero[6]. Condition (M.4) pertains to streams converging to zero; it requires that convergence to zero in terms of d certainly include the case of convergence of any given *finite number* of components to zero (when all other components are fixed at zero).

While conditions (M.1)-(M.4) do impose restrictions on the class of metrics studied, it will be noted that the most commonly used metrics in this literature satisfy all these conditions. Here are a few examples:

$$\left.\begin{array}{l}
\text{(i) } d_c(x, y) = \sum_{k=1}^{\infty}(|x_k - y_k| / 2^k) \\
\text{(ii) } d_u(x, y) = \sup_{k \in \mathbb{N}} |x_k - y_k| \\
\text{(iii) } d_p(x, y) = \min\{1, [\sum_{k=1}^{\infty} |x_k - y_k|^p]^{(1/p)}\} \; for \; p \in (1, \infty) \\
\text{(iv) } d_1(x, y) = \min\{1, \sum_{k=1}^{\infty} |x_k - y_k|\} \\
\text{(v) } d_q(x, y) = \min\{1, \sum_{k=1}^{\infty} |x_k - y_k|^q\} \; for \; q \in (0, 1)
\end{array}\right\} (D)$$

The distance function d_c generates the co-ordinatewise convergence or the product topology, d_u the uniform (or sup-norm) topology (both d_c and d_u were used by Diamond (1965)). The distance function d_p generates the

[5] In particular, this property makes the metric *translation invariant*: if $x, y \in X$ and $z \in \mathbb{R}^{\mathbb{N}}$, such that $(x + z), (y + z) \in X$, then $d(x + z, y + z) = d(x, y)$.

[6] Note that (M.3) is weaker than requiring *shift invariance* of the distance.

bounded ℓ_p metric topology (for $1 < p < \infty$), d_1 the bounded ℓ_1 metric topology (used by Svensson (1980)), and d_q the bounded ℓ_q metric topology (for $0 < q < 1$). The metrics have been listed in the order of the "strength" of the topologies induced by them, the product topology being the weakest (among the five). A weaker topology has fewer open sets, and therefore a continuity axiom in terms of such a topology is a more demanding axiom.

The class of metrics satisfying (M.1)-(M.4) will be denoted by Δ. Given a metric $d \in \Delta$, one can define a function $f : X \to \mathbb{R}_+$ by:

$$f(x) = d(x, 0) \quad for\ all\ x \in X$$

The function, f, satisfies some useful properties[7] which we note below:

(F.1) $f(x) = 0$ if and only if $x = 0$.

(F.2) For any $x, y \in X$, with $(x + y) \in X$, we have $f(x + y) \leq f(x) + f(y)$.

(F.3) For all $\lambda \in [0, 1]$ and $x \in X$, $f(\lambda x) \leq f(x)$.

(F.4) If $x^n \in X$ for $n \in \mathbb{N}$, and $f(x^n) \to 0$ as $n \to \infty$, then for every $\lambda \in \mathbb{R}_+$, such that $\lambda x^n \in X$ for $n \in \mathbb{N}$, $f(\lambda x^n) \to 0$ as $n \to \infty$.

(F.5) If $x \in X$, and $\lambda^n \in [0, 1]$ for $n \in \mathbb{N}$, with $\lambda^n \to 0$ as $n \to \infty$, then $f(\lambda^n(x(M), 0[M])) \to 0$ as $n \to \infty$, for every $M \in \mathbb{N}$.

Without further mention, all metrics discussed henceforth will be taken to belong to Δ. We are now ready to state the Continuity axiom.

Axiom 3 (d-Continuity): *The set $LC(x)$ is a closed subset of the metric space (X, d) for every $x \in X$.*

Note that the continuity axiom is stated in a weak form: the usual continuity axiom asserts that both the upper contour set and the lower contour set of each point $x \in X$ is closed (see, for example, Diamond (1965) or Svensson (1980)).

[7] The properties make the function, f, behave somewhat like a F-norm. However, f *does not* satisfy the following property of F-norms: if $x \in X$, and $\lambda^n \in [0, 1]$ for $n \in \mathbb{N}$, with $\lambda^n \to 0$ as $n \to \infty$, then $f(\lambda^n x) \to 0$ as $n \to \infty$. For a comprehensive discussion of F-norms, see Köthe (1969, p.163).

