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A General Equilibrium Analysis of the Supply of Capital

by

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Abstract

The point of this paper is that if output is durable then optimal behavior of a supplier is characterized by production smoothing. Durability of goods (such as capital) has opposite effects on the supply of the goods. Higher durability on the one hand raises the variability of investment demand for the goods by lowering the user's cost, which tends to raise the variability of supply; on the other hand it lowers the expected future demand for the goods, which tends to reduce the variability of supply. These opposite effects of durability manifest in economies where suppliers of durable goods opt to use inventories to buffer demand shocks. Due to inventory adjustment and rational expectation, the variability of production can be reduced both absolutely and relative to sales if output is durable.

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1 Introduction

The demand of fixed capital, as well as its associated supply and inventory investment, are some of the most volatile economic variables in the United States and in many industrial economies. Understanding why this is the case is important for understanding the business cycle. Although it seems that the volatility associated with capital has to do with the fact that capital is a type of highly durable goods (business investment in durable structures is far more volatile than investment in the relatively less durable equipments), yet the exact mechanisms as to how the durability of goods affect their demand and supply is still an issue for investigation.[1] Standard textbook theory suggests that demand of durable goods is positively related to durability of the goods (i.e., it is negatively associated with the rate of depreciation) due to user's cost effect, hence production (supply) of durable goods should also be positively affected by durability. This argument, however, is incomplete and potentially misleading since it misses another important effect of durability on the supply of durable goods: when goods are durable, a higher current demand also implies a lower potential future demand for the goods. This intertemporal substitution effect of durability may mitigate the volatility of production and render production less variable than sales.

This point is demonstrated here in a general equilibrium framework in which capital suppliers (e.g., upstream firms) produce, store and sell capital goods to a competitive market to meet the investment demand of capital buyers (e.g., downstream firms). Due to production lags (e.g., time-to-built) and uncertainty in sales, the suppliers base production plans on expected future demand and opt to hold inventories to avoid possible stoucko uts. It is shown in this environment that despite durability raises the volatility of demand, it reduces the volatility of production, even if the cost of production is linear (so that the conventional

[1] The literature on the lumpiness of investment behavior deals with volatility of capital from the demand side. But this literature has left out the issue of capital supply with respect to capital goods production and its associated inventory behavior. See for example, Thomas (2002) and Kahn and Thomas (2002) and the reference therein.

cost-saving motive for production smoothing is absent).[2]

The rest of the paper is organized as follows. In section 2 a general equilibrium model of capital is presented. In section 3 closed-form policies for optimal demand, supply, inventory investment and equilibrium price of capital are derived. It is shown that if output is durable, then optimal behavior of firms will be characterized by production smoothing relative to sales. In section 4 two concrete examples are provided to show that the relationship between durability and absolute variance of production depends on the relative strength of the user's cost effect on demand. Finally, section 5 concludes the paper.

2 The Model

Downstream Firms: A representative buyer purchases capital goods (investment) and produces output according to the production technology,

$$f(k_t, \theta_t),$$

where k represents capital, θ is an $i.i.d$ random variable representing shocks to the firm's demand or productivity, and $f()$ satisfies

$$f'_k > 0, f''_{kk} < 0, f'_\theta > 0.$$

The market price for purchasing capital goods (cost of investment I) is λ_t which the firm takes as given. Assume full capacity utilization, the firm chooses sequences of either the capital stock, $\{k_{t+j}\}_{j=0}^{\infty}$, or the rate of investment, $\{I_{t+j}\}_{j=0}^{\infty}$, to maximize the discounted expected profit,

$$\max E_t \sum_{j=0}^{\infty} \beta^j \left[f(k_{t+j}, \theta_{t+j}) - \lambda_{t+j} I_{t+j} \right]$$

subject to

$$k_{t+j} = I_{t+j} + (1-\delta) k_{t+j-1};$$

where $\beta \in (0,1)$ is the discount factor and δ is the depreciation rate of capital.

[2] The stockout-avoidance motive for holding inventories in the model is similar to that studied by Abel (1985), Reagan (1982), and Kahn (1987). These authors, however, do not study durable goods and they all use partial equilibrium analysis in which price is exogenous.

Upstream Firms: A representative supplier produces capital goods (y_t) according to a linear production technology. This implies that the cost function is linear in output, ay_t, where a is a positive constant. Assume that there is a one period production lag between the commitment of input and the availability of output for sale (i.e., the firm must make production plans (y_t) one period in advance before demand for capital in period t is known), hence the total output available for sale in period t is the existing stock of inventories carried from last period (s_{t-1}) plus the current output (y_t) that was committed last period, $s_{t-1} + y_t$. Without loss of generality the depreciation rate for inventories is assumed to be zero and there is no other costs for holding inventories except the cost associated with time discounting, β. The firm takes expected output price (λ_t) and expected investment demand from buyers (I_t) as given and chooses sequences of production plans (y_t) and inventory investment ($s_t - s_{t-1}$) to maximizes's discounted expected profit,

$$\max_{\{y_{t+j}\}} E_{t-1} \left\{ \max_{\{s_{t+j}\}} \left\{ E_t \sum_{j=0}^{\infty} \beta^j \left[\lambda_{t+j} I_{t+j} - a y_{t+j} \right] \right\} \right\}$$

subject to

$$I_{t+j} + s_{t+j} = s_{t+j-1} + y_{t+j},$$

and

$$s_{t+j} \geq 0,$$
$$y_{t+j} \geq 0,$$

where the expectation operators, $\{E_{t-1}, E_t\}$, indicate the relevant information sets when decisions are made.

Competitive Equilibrium: A competitive equilibrium is a set of decision rules for capital sales (I_t), capital production (y_t), inventory holdings (s_t) and the price of capital (λ_t) such that the following first order conditions hold:

$$f'_k(k_t, \theta_t) = \lambda_t - \beta(1-\delta) E_t \lambda_{t+1} \tag{1}$$

$$a = E_{t-1} \lambda_t \tag{2}$$

$$\lambda_t = \beta E_t \lambda_{t+1} + \pi_t \tag{3}$$

$$[k_t - (1-\delta)k_{t-1}] + s_t = s_{t-1} + y_t \tag{4}$$

$$\pi_t s_t = 0 \tag{5}$$

where equation (1) determines the buyer's optimal demand for capital, equation (2) determines the supplier's optimal production of capital, equation (3) determines the supplier's optimal inventory holdings, equation (4) is the capital goods market clearing condition, and equation (5) is the Kuhn-Tucker condition for the nonnegativity constraint on the supplier's inventories (hence π is the complementarity slackness multiplier).[3]

Equation (1) shows that the optimal demand for capital decreases when δ increases (i.e., when the durability of goods decreases), holding capital prices constant. This is the familiar user's cost effect of durability on demand. Equation (2) shows that the optimal supply of capital goods is chosen to the point such that the marginal cost of production (a) equals the expected value of capital in the goods market (λ_t). Equation (3) shows that the optimal level of inventories held by the supplier is determined by the point where the cost of increasing one extra unit of inventory holdings, which is the opportunity cost for not selling the good (λ_t), equals the discounted expected benefit of having one more unit of inventories available for sale next period (λ_{t+1}) plus the benefit of relaxing the slackness constraint by one unit (π_t), which is zero if the constraint does not bind. The intertemporal substitution effect of durability on future demand lies in the relationship,

$$I_t = k_t - (1-\delta)k_{t-1},$$

where purchase of the capital stock last period reduces the current investment demand for capital. The more durable is the good, the larger such effect is.

The intriguing question is, how these two opposite effects of durability, the user's cost effect and the intertemporal substitution effect, affect the supply of capital goods in general equilibrium? It is shown below that despite durability raises the volatility of demand for capital, it nonetheless can reduce the

[3]Given that investment demand is always positive (since $f'_k > 0$), the nonnegativity constraint on production will never bind. Hence the constraint, $y \geq 0$, is ignored.

volatility of capital production. This mechanism of production smoothing differs fundamentally from that arising from increasing marginal cost of production (see Blinder 1986 for production smoothing behaviors under convex cost of production).

3 Optimal Supply of Capital

To characterize equilibrium decision rules of the model, consider two possibilities: the realized value of the shock (θ) and the associated investment demand for capital are either below "normal" or above "normal", such that the nonnegativity constraint on inventory is either non-binding or binding.