3 The Characterization Result

3.1 Statement and Discussion

In this section, we state and prove our characterization result, which provides a relatively easy to check condition that is both necessary and sufficient for the Anonymity, Pareto and Continuity axioms to be simultaneously satisfied by any social welfare ordering.

Given any metric $d \in \Delta$, we define the distance between a point $\bar{x} \in X$ and a set $A \subset X$ in the usual way:

$$d(\bar{x}, A) = \inf_{x \in A} d(x, \bar{x})$$

The basic condition involved in the characterization result can now be stated as follows (recalling that S is the unit simplex in X).

Condition S *(Simplex Condition)* $d(0, S) > 0$.
This means that utility sequences lying in the unit simplex of X (and therefore bounded away from zero in the bounded ℓ_1 metric) must be bounded away from zero in terms of the chosen metric, d.

Theorem 1 *A social welfare ordering $\succsim$ on X can satisfy the Anonymity, Pareto and d -Continuity axioms simultaneously (where $d \in \Delta$) if and only if d satisfies Condition S.*

Just as a quick application of this characterization result, it is instructive to investigate what it implies for the five distance functions listed in (D) above. It is known from Diamond (1965) that ethical preferences cannot satisfy the Continuity axiom in terms of the metrics d_c and d_u, and these metrics clearly violate Condition S, as they should according to the theorem. At the other end of the spectrum, it is known from Svensson (1980) that there exist ethical preferences which are continuous in the metric d_1 (and therefore in any metric, like d_q, which induces a stronger topology than d_1 does), and the metrics d_1 and d_q clearly satisfy Condition S, as they should according to the theorem. This leaves us with the class of bounded ℓ_p metrics (for $1 < p < \infty$). The literature has little to say about this case. However, these metrics clearly also violate Condition S, and so we can infer from the theorem that ethical preferences cannot be continuous in terms of the bounded ℓ_p metrics.

Actually, it is not a coincidence that Svensson's bounded ℓ_1 metric provides the appropriate dividing line between possibility and impossibility results among the metrics listed in (D). In fact, it provides such a dividing line among all metrics $d \in \Delta$, a result which is worth noting formally.

Theorem 2 *Among all metrics $d \in \Delta$, Svensson's bounded ℓ_1 metric induces the weakest topology under which there exist continuous ethical preferences.*

3.2 Proofs

Before coming to the proofs of the two theorems stated above, we note a useful lemma, relating the behavior of a metric with respect to the unit simplex, to its behavior with respect to α -simplexes, where $\alpha \in (0, 1)$. It is used in the proofs of both the theorems.

Lemma 1 *Let d be a metric in Δ. Then, for each $M \in \mathbb{N}$, we have:*

$$d(0, S(1/M)) \geq d(0, S)/M \tag{1}$$

where:

$$S(1/M) = \{x \in X : \sum_{k=1}^{\infty} x_k = (1/M)\} \;\; for\ M \in \mathbb{N} \tag{2}$$

Proof. Let $M \in \mathbb{N}$ be given, and let z be an arbitrary element in $S(1/M)$. Then, by (2), $Mz \in X$, and by property (F.2), we have $f(Mz) \leq Mf(z)$. Also, by (2), $Mz \in S$, and so $d(0, S) \leq f(Mz)$. Thus, we have $f(z) \geq d(0, S)/M$. Since z was an arbitrary element in $S(1/M)$, (1) must hold. ■

It is convenient to break up the proof of the characterization result (Theorem 1) into three steps. First, we follow the technique of Diamond (1965) to prove the impossibility result. Second, we establish a lemma, which is useful in addressing the possibility result. Third, we use the lemma and follow the method of Svensson (1980) to prove the possibility result.

Proposition 1 *A social welfare ordering $\succsim$ on X cannot satisfy the Anonymity, Pareto and d -Continuity axioms simultaneously (where $d \in \Delta$) if d violates Condition S.*

Proof. Suppose, contrary to what has to be proved, that there exists a SWO $\succsim$ satisfying the Anonymity, Pareto and d-Continuity axioms (where $d \in \Delta$), even though d violates Condition S. Then, there is a sequence $\{z^N\}_{N=1}^{\infty}$ with $z^N \in S$ for all $N \in \mathbb{N}$, and $d(z^N, 0) \to 0$ as $N \to \infty$.