Case A: If θ is below normal, suggesting that the investment demand for capital is low, hence the nonnegativity constraint on inventories does not bind. Hence $\pi_t = 0$ and $s_t \geq 0$. Equations (2) and (3) imply that the competitive price of capital is constant[4],

$$\lambda_t = \beta a.$$

Hence equation (1) implies

$$f_k'(k_t, \theta_t) = \beta \delta a,$$

which gives the optimal capital demand under case A as an increasing function of θ,

$$k_t = k^*(\theta_t), \quad \text{where } \frac{\partial k^*(\theta)}{\partial \theta} > 0.$$

The market clearing condition (4) then implies

$$s_t = y_t + s_{t-1} + (1-\delta)k_{t-1} - k^*(\theta_t).$$

The threshold value for θ is determined by the constraint, $s_t \geq 0$, which implies

$$k_t^*(\theta_t) \leq y_t + s_{t-1} + (1-\delta)k_{t-1}, \tag{6}$$

or

$$\begin{aligned} \theta_t &\leq (k^*)^{-1}\left(y_t + s_{t-1} + (1-\delta)k_{t-1}\right) \\ &\equiv z(y_t), \end{aligned} \tag{7}$$

[4] This implies that goods price is downward sticky in an inventory economy. See Blinder (1982), Amihud and Mendelson (1983) and Reagan (1982) for more discussions on this issue.

where $z(y)$ denotes the optimal cutoff point for θ such that there is a stockout if $\theta > z$. Namely, z_t is defined as

$$k^*(z_t) \equiv y_t + s_{t-1} + (1-\delta)k_{t-1}. \tag{8}$$

Since $k^*(\theta)$ is a monotonically increasing function, we have

$$\frac{\partial k^*(z)}{\partial z} > 0 \text{ and } \frac{\partial z(y)}{\partial y} > 0. \tag{8'}$$

Case B: If investment demand is above normal due to a large shock, then the nonnegativity constraint on inventories binds. Hence $\pi_t > 0$ and $s_t = 0$. The market-clearing condition (4) implies that the investment demand is met with the entire existing stock of goods,

$$k_t - (1-\delta)k_{t-1} = y_t + s_{t-1}. \tag{9}$$

Clearly, the probabilities of case A an case B depend on the production level committed last period, y_t. To determine the optimal production policy, we can utilize equation (2). Denote $\phi()$ as the probability density function of θ with support $[A, B]$, then equation (2) can be expanded as

$$\begin{aligned} a &= E_{t-1}\lambda_t \qquad (10) \\ &= \int_A^{z(y_t)} \beta a \phi(\theta) d\theta + \int_{z(y_t)}^{B} \left[f_k' (k_t, \theta_t) + \beta(1-\delta)a \right] \phi(\theta) d\theta \end{aligned}$$

where the cutoff point that determines the probability of stocking out, $z(y)$, is defined in (8).

The interpretation of (10) is straightforward. The expected value of λ is a probability distribution of two terms: $\lambda = \beta a$ if the realized shock is small so that there is no stockout ($\pi = 0$); or $\lambda = f_k'(k, \theta) + \beta(1-\delta)a$ if the realized shock is large so that there is a stockout ($\pi > 0$). In the later case the optimal level of capital demand (k_t) is given by (9). In other words, the left-hand side of (10) is the cost of producing one extra unit of capital goods today, a. The marginal benefit of having one extra unit of capital goods available next period is given by the right-hand side of (10) with two possibilities. First, in the event of no stockout due to a low demand, the firm gets to save on the marginal cost of production by postponing production for one period. The present value of this term is βa and this event happens with probability $\int_A^{z(y)} \phi(\theta)d\theta$. Second, in the event of a stockout due to a high demand, the firm can sell the product at the competitive market price, λ_t, which equals the marginal product of capital plus the present market value of the nondepreciated part, $f'_k(k,\theta) + \beta(1-\delta)a$, where k is determined by (9) under case B. This event happens with probability $\int_{z(y)}^B \phi(\theta)d\theta$.

Clearly, the probability of stocking out in period t, $\int_{z(y)}^B \phi(\theta)d\theta$, is determined by the level of production (y) committed in period $t-1$. The larger is y, the more inventory the firm has (i.e., the larger $z(y)$ is), hence the smaller the probability of stocking out is. Since holding inventories is costly due to time discounting, and stocking out is also costly due to loss of opportunities for sale, the level of production is determined to the point where the expected marginal revenue ($E_{t-1}\lambda_t$) equals marginal cost (a).

Proposition 1 *An optimal cutoff point (which is also the optimal inventory target of the supplier),* $z(y) \in [A, B]$*, exists and it is unique and also constant,* $z(y) = \bar{z}$*. Furthermore,* $\bar{z}$ *positively depends on the variance of* θ*.*

Proof. Rewrite (10) as (by substituting out k_t using equation 9):

$$\begin{aligned} a &= \int_A^{z(y_t)} \beta a\phi(\theta)d\theta + \int_{z(y_t)}^B \left[f'_k(k_t,\theta_t) + \beta(1-\delta)a\right]\phi(\theta)d\theta \\ &= \int_A^{z(y_t)} \beta a\phi(\theta)d\theta + \int_{z(y_t)}^B \left[f'_k\left((y_t + s_{t-1} + (1-\delta)k_{t-1}),\theta_t\right) + \beta(1-\delta)a\right]\phi(\theta)d\theta \\ &= \int_A^{z(y_t)} \beta a\phi(\theta)d\theta + \int_{z(y_t)}^B \left[f'_k\left(k^*(z_t),\theta_t\right) + \beta(1-\delta)a\right]\phi(\theta)d\theta, \end{aligned}$$

where the last equality utilized the definition of $z(y)$. The above equation can be simplified (after rearranging terms) to:

$$(1-\beta)a = \int_{z(y_t)}^{B} \left[f_k'\left(k^*(z_t),\theta_t\right) - \beta\delta a\right]\phi(\theta)d\theta \tag{11}$$

$$\equiv \int_{z_t}^{B} g(z_t,\theta_t)\phi(\theta)d\theta.$$

Notice that $k^*(z)$ is an increasing function of z (see equation $8'$), hence f_k' is a decreasing function of z. Thus, $g_z' = f_{kk}'' \frac{\partial k^*(z)}{\partial z} < 0$. Since $g > 0$ (by equation 1, $f_k' > \beta\delta a$ under case B)[5], hence clearly, the right-hand side of (11) is monotonically decreasing in z :

$$\frac{\partial \int_{z_t}^{B} g(z_t,\theta_t)\phi(\theta)d\theta}{\partial z} = -g(z,z)f(z) + \int_{z}^{B} g_z'\phi(\theta)d\theta < 0.$$

It is easy to see that the minimum of the right-hand side of (11) is zero when $z = B$ and the maximum is greater than $(1-\beta)\,a$ when $z = A$ (since $f_k'(k^*(A),\theta_t)$ can be made arbitrarily large as $A \to -\infty$ by assuming that f_k' is sufficiently diminishing in k). Hence a unique solution for z_t exists. Furthermore, since θ is $i.i.d$, the right-hand side of (11) after integration is an implicit function in the form, $G(z_t,\Omega) = 0$, where Ω is a set of constant parameters. Hence, z_t must be a constant, $z_t = \bar{z}$, which solves $G(\bar{z},\Omega) = 0$ or

$$(1-\beta)a = \int_{\bar{z}}^{B} g(\bar{z},\theta_t)\phi(\theta)d\theta. \tag{12}$$

Now, consider an increase in the variance of θ that preserves the mean (i.e., an increase in the value of B by a symmetric expansion of the interval $[A,B]$). (12) indicates that $\bar{z}$ must also increase in order to maintain the equality.■

Proposition 2 *The equilibrium decision rules for demand, supply, inventory investment and market price of capital are given by*

$$k_t = \begin{cases} k^*(\theta_t)\,, \; if\ \theta_t \le \bar{z} \\ k^*(\bar{z})\;\;, \; if\ \theta_t > \bar{z} \end{cases}$$

[5] $E_t\lambda_{t+1} = a$ by equation (2).