For each $N \in \mathbb{N}$, using the fact that $I(z^N) = 1$, we can choose $k(N) \in \mathbb{N}$, such that:

$$\alpha(N) \equiv \sum_{n=1}^{k(N)} z_n^N \geq [(N-1)/N] \tag{3}$$

and define $y^N = (y_1^N, \ldots, y_{k(N)}^N)$ as follows:

$$\left.\begin{array}{lcl} y_1^N & = & z_1^N \\ y_2^N & = & z_1^N + z_2^N \\ \cdots & \cdots & \cdots \\ y_{k(N)}^N & = & z_1^N + z_2^N + \cdots + z_{k(N)}^N \end{array}\right\} \tag{4}$$

Define $x \in X$ as follows:

$$x = (0, y^1, 0, y^2, 0, y^3, \ldots) \tag{5}$$

Similarly, define $\bar{x} \in X$ (by replacing the first zero in x by 1) as follows:

$$\bar{x} = (1, y^1, 0, y^2, 0, y^3, \ldots) \tag{6}$$

We now construct a sequence of points (x^N), where $x^N \in X$ for $k \in \mathbb{N}$, such that (i) each x^N is indifferent to x, and (ii) x^N converges to $\bar{x}$ in terms of the d metric. To this end, define, for each $N \in \mathbb{N}$:

$$x^N = (\alpha(N), y^1, 0, y^2, \ldots, 0, 0, y_1^N, y_2^N, ..., y_{k(N)-1}^N, 0, y_1^{N+1}, 0, ...) \tag{7}$$

Note that x^N is obtained from x, by interchanging the $\alpha(N)$ appearing as the last term in the vector y^N in (4) with the first 0 in (5), and then interchanging $(y_1^N, y_2^N, ..., y_{k(N)-1}^N, 0)$ in the resulting sequence with $(0, y_1^N, y_2^N, ..., y_{k(N)-1}^N)$. By the Anonymity axiom, we must therefore have $x^N \sim x$ for all $N \in \mathbb{N}$.

Note that for $N \in \mathbb{N}$, we have:

$$\begin{array}{rcl} d(x^N, \bar{x}) & = & f(1-\alpha(N), 0, ..., 0, z_1^N, z_2^N, ..., z_{k(N)}^N, 0, 0, ...) \\ & \leq & f(1-\alpha(N), 0, 0, ...) + f(0, 0, ..., 0, z_1^N, z_2^N, ..., z_{k(N)}^N, 0, 0, ...) \\ & \leq & f((1/N), 0, 0, ...) + f(0, 0, ..., 0, z_1^N, z_2^N, ..., z_{k(N)}^N, 0, 0, ...) \\ & \leq & f((1/N), 0, 0, ...) + f(z_1^N, z_2^N, ..., z_{k(N)}^N, 0, 0, ...) \\ & \leq & f((1/N), 0, 0, ...) + f(z^N) \end{array} \tag{8}$$

the first line in (8) following from (M.1), the second line from (F.2), the third line from (M.2) and (3), the fourth line from (M.3), and the last line from (M.2). Thus, using (M.4), we have:

$$d(x^N, \bar{x}) \to 0 \; as \; N \to \infty \tag{9}$$

Since $x^N \in LC(x)$ for all $N \in \mathbb{N}$, (9) implies that $\bar{x} \in LC(x)$, by the d-Continuity axiom. But, from (5) and (6), it is clear that $\bar{x} \succ x$ by the Pareto axiom, and this contradiction establishes the result. ■

Lemma 2 *Suppose $d \in \Delta$ satisfies Condition S. Suppose $x \in X$, and $\{x^N\}_{N=1}^{\infty}$ is a sequence, satisfying $x^N \in X$ for all $N \in \mathbb{N}$ and:*

$$\lim_{N \to \infty} d(x^N, x) = 0 \tag{10}$$

then:

$$\sum_{k=1}^{\infty} \left| x_k^N - x_k \right| \to 0 \; as \; N \to \infty \tag{11}$$