$$I_t = \begin{cases} k^*(\theta_t) - (1-\delta)k^*(\theta_{t-1}) \, , \textit{if } \theta_t \leq \bar{z} \;\&\; \theta_{t-1} \leq \bar{z} \\ k^*(\theta_t) - (1-\delta)k^*(\bar{z}) \quad \, , \textit{if } \theta_t \leq \bar{z} \;\&\; \theta_{t-1} > \bar{z} \\ k^*(\bar{z}) - (1-\delta)k^*(\theta_{t-1}) \, , \textit{if } \theta_t > \bar{z} \;\&\; \theta_{t-1} \leq \bar{z} \\ \delta k^*(\bar{z}) \quad\quad\quad\quad\quad\quad\quad \, , \textit{if } \theta_t > \bar{z} \;\&\; \theta_{t-1} > \bar{z} \end{cases}$$

$$y_t = \begin{cases} \delta k^*(\theta_{t-1}) \, , \textit{if } \theta_{t-1} \leq \bar{z} \\ \delta k^*(\bar{z}) \quad\;\; , \textit{if } \theta_{t-1} > \bar{z} \end{cases}$$

$$s_t = \begin{cases} k^*(\bar{z}) - k^*(\theta_t) \, , \textit{if } \theta_t \leq \bar{z} \\ 0 \quad\quad\quad\quad\quad\;\; , \textit{if } \theta_t > \bar{z} \end{cases}$$

$$\lambda_t = \begin{cases} \beta a \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\;\; , \textit{if } \theta_t \leq \bar{z} \\ [f_k'(k^*(\bar{z}), \theta_t) + \beta(1-\delta)a] \, , \textit{if } \theta_t > \bar{z} \end{cases}$$

where the constant $\bar{z}$ is the optimal inventory target set by the supplier of capital goods.

Proof. By proposition (1) and equation (8), the optimal production policy is given by

$$y_t = k^*(\bar{z}) - s_{t-1} - (1-\delta)k_{t-1}.$$

Substituting this into the values of inventory (s_t) discussed above under case A and case B respectively gives

$$s_t = \begin{cases} k^*(\bar{z}) - k^*(\theta_t) \text{ if } \theta_t \leq \bar{z} \\ 0 \quad\quad\quad\quad\quad\;\; \text{if } \theta_t > \bar{z} \end{cases}.$$

Similarly, we have

$$k_t = \begin{cases} k^*(\theta_t) \text{ if } \theta_t \leq \bar{z} \\ k^*(\bar{z}) \;\; \text{if } \theta_t > \bar{z} \end{cases}.$$

Shifting the time subscribe backward by one period for s_t and k_t and then substituting them into the production policy give

$$y_t = \begin{cases} \delta k^*(\theta_{t-1}) \text{ if } \theta_{t-1} \leq \bar{z} \\ \delta k^*(\bar{z}) \quad\; \text{if } \theta_{t-1} > \bar{z} \end{cases}.$$

The other decision rules follow straightforwardly.■

Notice that the competitive market price of capital, λ_t, has the property described by Reagan (1982). Namely, it is downward sticky when demand is low

$(\lambda_t = \beta a)$, because firms opt to hold inventories rather than to sell them at a price below marginal cost, speculating that demand may be stronger in the future. Such rational behavior attenuates downward pressure on price. When realized demand is high, on the other hand, the firm draws down its inventories until a stockout occurs and price rises to clear the market $(\lambda_t = [f_k'(k^*(\bar{z}), \theta) - \beta\delta a] + \beta a > \beta a$ is an increasing function of $\theta)$.

Proposition 3 *The volatility of production relative to that of sales decreases as the durability of the goods increases. Furthermore, the variance ratio of production to sales is always less than one as long as $\delta < 1$.*

Proof. Denote $P \equiv \Pr[\theta \leq \bar{z}]$ and denote σ_k^2 as the variance of capital. Then the variance of production and sales (investment demand) are given respectively by

$$\sigma_y^2 = P\delta^2\sigma_k^2$$

$$\sigma_I^2 = P^2\left[1 + (1-\delta)^2\right]\sigma_k^2 + P(1-P)\left[1 + (1-\delta)^2\right]\sigma_k^2 = P\left[1 + (1-\delta)^2\right]\sigma_k^2$$

Hence the variance ratio of production to sales is given by

$$\frac{\sigma_y^2}{\sigma_I^2} = \frac{\delta^2}{1 + (1-\delta)^2}$$

which is strictly less than one (unless $\delta = 1$) and strictly increasing in δ.■

The intuition is as follows. As plans for current production cannot be altered, any rise in current sales must be satisfied entirely by a reduction in inventories. On its own, this implies a one-for-one rise in the production committed for the next period to replenish the depleted inventory stock. However, if goods are durable, increased purchase in the current period raises buyers' stock of goods available for subsequent periods, reducing the anticipated increase in future sales, and hence the response in production as well.

4 Two Examples

Proposition 3 deals only with the relative volatility of production to sales. What happens to the absolute variance of production, however, depends on the details of the model, in particular, the specific functional forms of $f(k,\theta)$. This is so not only because a higher value of δ increases the user's cost of capital, lowering the optimal demand for capital and reducing the volatility of k^*, hence $\frac{\partial \sigma_k^2}{\partial \delta} < 0$; but because the optimal inventory target ($\bar{z}$) may also be affected by δ, causing the value of $P \equiv \Pr[\theta \leq \bar{z}]$ to change as δ changes. In other words, the total effect of a change in δ on the volatility of production is given by three terms,

$$\frac{\partial \sigma_y^2}{\partial \delta} = 2\delta P \sigma_k + \delta^2 \frac{\partial \sigma_k^2}{\partial \delta} P + \delta^2 \sigma_k^2 \frac{\partial P}{\partial \delta},$$

where the first terms shows a direct positive effect of δ on the volatility of y due to the intertemporal substitution effect of durability on future demand (i.e., a higher δ raises the anticipated future demand for capital), the second term shows a negative effect of δ on the volatility of y due to the user's cost effect (i.e., a higher δ lowers the current demand for capital), and the third term shows the effect of δ on the firm's inventory target policy ($\bar{z}$), which is likely positive but is not clear-cut unless the demand function of capital, $k^*()$, and the probability distribution function of θ, $\phi()$, are fully specified. In what follows I give two examples where the first example shows clearly that the absolute volatility of production decreases as the durability of goods increases, and the second example shows the possibility that the opposite may be true.

Economy 1: The production function for the buyer is given by the quadratic form,

$$f(k_t, \theta_t) = \theta_t k_t - \frac{1}{2} k_t^2.$$

Proposition 4 *In this economy the equilibrium decision rules for demand, supply, inventory investment and market price of capital are given by*

$$k_t = \begin{cases} \theta_t - \beta\delta a \;\textit{if}\; \theta_t \leq \bar{z} \\ \bar{z} - \beta\delta a \;\; \textit{if}\; \theta_t > \bar{z} \end{cases}$$

$$I_t = \begin{cases} \theta_t - (1-\delta)\theta_{t-1} - \beta\delta^2 a \;\textit{if}\; \theta_t \leq \bar{z} \;\&\; \theta_{t-1} \leq \bar{z} \\ \theta_t - (1-\delta)\bar{z} - \beta\delta^2 a \quad \textit{if}\; \theta_t \leq \bar{z} \;\&\; \theta_{t-1} > \bar{z} \\ \bar{z} - (1-\delta)\theta_{t-1} - \beta\delta^2 a \;\textit{if}\; \theta_t > \bar{z} \;\&\; \theta_{t-1} \leq \bar{z} \\ \delta\bar{z} - \beta\delta^2 a \qquad\qquad\quad \textit{if}\; \theta_t > \bar{z} \;\&\; \theta_{t-1} > \bar{z} \end{cases}$$

$$y_t = \begin{cases} \delta\theta_{t-1} - \beta\delta^2 a \; if \; \theta_{t-1} \leq \bar{z} \\ \delta\bar{z} - \beta\delta^2 a \quad\; if \; \theta_{t-1} > \bar{z} \end{cases}$$

$$s_t = \begin{cases} \bar{z} - \theta_t \; if \; \theta_t \leq \bar{z} \\ 0 \qquad\; if \; \theta_t > \bar{z} \end{cases}$$

$$\lambda_t = \begin{cases} \beta a \qquad\qquad if \; \theta_t \leq \bar{z} \\ \theta_t - \bar{z} + \beta a \; if \; \theta_t > \bar{z} \end{cases}.$$