Proof. For each $N \in \mathbb{N}$, denote $\left| x_k^N - x_k \right|$ by z_k^N for all $k \in \mathbb{N}$. Suppose, contrary to the Lemma, that there is $M \in \mathbb{N}$ and a subsequence of (z^N) [retain notation], such that:

$$\sum_{n=1}^{\infty} z_n^N > (1/M) \;\; for \; all \; N$$

Denote $(1/M)$ by β. Then, for each N, one can pick $k(N)$, such that:

$$\beta(N) \equiv \sum_{n=1}^{k(N)} z_n^N \geq \beta \;\; for \; all \; N \tag{12}$$

Since Condition S holds, we have:

$$\gamma \equiv d(0, S) > 0 \tag{13}$$

Using Lemma 1 and (13), we obtain:

$$d(0, S(\beta)) \geq \beta\gamma \tag{14}$$

We now write:

$$\begin{aligned} d(x^N, x) &\geq f(z_1^N, ..., z_{k(N)}^N, 0[k(N)]) \\ &\geq f(\beta z_1^N/\beta(N), ..., \beta z_{k(N)}^N/\beta(N), 0[k(N)]) \\ &\geq \beta\gamma \end{aligned} \tag{15}$$

the first line of (15) following from (M.2), the second line from the fact that (using (12)) for each $N \in \mathbb{N}$, $\beta(N) \geq \beta$, and (M.2), and the last line from the fact that (using (10) again) for each $N \in \mathbb{N}$, $(\beta z_1^N/\beta(N), ..., \beta z_{k(N)}^N/\beta(N), 0[k(N)]) \in S(\beta)$, and (14). But this contradicts (10), establishing the result. ■

Proposition 2 *Suppose $d \in \Delta$ satisfies Condition S. Then, there is a social welfare ordering $\succsim$ on X, satisfying the Anonymity, Pareto and d-Continuity axioms simultaneously.*

Proof. The social welfare ordering $\succsim$ is specified exactly along the lines of Svensson (1980, p.1254-55). We specify it here in terms of slightly different notation.

(i) Define a relation E on X as follows:

$$xEy \; iff \; \sum_{n=1}^{\infty} |x_n - y_n| < \infty$$

It can be checked that E is an equivalence relation. Now, let:

$$X = \cup_{j \in J} X_j$$

be the partition of X in equivalence classes (X_j) corresponding to E. Let M be a set of representatives, exactly one from each set X_j; thus,

$$M = \{x^j : x^j \in X_j, \, j \in J\}$$

This can be done by the axiom of choice.

(ii) Define a relation R' on M as follows:

$$xR'y \; iff \; there \; is \; \bar{N} \in \mathbb{N}, such \; that \; \sum_{n=1}^{N} (x_n - y_n) \geq 0 \; for \; all \; N \geq \bar{N}$$

It can be checked that R' is a quasi-ordering on M.

(iii) By Szpilrajn's lemma, there is an ordering R'' on M such that R' is a subrelation to R''.

(iv) For any $j \in J$, define the function F_j by:

$$F_j(x) = \sum_{n=1}^{\infty} (x_n - x_n^j) \ \ for\ all\ x \in X_j$$

and define the relation R_j on X_j by:

$$xR_j y\ iff\ F_j(x) \geq F_j(y)$$

Clearly, R_j is an ordering on X_j, since F_j is well-defined on X_j.

(v) Define a relation $\succsim$ on X as follows. If $x \in X_j$ and $y \in X_i$, $x^j, x^i \in M$, then:

$$\begin{array}{l} (i)\ If\ x^j P'' x^i,\ then\ x \succ y \\ (ii)\ If\ x^j I'' x^i,\ then\ x \succsim y\ iff\ F_j(x) \geq F_i(y) \end{array}$$

One can follow Svensson (1980, p.1255) to check that $\succsim$ is an ordering on X, which satisfies the Anonymity and Pareto axioms.

It remains to check that $\succsim$ satisfies the d-Continuity axiom. To this end, let x, y be points in X, and let (x^N) be an arbitrary sequence of points in $LC(y)$, such that:

$$d(x^N, x) \rightarrow 0\ as\ N \rightarrow \infty \tag{16}$$

We have to show that $y \succsim x$. Let $x \in X_j$ and let $y \in X_i$. Using Lemma 2 and (16), we have:

$$\sum_{n=1}^{\infty} |x_n^N - x_n| \rightarrow 0\ as\ N \rightarrow \infty \tag{17}$$

Thus, there is some $\hat{N} \in \mathbb{N}$, such that $x^N \in X_j$ for all $N \geq \hat{N}$. If $x^j P'' x^i$, then by definition of $\succsim$, we must have $x^N \succ y$ for $N \geq \hat{N}$, which contradicts the fact that $x^N \in LC(y)$ for all $N \in \mathbb{N}$. Thus, we have $x^i R'' x^j$.