Proof. The marginal product of capital is given by $\theta_t - k_t$ and the capital demand function $k^*()$ is given by

$$k^*(x) = x - \beta\delta a$$

where $x = \theta$ in case there is no stockout ($\theta \leq \bar{z}$) and $x = \bar{z}$ in case there is a stockout ($\theta > \bar{z}$). Substituting $k^*(x)$ into the decision rules in proposition 2 gives the desired results.■

Proposition 5 *In this economy the inventory target, $\bar{z}$, is independent of δ.*

Proof. Applying the decision rule for k_t, equation (11) now becomes,

$$\begin{aligned}(1-\beta)a &= \int_z^B \left[\theta_t - k^*(z) - \beta\delta a\right]\phi(\theta)d\theta \\ &= \int_{\bar{z}}^B \left[\theta - \bar{z}\right]\phi(\theta)d\theta.\end{aligned}$$

Clearly, $\bar{z}$ is independent of δ.■

Thus, the parameter $P \equiv \Pr\left[\theta \leq \bar{z}\right]$ is also independent of δ. Based on the decision rules, the variances of demand and production can be found as:

$$\sigma_I^2 = P\left[1 + (1-\delta)^2\right]\sigma_\theta^2$$

$$\sigma_y^2 = P\delta^2\sigma_\theta^2$$

Since P is independent of δ, we have

$$\frac{\partial\sigma_I^2}{\partial\delta} < 0 \text{ and } \frac{\partial\sigma_y^2}{\partial\delta} > 0.$$

Namely, despite the variance of capital demand increases as the durability increases (indicating a user's cost effect), the variance of production decreases nonetheless, indicating that the intertemporal substitution effect dominates the user's cost effect on production. However, note that the relative volatility ratio of production to sales is still given by $\frac{\delta^2}{1+(1-\delta)^2} < 1$.

Economy 2: The production function for the buyer is given by the constant elasticity form,

$$f(k_t, \theta_t) = \frac{\theta_t^{\gamma} k_t^{1-\gamma}}{1-\gamma}, \quad 1 \geq \gamma \geq 0.$$

Proposition 6 *In this economy the equilibrium decision rules for demand, supply, inventory investment and market price of capital are given by*

$$k_t = \begin{cases} \left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \theta_t \text{ if } \theta_t \leq \bar{z} \\ \left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \bar{z} \text{ if } \theta_t > \bar{z} \end{cases}$$

$$I_t = \begin{cases} \left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \theta_t - (1-\delta)\left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \theta_{t-1} & \text{if } \theta_t \leq \bar{z} \;\&\; \theta_{t-1} \leq \bar{z} \\ \left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \theta_t - (1-\delta)\left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \bar{z} & \text{if } \theta_t \leq \bar{z} \;\&\; \theta_{t-1} > \bar{z} \\ \left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \bar{z} - (1-\delta)\left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \theta_{t-1} & \text{if } \theta_t > \bar{z} \;\&\; \theta_{t-1} \leq \bar{z} \\ \delta\left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \bar{z} & \text{if } \theta_t > \bar{z} \;\&\; \theta_{t-1} > \bar{z} \end{cases}$$

$$y_t = \begin{cases} \delta\left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \theta_{t-1} & \text{if } \theta_{t-1} \leq \bar{z} \\ \delta\left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \bar{z} & \text{if } \theta_{t-1} > \bar{z} \end{cases}$$

$$s_t = \begin{cases} \left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} (\bar{z} - \theta_t) & \text{if } \theta_t \leq \bar{z} \\ 0 & \text{if } \theta_t > \bar{z} \end{cases}$$

$$\lambda_t = \begin{cases} \beta a & \text{if } \theta_t \leq \bar{z} \\ \left(\left[\frac{\theta_t}{\bar{z}}\right]^{\gamma} - 1\right)\beta\delta a + \beta a & \text{if } \theta_t > \bar{z} \end{cases}.$$

Proof. The marginal product of capital is given by $\left(\frac{\theta}{k}\right)^{\gamma}$ and the capital demand function is given by

$$k^*(x) = \left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} x$$

where $x = \theta$ in case there is no stockout $(\theta \leq \bar{z})$ and $x = \bar{z}$ in case there is a stockout $(\theta > \bar{z})$. Substituting $k^*(x)$ into the decision rules in proposition 2 gives the desired results.■

Proposition 7 *In this economy the inventory target, $\bar{z}$, positively depends on δ.*

Proof. Applying the decision rule for k_t in this economy to equation (11) gives,

$$\begin{aligned}(1-\beta)a &= \int_z^B \left[\left(\frac{\theta_t}{k^*(\bar{z})}\right)^{\gamma} - \beta\delta a\right]\phi(\theta)d\theta \\ &= \int_z^B \left[\left(\frac{\theta_t}{\left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}}\bar{z}}\right)^{\gamma} - \beta\delta a\right]\phi(\theta)d\theta \\ &= \int_z^B \beta\delta a\left[\left(\frac{\theta_t}{z}\right)^{\gamma} - 1\right]\phi(\theta)d\theta,\end{aligned}$$

which can also be expressed as

$$\frac{(1-\beta)}{\beta\delta} = \int_{\bar{z}}^B \left[\left(\frac{\theta_t}{\bar{z}}\right)^{\gamma} - 1\right]\phi(\theta)d\theta.$$

Since the right hand side is decreasing in $\bar{z}$, hence $\bar{z}$ positively depends on δ.■

Thus, the probability measure, $P \equiv \Pr[\theta \leq \bar{z}]$, also positively depends on δ. Based on the decision rules, the variances of investment demand and production can be found as:

$$\sigma_I^2 = \left[\frac{1}{\delta}\right]^{\frac{2}{\gamma}}\left[\frac{1}{\beta a}\right]^{\frac{2}{\gamma}}\left[1+(1-\delta)^2\right]P\sigma_\theta^2$$

$$\sigma_y^2 = \left[\frac{1}{\delta}\right]^{\frac{2(1-\gamma)}{\gamma}}\left[\frac{1}{\beta a}\right]^{\frac{2}{\gamma}}P\sigma_\theta^2$$

Clearly, holding P constant, we have

$$\frac{\partial \sigma_I^2}{\partial \delta} < 0 \text{ and } \frac{\partial \sigma_y^2}{\partial \delta} < 0.$$

Hence, as long as P does not increase too fast when δ increases, the volatility of both investment demand and capital production may both increase as the durability increases (i.e., as the rate of depreciation decreases), provided that the inventory target does not move substantially with δ and/or the cumulative density function for θ is sufficiently flat near $\bar{z}$). This situation is certainly a possibility. Nonetheless, the volatility ratio of production to sales is still given by $\frac{\delta^2}{1+(1-\delta)^2} < 1$, hence the relative volatility of production to sales will still be a decreasing function of the durability because the volatility of sales increases faster than that of production as the durability increases (proposition 3).

5 Conclusion

This paper uncovers a different mechanism of production smoothing arising from durability of capital goods.[6] Closed form decision rules for demand, supply, inventory investment and competitive price of capital goods are characterized in general equilibrium. It is shown that although higher durability on the one hand raises the volatility of capital demand by lowering the user's cost, which in turn raises the volatility of capital production, on the other hand it reduces the volatility of production relative to sales by lowering the anticipated future demand of capital. Under a stockout-avoidance motive for holding inventories these effects lead to production smoothing in spite of linear cost in production. Thus, in order to explain why capital goods production is more variable than sales and why the supply of more durable capital goods is more volatile than the supply of less durable capital goods in the real world (e.g., Blinder and Maccini 1991), one may have to rely on other economic mechanisms to overcome the production smoothing effect of durability, such as nonconvex costs (e.g., Ramey 1991), supply-side shocks (e.g., Blanchard 1983, Blinder 1986, Christiano 1988,

[6] This mechanism is different from that studied by Abel (1985). The model studied in this paper is closer to that in Kahn (1987). Kahn's model, however, is partial equilibrium and he does not consider durable goods.

Eichenbaum 1989, Kydland and Prescott 1982, and West 1986, among others), or the (S,s) model (e.g., see Caballero and Engel 1999, Fisher and Hornstein 2000, and Kahn and Thomas 2002). These implications of the model apply also to durable consumption goods, since one can reinterpret the production function in the model, $f(k, \theta)$, as the utility function and the stock of capital (k) as the stock of consumption goods (see Wen 2003 for analysis of durable consumption goods in a similar framework).

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