If $x^i P'' x^j$, then by definition of $\succsim$, we have $y \succ x$. Thus, it only remains to examine the case in which $x^i I'' x^j$. Since $x^N \in X_j$ for all $N \geq \hat{N}$ and $y \in X_i$, and $x^N \in LC(y)$ for all $N \in \mathbb{N}$, we must have by definition of $\succsim$,

$$F_j(x^N) \leq F_i(y) \ \ for\ all\ N \geq \hat{N} \tag{18}$$

Thus, we have for $N \geq \hat{N}$:

$$\begin{aligned} F_j(x) - F_i(y) &= F_j(x) - F_j(x^N) + F_j(x^N) - F_i(y) \\ &= \sum_{n=1}^{\infty} (x_n - x_n^N) + F_j(x^N) - F_i(y) \\ &\leq \sum_{n=1}^{\infty} (x_n - x_n^N) \end{aligned} \tag{19}$$

the inequality in the third line of (19) following from (18). Now, using (17), we obtain:

$$F_j(x) \leq F_i(y)$$

Thus, $y \succsim x$ by definition of $\succsim$, establishing the result. ■

The proof of the second theorem uses Lemma 1 to establish a proposition (Proposition 3) which shows that the topology induced by any metric $d \in \Delta$, satisfying the simplex condition, is stronger than the topology induced by the bounded ℓ_1 metric. Theorem 2 follows directly from the characterization result in Theorem 1 and this proposition.

Proposition 3 *Let d_1 denote the bounded ℓ_1 metric. Let $d \in \Delta$ be any metric satisfying Condition S. Then the topology induced by d is stronger than the topology induced by d_1.*

Proof. Let $x \in X$ and $\varepsilon > 0$ be given. We have to show that there is $\delta > 0$ such that:

$$B_d(x, \delta) \subset B_{d_1}(x, \varepsilon) \tag{20}$$

where $B_d(x, \delta)$ is the open ball with center x and radius δ in terms of the metric d, and $B_{d_1}(x, \varepsilon)$ is the open ball with center x and radius ε in terms of the metric d_1. [Note that, when (20) is valid, every open set in the d_1 metric topology is also open in the d metric topology].

Denote $d(0, S)$ by θ. Since $d \in \Delta$ satisfies Condition S, we know that $\theta > 0$. Choose $N \in \mathbb{N}$ such that $(1/N) < \varepsilon$. Then, choose $\delta > 0$ such that $\delta < (\theta/N)$.

Let $y \in B_d(x, \delta)$. We first establish that:

$$\sum_{k=1}^{\infty} |x_k - y_k| < \infty \tag{21}$$

Suppose that (21) is violated. Then, one can choose $n \in \mathbb{N}$, such that:

$$\sum_{k=1}^{n} |x_k - y_k| > 1$$

Now, defining $z \in X$ by: $z_k = y_k$ for $k = 1, ..., n$ and $z_k = x_k$ for $k > n$, we have:

$$A = \sum_{k=1}^{\infty} |x_k - z_k| = \sum_{k=1}^{n} |x_k - y_k| > 1 \tag{22}$$

Further, we have $d(x, y) = d(|x - y|, 0) \geq d(|x - z|, 0) = f(|x - z|) \geq f((1/A)|x - z|) = d((1/A)|x - z|, 0) \geq \theta$, since $(1/A)|x - z| \in S$, using (22). But, since $y \in B_d(x, \delta)$, we have $d(x, y) < \delta$. Thus, we must have $\delta > \theta$, which contradicts our choice of δ, and establishes the validity of (21). Denote the sum in (21) by C.

If $C = 0$, then $y = x$, and so $y \in B_{d_1}(x, \varepsilon)$, establishing (20). Thus, the only non-trivial case to consider is where $C > 0$. We claim that $C < \varepsilon$. For, if $C \geq \varepsilon$, we have:

$$d(|x - y|, 0) \geq d([(1/N)/C]|x - y|, 0) \tag{23}$$

since $(1/N) < \varepsilon \leq C$. Clearly, $[(1/N)/C]|x - y| \in S(1/N)$, and so by Lemma 1, we have:

$$d([(1/N)/C]|x - y|, 0) \geq (\theta/N) \tag{24}$$

Combining (23) and (24), we have $d(|x - y|, 0) \geq (\theta/N)$. But, since $y \in B_d(x, \delta)$, we have $d(x, y) < \delta$. Thus, we get $\delta > (\theta/N)$, a contradiction. This establishes our claim that $C < \varepsilon$. Thus, by (21), and the definition of the bounded ℓ_1 metric, we have $d_1(x, y) < \varepsilon$, so that $y \in B_{d_1}(x, \varepsilon)$, establishing (20). ∎

4 Concluding Remarks

We conclude with a couple of observations. First, the scope of our two main results (Theorems 1 and 2) are restricted to the class of metrics, satisfying conditions (M.1)-(M.4). Of these, (M.1), (M.2) and (M.4) appear to be very natural restrictions in this context. Condition (M.3), relating to shifts in time, is somewhat less obvious, although it is satisfied by a number of metrics, used in this literature. It is possible that (M.3) can be weakened somewhat, while preserving the results of the paper. This is left as an open question.

Second, there appears to be a close connection between the problem of characterization of metrics under which ethical preferences can be continuous, and the problem of characterization of metrics such that every Paretian and continuous social welfare ordering can be *represented* by a social welfare function. The connection can be seen informally as follows: we know from Basu and Mitra (2003) that there is no social welfare function which simultaneously satisfies the Anonymity and Pareto axioms. Thus, when the continuity requirement is strong (as in the sup-metric continuity of Diamond (1965)), every Paretian and continuous social welfare ordering can be represented by a social welfare function, so that by the Basu-Mitra result, Pareto, Anonymity and sup-metric Continuity cannot be simultaneously satisfied by any social welfare ordering (which is Diamond's result). On the other hand, when the continuity requirement is weak (as in the bounded ℓ_1 metric continuity of Svensson (1980)), we know that Pareto, Anonymity and ℓ_1 metric Continuity can be simultaneously satisfied by the social welfare order constructed by Svensson, so that by the Basu-Mitra result, such a social welfare order cannot be represented by a social welfare function. It would be useful to explore more formally the connection between the two problems.

References

[1] Basu, K. and Mitra, T.: Aggregating Infinite Utility Streams with Intergenerational Equity: the Impossibility of Being Paretian, *Econometrica* **71** (2003), 1557-1563.

[2] Brown, D.J. and Lewis, L.M.: Myopic Economic Agents, *Econometrica* **49** (1981), 359- 368.

[3] Campbell, D.: Impossibility Theorems and Infinite Horizon Planning, *Social Choice and Welfare* **2** (1985), 283-293.

[4] Diamond, P.: The Evaluation of Infinite Utility Streams, *Econometrica* **33** (1965), 170-177.

[5] Fleurbaey, M. and Michel, P.: Intertemporal Equity and the Extension of the Ramsey Criterion, *Journal of Mathematical Economics* **39** (2003), 777-802.

[6] Köthe, G.: *Topological Vector Spaces-I*, New York: Springer-Verlag, 1969.

[7] Lauwers, L.: Continuity and Equity with Infinite Horizons, *Social Choice and Welfare* **14** (1997), 345-356.

[8] Ramsey, F.P.: A Mathematical Theory of Savings, *Economic Journal*, **38** (1928), 543-559.

[9] Sakai, T.: An Axiomatic Approach to Intergenerational Equity, *Social Choice and Welfare* **20** (2003), 167-176.

[10] Sakai, T.: Equitable Intergenerational Preferences on Restricted Domain, forthcoming in *Social Choice and Welfare*, (2004).

[11] Shinotsuka, T.: Equity, Continuity, and Myopia : A Generalization of Diamond's Impossibility Theorem, *Social Choice and Welfare* **15** (1997), 21-30.

[12] Svensson, L.-G.: Equity among Generations, *Econometrica* **48** (1980), 1251-1256.